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BOUNDEDNESS OF SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATION SYSTEMS

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There is a theorem in [1] concerning boundedness of solutions of a non-linear differential equation of order two

$$x'' + a(t)f(x)) = 0$$

and a generalization of this theorem to a system

$$x_i'' + a_i(t)\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, \dots n .$$

In [2] this result is generalized to the system

$$x_i'' + a_i(t)\sum_{k=1}^{n} b_{i,k}(t)x_k' + a_i(t)\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, \dots, n,$$

where the function F is, among other conditions, assumed to be a function of $x_1, \dots, x_n$ and therefore independent of t. In [3] some results are proved concerning boundedness, oscillatoriness and extension of solutions of several types of nonlinear differential equations of order two.

The aim of the present paper is the investigation of boundedness of solutions of non-linear differential equation systems. Some results are given which are generalizations of those appearing in [1], [2] and [3].

Consider a non-linear differential equation system of the form

$$x_i'' + f_i(t, x_1, \dots, x_n) = 0, \quad i = 1, 2, \dots, n , \tag{1}$$

where $f_i(t, x_1, \dots, x_n)$ and $\dfrac{\partial f_i}{\partial t}$ are defined and continuous for $t \geqq t_0 \geqq 0$, $\sum\limits_{i=1}^{n} |x_i| < \infty$. Suppose further that $f_i(t, x_1, \dots, x_n)$ are such that

$$\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_k} = 0 \quad \text{for} \quad k \neq i,\ k = 1, 2, \dots, n ,$$

where $F_i\ (t,x_1,\ldots,\ x_n) = \int\limits_0^{x_i} f_i\ (t,x_1,\ldots,\ x_{i-1},\ s,\ x_{i+1},\ldots,x_n)\,\mathrm{d}s$. Let $F(t,\mathbf{x}) =$ $= F(t, x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} F_i(t, x_1, \ldots, x_n)$.

Theorem 1. *Suppose that for every continuously differentiable vector function* $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ *which is defined on the interval* $(t_0, \bar{t})$, $\bar{t} \leqq \infty$ *and unbounded for* $t \to \bar{t}_-$, *there exists a sequence* $\{t_k\}_{k=1}^{\infty}$, *such that*

$$\frac{\partial F(t, \mathbf{x}(t))}{\partial t} \leqq \frac{\partial F(t, \mathbf{x}(t_k))}{\partial t}, \quad t_0 \leqq t \leqq t_k \tag{2}$$

and

$$\lim_{k\to\infty} F(t_0, \mathbf{x}(t_k)) = F, \tag{3}$$

where $F \leqq \infty$ *is independent of* $x(t)$.

Then every solution $\mathbf{x}(t)$ *of the system* (1) *,which satisfies the relation*

$$\tfrac{1}{2}\, \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) < F, \tag{4}$$

is bounded on its domain $\langle t_0, \infty)$.

($\|\,.\,\|$ stands for the Euclidean norm).

Proof. Suppose that a vector function $\mathbf{x}(t)$ is a solution of the system (1), satisfies the relation (4) and is nevertheless unbounded for $t \to \bar{t}_-$, where $\langle t_0, \bar{t})$ is an interval on which this solution is defined. This means that there exists a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k \to \bar{t}_-$ for $k \to \infty$ such that $\lim\limits_{k\to\infty} \|\mathbf{x}(t_k)\| = +\infty$.

By multiplying the i-th equation of the system (1) by the function $x_i'(t)$, summing over $i = 1, 2, \ldots, n$ and then integrating over the interval (t_0, t), where $t \in (t_0, \bar{t})$. We get

$$\tfrac{1}{2}\, \|\mathbf{x}'(t)\|^2 + \int\limits_{t_0}^{t} \sum_{i=1}^{n} f_i(s, x_1(s), \ldots, x_n(s)) x_i'(s)\, \mathrm{d}s = \tfrac{1}{2}\, \|\mathbf{x}'(t_0)\|^2$$

and therefore, since

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_k} \right) x_k'(t) = \frac{\partial F}{\partial t} + \sum_{k=1}^{n} f_k(t, x_1, \ldots, x_n) x_k',$$

$$\tfrac{1}{2}\, \|\mathbf{x}'(t)\|^2 + F(t, \mathbf{x}(t)) = \tfrac{1}{2}\, \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) + \int\limits_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\, \mathrm{d}s. \tag{5}$$

From this, taking into account (2), we get

$$F(t_k, \mathbf{x}(t_k)) \leqq \tfrac{1}{2}\,\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) + \int_{t_0}^{t_k} \frac{\partial F(s, \mathbf{x}(t_k)}{\partial s}\,\mathrm{d}s =$$

$$= \tfrac{1}{2}\,\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) + F(t_k, \mathbf{x}(t_k)) - F(t_0, \mathbf{x}(t_k))\,,$$

or

$$F(t_0, \mathbf{x}(t_k)) \geqq \tfrac{1}{2}\,\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0))\,,$$

which means that for $k \to \infty$ we have

$$F \leqq \tfrac{1}{2}\,\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0))\,,$$

which contradicts the assumption that $\mathbf{x}(t)$ satisfies the relation (4).

It is now necessary to prove that $\bar{t} = +\infty$, or that every solution satisfying the condition (4) can be extended to $\langle t_0, \infty)$.

Let $\bar{t} < \infty$. It is enough to prove that there exist finite limits $\lim\limits_{t\to\bar{t}-} \mathbf{x}(t)$ and $\lim\limits_{t\to\bar{t}-} \mathbf{x}'(t)$. As $\mathbf{x}(t)$ is bounded on $\langle t_0, \bar{t})$, clearly every component $x_i(t)$ of the vector $\mathbf{x}(t)$ is bounded. If the $\lim\limits_{t\to\bar{t}-} \mathbf{x}(t)$ does not exist, the same must be true for at least one component limit $\lim\limits_{t\to\bar{t}-} x_i(t)$. By the corresponding Lemma in [3] $\limsup\limits_{t\to\bar{t}-} x_i'(t) = +\infty$ and $\liminf\limits_{t\to\bar{t}-} x_i'(t) = -\infty$, so that there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \bar{t}_-$ for $k \to \infty$ and $\lim\limits_{k\to\infty} x_i'(t_k) = +\infty$.

For $t = t_k$ we get from (1)

$$x_i'(t_k) = x_i'(t_0) - \int_{t_0}^{t_k} f_i(s, x_1(s), \ldots, x_n(s))\,\mathrm{d}s$$

and therefore

$$\lim_{k\to\infty} \int_{t_0}^{t_k} f_i(s, x_1(s), \ldots, x_n(s))\,\mathrm{d}s = -\infty\,.$$

But this contradicts the assumption that $\bar{t} < \infty$, as $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ is bounded for $t \in \langle t_0, \bar{t})$ and the functions $f_i(t, x_1, \ldots, x_n)$ are continuous for $t \geqq t_0 \geqq 0$, $\sum\limits_{i=1}^{n} |x_i| < \infty$. This completes the proof.

Remark 1. Evidently if $F = +\infty$, then every solution of (1) is bounded on $\langle t_0, \infty)$.

Theorem 2. *Suppose that for every* $t \geqq t_0 \geqq 0$, $\sum_{i=1}^{n} |x_i| < \infty$ *we have*

$$\frac{\partial F(t, \mathbf{x})}{\partial t} \leqq 0 . \tag{6}$$

If for every sequence $\{t_k\}_{k=1}^{\infty}$ *such that* $t_k \to \infty$ *for* $k \to \infty$ *and every sequence* $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ *such that* $\|\mathbf{x}^{(k)}\| \to \infty$ *for* $k \to \infty$ *we have*

$$\lim_{k\to\infty} F(t_k, \mathbf{x}^{(k)}) = F , \tag{7}$$

then every solution of (1) *which satisfies the condition*

$$\tfrac{1}{2} \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) < F , \tag{8}$$

is bounded on $\langle t_0, \infty)$.

Proof: If the solution $\mathbf{x}(t)$ satisfies the condition (8) and is defined on $\langle t_0, \infty)$, the proof is simple. Using (5) and (6) we get

$$\tfrac{1}{2} \|\mathbf{x}'(t)\|^2 + F(t, \mathbf{x}(t)) \leqq \tfrac{1}{2} \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) . \tag{9}$$

If $\mathbf{x}(t)$ were unbounded for $t \to \infty$, there would exist a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ for $k \to \infty$ and $\lim_{k\to\infty} \|\mathbf{x}(t_k)\| = \infty$. By (9) and (7) we get

$$F \leqq \tfrac{1}{2} \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) ,$$

which contradicts the assumption (8).

Now let $\mathbf{x}(t)$ be a solution of (1) satisfying the condition (8) which is defined on $\langle t_0, \bar{t})$, $\bar{t} < \infty$ and suppose that for $t \to \bar{t}_- \mathbf{x}(t)$ is unbounded. In that case there exists a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k \to \bar{t}_-$ such that for $k \to \infty$, $t_k \to \bar{t}_-$ and $\lim_{k\to\infty} \|\mathbf{x}(t_k)\| = +\infty$. Let $\{\tilde{t}_k\}_{k=1}^{\infty}$ be any sequence such that

$$t_k \leqq \tilde{t}_k \quad (k = 1, 2, \ldots, n, \ldots),\ \tilde{t}_k \to \infty \text{ for } k \to \infty .$$

By (6) and (9)

$$F(\tilde{t}_k, \mathbf{x}(t_k)) \leqq F(t_k, \mathbf{x}(t_k)) \leqq \tfrac{1}{2} \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) ,$$

which again contradicts the assumption (8).

The proof that any solution satisfying the condition (8) can be extended to $\langle t_0, \infty)$ is completely analogous to that of Theorem 1.

Theorem 3. *In addition to the hypotheses of Theorem* 2, *suppose that* $F(t, \mathbf{x}) \geqq 0$ *for* $t \geqq t_0 \geqq 0$, $\sum_{i=1}^{n} |x_i| < \infty$.

Then any solution of (1) *which satisfies the condition* (8) *as well as the first* *derivative of any solution, are bounded on their domain which is* $\langle t_0, \infty)$ *if the said solution satisfies the condition* (8).

Proof. The boundedness of a solution satisfying (8) is ensured by Theorem 2.

In view of the assumption $F(t, \mathbf{x}) \geqq 0$ for $t \geqq t_0 \geqq 0, \sum_{i=1}^{n} |x_i| < \infty$, we get from (9)

$$\tfrac{1}{2} \|\mathbf{x}'(t)\|^2 \leqq \tfrac{1}{2} \|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) ,$$

which means that the first derivative of any solution is bounded on its domain.

Consider the system

$$x_i'' + (1 + \varphi_i(t))f_i(t, x_1, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, n) , \tag{10}$$

where $f_i(t, x_1, \ldots, x_n)$ are the same functions as in (1), while $\varphi_i(t)$ and $\varphi_i'(t)$ are defined and continuous for all $t \geqq t_0 \geqq 0$.

Theorem 4. *In addition to the hypotheses of Theorem* 1 *suppose that*

$$1 + \varphi_i(t) \geqq k > 0, \quad \varphi_i'(t) \geqq 0 \tag{11}$$

for every $t \geqq t_0 \geqq 0$ *and* $i = 1, 2, \ldots, n$.

Then any solution of (10) *which satisfies the condition*

$$\tfrac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1 + \varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) < F \tag{12}$$

is bounded on $\langle t_0, \infty)$.

Proof. Suppose that $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ is again a solution of (10), defined on $\langle t_0, \bar{t})$ which satisfies the condition (12) and is not bounded for $t \to \bar{t}_-$.

By (10)

$$\sum_{i=1}^{n} \int_{t_0}^{t} \frac{x_i''(s)x_i'(s)}{1 + \varphi_i(s)} \mathrm{d}s + \sum_{i=1}^{n} \int_{t_0}^{t} f_i(s, x_1(s), \ldots, x_n(s))x_i'(s) \, \mathrm{d}s = 0 ,$$

where $t \in (t_0, \bar{t})$. Moreover,

$$\tfrac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2(t)}{1 + \varphi_i(t)} + \tfrac{1}{2} \sum_{i=1}^{n} \int_{t_0}^{t} \frac{x_i'^2(s)\varphi_i'(s)}{[1 + \varphi_i(s)]^2} \mathrm{d}s + F(t, \mathbf{x}(t)) = \tag{13}$$

$$= \frac{1}{2} \sum_{t=1}^{n} \frac{x_i'^2(t_0)}{1 + \varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) + \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s} \, \mathrm{d}s ,$$

or

$$F(t, \mathbf{x}(t)) \leqq \frac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1 + \varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) + \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s} \, \mathrm{d}s , \tag{14}$$

from which similarly as in the proof of Theorem 1 we get

$$F \leqq \frac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1 + \varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) ,$$

which contradicts the assumption (12).

The proof that any solution can be extended to $\langle t_0, \infty)$ is analogous to that of Theorem 1.

Theorem 5. *Suppose that, in addition to the hypotheses of Theorem* 2, (11) *holds. Then any solution of* (10) *satisfying* (12) *is bounded on* $\langle t_0, \infty)$.

Proof. The theorem can be proved using the relation (14). By using (6), we get

$$F(t, \mathbf{x}(t) \leqq \frac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1 + \varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) ,$$

which, by (7) contradicts the assumption (12).

Evidently the following theorem also holds:

Theorem 6. *Suppose that, in addition to the hypotheses of Theorem* 3, *the functions* $\varphi_i(t)$ *satisfy the condition* (11). *Then every solution of* (10) *which satisfies the condition* (12) *is bounded on* $\langle t_0, \infty)$.

If in addition to this for every i *and all* $t \geqq t_0$ $\varphi_i(t) < \infty$, *then also the first derivative of any solution is bounded on its domain.*

Theorem 7. *Suppose that the hypotheses of Theorem* 3 *are valid and that* $F = +\infty$ *in* (7). *If for all* $t \geqq t_0 \geqq 0$, $i = 1, 2, \ldots, n$

$$0 < \alpha \leqq 1 + \varphi_i(t) \leqslant \beta < \infty, \int_{t_0}^{\infty} |\varphi_i'(t)| \, \mathrm{d}t < \infty , \tag{15}$$

then every solution of (10) *and its first derivative are bounded on* $\langle t_0, \infty)$.

Proof. Suppose that a vector function $\mathbf{x}(t)$ is a solution of (10), is defined on $\langle t_0, \bar{t})$ and that $\limsup_{t\to\bar{t}} \|\mathbf{x}(t)\| = +\infty$.

Using (13), (6) and the assumption $F(t, \mathbf{x}) \geqq 0$, we get

$$\frac{1}{2}\sum_{i=1}^{n} \frac{x_i'^2(t)}{1+\varphi_i(t)} \leqq \frac{1}{2}\sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1+\varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0)) +$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{t_0}^{t} \frac{x_i'^2(s)}{[1+\varphi_i(s)]^2}\,|\varphi_i'(s)|\,\mathrm{d}s$$

and therefore

$$\|\mathbf{x}'(t)\|^2 \leqq 2\beta K_0 + \frac{\beta}{\alpha^2}\int_{t_0}^{t}\sum_{i=1}^{n} x_i'^2(s)\,|\varphi_i'(s)|\,\mathrm{d}s\,,$$

where

$$K_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{x_i'^2(t_0)}{1+\varphi_i(t_0)} + F(t_0, \mathbf{x}(t_0))\,.$$

Further

$$\|\mathbf{x}'(t)\|^2 \leqq 2\beta K_0 + \frac{\beta}{\alpha^2}\int_{t_0}^{t}\|\mathbf{x}'(s)\|^2\sum_{i=1}^{n}|\varphi_i'(s)|\,\mathrm{d}s\,.$$

Using Bellman's lemma [4] we get

$$\|\mathbf{x}'(t)\|^2 \leqq 2\beta K_0 \exp\left[\frac{\beta}{\alpha^2}\int_{t_0}^{t}\sum_{i=1}^{n}|\varphi_i'(s)|\,\mathrm{d}s\right] \leqq K_1 < \infty\,,$$

so that $\mathbf{x}'(t)$ is bounded.

We have still to prove that $\mathbf{x}(t)$ is also bounded. This can be done by using (13) again. We get

$$F(t, \mathbf{x}(t)) \leqq K_0 + \frac{1}{2}\sum_{i=1}^{n}\int_{t_0}^{t}\frac{x_i'^2(s)}{[1+\varphi_i(s)]^2}\,|\varphi_i'(s)|\,\mathrm{d}s$$

and therefore

$$F(t, \mathbf{x}(t)) \leqq K_0 + \frac{K_1}{2\alpha^2} \sum_{i=1}^{n} \int_{t_0}^{t} |\varphi_i'(s)| \, ds \leqq K_2 < \infty$$

for all $t \in \langle t_0, \bar{t})$. Suppose that $\{t_k\}_{k=1}^{\infty}$ is a sequence such that $t_k \to \bar{t}_-$ for $k \to \infty$ and $\lim_{k=1} \|\mathbf{x}(t_k)\| = +\infty$.

For this sequence we obtain, using the last inequality, a result which contradicts (7) with $F = +\infty$. This completes the proof.

Theorem 8. *Suppose that the hypotheses of Theorem 7 are all valid except* (15). *If*

$$\lim_{t\to\infty} \varphi_i(t) = 0, \quad \int_{t_0}^{\infty} |\varphi_i'(t)| \, dt < \infty, \quad i = 1, 2, \ldots, n,$$

then there exists t_1, *such that* $t_1 \geqq t_0 \geqq 0$ *and every solution of the system* (10) *is bounded on* $\langle t_1, \infty)$.

Proof. The proof of this theorem is evident and rests on that of Theorem 7. Namely the condition $\lim_{t\to\infty} \varphi_i(t) = 0$ ensures the existence of $t_1 \geqq t_0$ such that for $t \geqq t_1$

$$\tfrac{1}{2} \leqq 1 + \varphi_i(t) \leqq \tfrac{3}{2}.$$

Therefore the condition (15) is also satisfied and the conclusion of the theorem holds.

Remark 2. From this proof it is evident that in Theorems 4, 5 and 6 the condition (11) can be replaced by the following condition:

$$\lim_{t\to\infty} \varphi_i(t) = 0, \quad \varphi_i'(t) \geqq 0, \ t \geqq T \geqq t_0, \quad i = 1, 2, \ldots, n.$$

In that case in (12) we substitute for t_0 a number t_1 such that $t_1 \geqq T$ and that for $t \geqq t_1$ is $1 + \varphi_i(t) \geqq k > 0$, $i = 1, \ldots, n$.

Let us now investigate the boundedness of solutions of the system

$$x_i'' + f_i(t, x_1, \ldots, x_n) = c_i(t), \quad i = 1, \ldots, n, \tag{16}$$

where $f_i(t, x_1, \ldots, xq)$ are again the same as in (1) while $c_i(t)$, $c_i'(t)$ are defined and continuous for all $t \geqq t_0 \geqq 0$. Under such conditions the following theorem holds:

Theorem 9. *Suppose that the hypotheses of Theorem 7 are valid with the exception of* (15). *If the vector* $\mathbf{c}(t) = (c_1(t), \ldots, c_n(t))$ *is such that*

$$\int_{t_0}^{\infty} \|\mathbf{c}(t)\| \, dt < \infty,$$

then every solution of (16), *together with its first derivative is bounded on the interval* $\langle t_0, \infty)$.

Proof. By multiplying the system (16) by $x_i'(t)$, $i = 1, 2, \ldots, n$, where $x_i(t)$ are the components of a solution $\mathbf{x}(t)$ of the system (16), summing over i and integrating from t_0 to t ($t \in (t_0, \bar{t})$, where $\langle t_0, \bar{t})$ is the interval of definition of $\mathbf{x}(t)$) we get

$$\tfrac{1}{2}\|\mathbf{x}'(t)\|^2 + F(t, \mathbf{x}(t)) = \tfrac{1}{2}\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) +$$

$$+ \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\, \mathrm{d}s + \sum_{i=1}^{n} \int_{t_0}^{t} c_i(s) x_i'(s)\, \mathrm{d}s$$

and therefore

$$\tfrac{1}{2}\|\mathbf{x}'(t)\|^2 + F(t, \mathbf{x}(t)) \leqq K_0 + \int_{t_0}^{t} \sum_{i=1}^{n} |c_i(s) x_i'(s)|\, \mathrm{d}s\,. \tag{17}$$

Now suppose that $\mathbf{x}(t)$ is an arbitrary solution of (16). Then from (17) we get

$$\tfrac{1}{2}\|\mathbf{x}'(t)\|^2 \leqq K_0 + \int_{t_0}^{t} \sum_{i=1}^{n} |c_i(s) x_i'(s)|\, \mathrm{d}s\,,$$

which means

$$\|\mathbf{x}'(t)\| \leqq \tfrac{1}{2}\, \|\mathbf{x}'(t)\|^2 + \tfrac{1}{2} \leqq K_0 + \tfrac{1}{2} + \int_{t_0}^{t} \|\mathrm{c}(s)\| \,.\, \|\mathbf{x}'(s)\|\, \mathrm{d}s\,,$$

and therefore

$$\|\mathbf{x}'(t)\| \leqq K_1 \exp \int_{t_0}^{t} \|\mathbf{c}(s)\|\, \mathrm{d}s\,,$$

where $K_1 = K_0 + \tfrac{1}{2}$.

Thus $\mathbf{x}'(t)$ is bounded and there exists a constant K such that, for all $t \in \in \langle t_0, \bar{t})$, $\|\mathbf{x}'(t)\| \leqq K$.

From (17) we also get

$$F(t, \mathbf{x}(t)) \leqq K_0 + \int_{t_0}^{t} \Big(\sum_{i=1}^{n} x_i'^2(s)\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} c_i^2(s)\Big)^{\frac{1}{2}}\, \mathrm{d}s = K_0 + \int_{t_0}^{t} \|\mathbf{c}(s)\| \,.\, \|\mathbf{x}'(s)\|\, \mathrm{d}s\,,$$

and therefore

$$F(t, \mathbf{x}(t) \leqq K_0 + K \int_{t_0}^{t} \|\mathbf{c}(s)\|\, \mathrm{d}s\,,$$

which means that for all $t \in (t_0, \bar{t}\rangle$ $F(t, \mathbf{x}(t))$ is a bounded function. Thus, since in (7) F is equal to $+\infty$, $\|\mathbf{x}(t)\|$ is bounded.

If $\bar{t} < \infty$, then it is easy to prove that the solution $\mathbf{x}(t)$ can be extended to $\langle t_0, \infty)$. This completes the proof.

Remark 3. It is possible to generalize Theorems 1—9 by investigating, instead of the systems (1), (10) and (16), the following systems:

$$x_i'' + \sum_{k=1}^{n} b_{i,k}(t)x_k' + f_i(t, x_1, \ldots, x_n) = 0, \tag{18}$$

$$x_i'' + (1 + \varphi_i(t)) \sum_{k=1}^{n} b_{i,k}(t)x_k' + (1 + \varphi_i(t))f_i(t, x_1, \ldots, x_n) = 0, \tag{19}$$

$$x_i'' + \sum_{k=1}^{n} b_{i,k}(t)x_k' + f_i(t, x_1, \ldots, x_n) = c_i(t), \tag{20}$$

where it is further supposed that for any $t \geqq t_0 \geqq 0$, $\sum_{i=1}^{n} |x_i| < \infty$, $b_{i,k}(t)$ is a continuous function and $\sum_{i,k=1}^{n} b_{i,k}(t)x_ix_k \geqq 0$.

As an example, we shall prove the following:

Theorem 1a. *Suppose that the hypotheses of Theorem* 1 *are valid and that for all* $t \geqq t_0 \geqq 0$

$$\sum_{i,k=1}^{n} b_{i,k}(t)x_ix_k \geqq 0.$$

Then every solution of (18) *which satisfies the condition* (4) *is bounded on its domain.*

Proof. By multiplying the i-th equation of (18) by $x_i'(t)$, summing and integrating we get

$$\tfrac{1}{2}\|\mathbf{x}'(t)\|^2 + \int_{t_0}^{t} \sum_{i,k=1}^{n} b_{i,k}(s)x_k'(s)x_i'(s)\,\mathrm{d}s + F(t, \mathbf{x}(t)) =$$

$$= \tfrac{1}{2}\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0)) + \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\,\mathrm{d}s\,.$$

Thus

$$\tfrac{1}{2}\|\mathbf{x}'(t)\|^2 + F(t, \mathbf{x}(t)) \leqq K_0 + \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\,\mathrm{d}s\,,$$

where

$$K_0 = \tfrac{1}{2}\|\mathbf{x}'(t_0)\|^2 + F(t_0, \mathbf{x}(t_0))\,.$$

From here on the proof is similar to that of Theorem 1.

Remark 4. We shall now show how some results of [3] concerning the bounds of solutions of non-linear equations of order 2 can be generalized to systems.

Theorem 1b. *Suppose that, in addition to the hypotheses of Theorem* 1, *the following conditions hold*:

a) $g_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $i = 1, 2, \ldots, n$ *are continuous for every* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *and there exist nonnegative constants* k_i *such that*

$$g_i(x_1, \ldots, x_n, y_1, \ldots, y_n)y_i \geqq k_i y_i^2$$

for all $\mathbf{x}$ *and* $\mathbf{y}$;

b) $a_i(t)$, $b_i(t)$ *are continuous nonnegative functions for* $t \geqq t_0 \geqq 0$ *and* $2k_i b_i(t) \geqq a_i'(t)$.

Then every solution of the system

$$a_i(t)x_i'' + b_i(t)g_i(x_1, \ldots, x_n, x_1', \ldots, x_n') + f_i(t, x_1, \ldots, x_n) = 0, \tag{21}$$

which satisfies the inequality

$$K_0 = \tfrac{1}{2}\sum_{i=1}^{n} a_i(t_0)x_i'^2(t_0) + F(t_0, \mathbf{x}(t_0)) < F, \tag{22}$$

is bounded on its domain.

Proof. By multiplying the i-th equation of (21) by $x_i'(t)$, summing and integrating from t_0 to t, $t \in (t_0, \bar{t})$, where $\langle t_0, \bar{t})$ is the domain of the solution $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$. We obtain

$$\tfrac{1}{2}\sum_{i=1}^{n}\int_{t_0}^{t} a_i(s)\frac{\mathrm{d}}{\mathrm{d}s}x_i'^2(s)\,\mathrm{d}s + \sum_{i=1}^{n}\int_{t_0}^{t} b_i(s)g_i(x_1, \ldots, x_n, x_1', \ldots, x_n')x_i'(s)\,\mathrm{d}s +$$
$$+ \sum_{i=1}^{n}\int_{t_0}^{t} f_i(s, x_1(s), \ldots, x_n(s))x_i'(s)\,\mathrm{d}s = 0\,.$$

Since $f_i(t, x_1, \ldots, x_n)$ are the same as in (1), we have

$$\tfrac{1}{2}\sum_{i=1}^{n} a_i(t)x_i'^2(t) + \sum_{i=1}^{n}\int_{t_0}^{t} [b_i(s)g_i(x_1, \ldots, x_n, x_1', \ldots, x_n')x_i'(s) -$$
$$- \tfrac{1}{2}a_i'(s)x_i'^2(s)]\,\mathrm{d}s + F(t, \mathbf{x}(t)) = \tfrac{1}{2}\sum_{i=1}^{n} a_i(t_0)x_i'^2(t_0) +$$
$$+ F(t_0, \mathbf{x}(t_0)) + \int_{t_0}^{t}\frac{\partial F(s, \mathbf{x}(s))}{\partial s}\,\mathrm{d}s\,.$$

Taking into account assumptions a) and b), we have

$$\frac{1}{2}\sum_{i=1}^{n} a_i(t)x_i'^2(t) + F(t, \mathbf{x}(t)) \leqq K_0 + \int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\, \mathrm{d}s\ .$$

and from here on the proof proceeds similarly as in Theorem 1.

Clearly this theorem is a generalization of our Theorem 1 as well as of Theorem (1) in [3]. Moreover, by adding to the hypotheses of any theorem dealing with the boundedness of solutions of (1) and their derivatives the assumptions a) and b) and substituting the condition (22) for (4) we obtain a valid theorem which, however, states that the solutions, and sometimes their derivatives, are bounded on their interval of definition.

Analogously the theorem concerning the boundedness of the solution of (16) can be generalized to solutions of the system

$$a_i(t)x_i'' + b_i(t)g_i(x_1, \ldots, x_n, x_1', \ldots, x_n') + f_i(t, x_1, \ldots, x_n) = c_{i(t)}\ ,$$

where $i = 1, \ldots, n$.

The following theorem is a generalization of Theorem 3 in [1].

Theorem 10. *Suppose that $F(\mathbf{x}) = F(x_1, \ldots, x_n)$ satisfies the hypotheses of Theorem 3 in [1], i. e. that it is a continuous, twice differentiable function and*

$$\min_{|\mathbf{x}|=r} F(x_1, \ldots, x_n) = m(r) \to \infty, \quad \text{for} \quad r \to \infty .$$

Suppose further that $a_i(t) > 0$, $a_i'(t) \geqq 0$, $g_i(y_1, \ldots, y_n) > 0$ are defined and continuous for $t \geqq t_0 \geqq 0, \sum_{i=1}^{n} |y_i| < \infty$, $i = 1, \ldots, n$. If $\dfrac{\partial G_i}{\partial y_k} = 0$, $i \neq k$

$i, k = 1, \ldots, n$, where $$G_i(y_1, \ldots, y_n) = \int_0^{y_i} \frac{s}{g_i(y_1, \ldots, y_{i-1}, s, y_{i+1}, \ldots, y_n)}\, \mathrm{d}s\ ,$$

then every solution $\mathbf{x}(t)$ of the system

$$x_i'' + a_i(t)\frac{\partial F}{\partial x_i} g_i(x_1', \ldots, x_n') = 0,\ i = 1, \ldots, n \tag{23}$$

is bounded on its domain.

Proof. By multiplying the i-th equation of (23) by $\dfrac{x_i'(t)}{a_i(t)g_i(x')}$, $g_i(\mathbf{x}') = g_i(x_1', \ldots, x_n')$, summing and integration we get

$$\int_{t_0}^{t} \sum_{i=1}^{n} \frac{x_i'' x_i'}{a_i(s) g_i(x_1', \ldots, x_n')} \, \mathrm{d}s + F(\mathbf{x}(t)) = F(\mathbf{x}(t_0)) ,$$

which gives us the relation

$$\int_{t_0}^{t} \sum_{i=1}^{n} \frac{1}{a(s)} \frac{\mathrm{d}}{\mathrm{d}s} G_i(\mathbf{x}'(s)) \, \mathrm{d}s + F(\mathbf{x}(t)) = F(\mathbf{x}(t_0)) ,$$

where $\mathbf{x}'(s) = (x_1'(s), \ldots, x_n'(s))$. Therefore

$$\sum_{i=1}^{n} \frac{1}{a_i(t)} G_i(\mathbf{x}'(t)) + F(\mathbf{x}(t)) \leqq F(\mathbf{x}(t_0)) + \sum_{i=1}^{n} \frac{1}{a_i(t_0)} G_i(\mathbf{x}'(t_0)) = K_0 , \tag{24}$$

so that

$$F(\mathbf{x}(t)) \leqq K_0 , \tag{25}$$

and consequently, $\|\mathbf{x}(t)\| < \infty$ for every t in the interval of definition of $\mathbf{x}(t)$.

Theorem 11. *Suppose that, under the assumptions made in Theorem* 10, $a_i(t) \leqq k$ *for* $t \geqq t_0 \geqq 0$ *and* $i = 1, 2, \ldots, n$. *If*

$$\min_{|\mathbf{y}|=r} G(y_1, \ldots, y_n) \to \infty \quad \text{for} \quad r \to \infty ,$$

where $G(\mathbf{y}) = G(y_1, \ldots, y_n) = \sum_{i=1}^{n} G_i(y_1, \ldots, y_n)$ *then every solution of* (23) *and its first derivative are bounded on* $\langle t_0, \infty)$.

Proof. Let $\mathbf{x}(t)$ be a solution of (23) which is defined on $\langle t_0, \bar{t})$. The boundedness of its first derivative can be deduced from (24) and (25). In fact

$$\sum_{i=1}^{n} \frac{1}{a_i(t)} G_i(\mathbf{x}'(t)) \leqq K_0 - F(\mathbf{x}(t)) ,$$

so that

$$G(\mathbf{x}'(t)) \leqq k(K_0 - F(\mathbf{x}(t))) ,$$

which means that for all $t \in \langle t_0, \bar{t})$ we have

$$G(\mathbf{x}'(t)) \leqq k(|K_0| + |K_0|)$$

and therefore $\|\mathbf{x}'(t)\| < \infty$.

It remains to be proved that $\bar{t} = +\infty$ or that any solution can be extended to $\langle t_0, \infty)$. To do this we shall show that if $\bar{t} < \infty$, then there exist finite limits $\lim_{t\to\bar{t}} \mathbf{x}(t)$ and $\lim_{t\to\bar{t}} \mathbf{x}'(t)$.

If $\lim_{t\to\bar{t}} \mathbf{x}(t)$ does not exist, then for at least one i $\lim_{t\to\bar{t}} x_i(t)$ does not exist. In this case, however, according to the lemma in [3], $\limsup_{t\to\bar{t}_-} x_i'(t) = +\infty$ and $\liminf_{t\to\bar{t}_-} x_i'(t) = -\infty$. This contradicts the assumption that $\mathbf{x}'(t)$ is bounded.

Suppose now that $\lim_{t\to\bar{t}_-} \mathbf{x}'(t)$ does not exist. Using the same lemma as before, we conclude that $\limsup_{t\to\bar{t}_-} x_i''(t) = +\infty$ and $\liminf_{t\to\bar{t}_-} x_i''(t) = -\infty$ for at least one i. Consider this i and the corresponding $x_i(t)$. If $\limsup_{t\to\bar{t}_-} x_i''(t) = +\infty$, then there exists a sequence $\{t_k\}_{k=1}^{\infty}$, such that for $k \to \infty$, $t_k \to \bar{t}_-$ and $\lim_{k\to\infty} x_i''(t_k) = +\infty$. For this sequence we get (using (23))

$$\lim_{k\to\infty} \left[a_i(t_k) \frac{\partial F(\mathbf{x}(t_k))}{\partial x_i} g_i(x_1'(t_k), \ldots, x_n'(t_k)) \right] = -\infty ,$$

which contradicts the assumptions that a_i, $\partial F/\partial x_i$ and g_i are continuous, $\bar{t} < \infty$, $\|\mathbf{x}(t)\| < \infty$ and $\|\mathbf{x}'(t)\| < \infty$. Thus we have proved that there exist finite limits $\lim_{t\to\bar{t}_-} \mathbf{x}(t)$ and $\lim_{t\to\bar{t}_-} \mathbf{x}'(t)$ and completed the proof.

A further generalization of this theorem and of Theorem 18 in [3] is the following theorem which deals with boundedness of solutions of the system

$$\text{(26)} \qquad x_i'' + f_i(t, x_1, \ldots, x_n) g_i(x_1', \ldots, x_n') = 0 , \quad i = 1, \ldots, n,$$

where $f_i(t, x_1, \ldots, x_n)$ are the same functions as those in (1) and where $g_i(y_1, \ldots, y_n)$ and $G_i(y_1, \ldots, y_n)$ satisfy the assumptions of Theorem 10.

Theorem 12. *Assuming the validity of the hypotheses of Theorem* 1, *any solution of* (26) *which satisfies the inequality*

$$\text{(27)} \qquad K_0 = G(\mathbf{x}'(t_0)) + F(t_0, \mathbf{x}(t_0)) < F ,$$

is bounded on its domain.

Proof. From (26) we get

$$\frac{x_i'' x_i'}{g_i(x_1', \ldots, x_n')} + f_i(t, x_1, \ldots, x_n) x_i' = 0$$

and therefore

$$\text{(28)} \qquad G(\mathbf{x}'(t)) + F(t, \mathbf{x}(t)) = G(\mathbf{x}'(t_0)) + F(t_0, \mathbf{x}(t_0)) +$$

$$+\int_{t_0}^{t} \frac{\partial F(s, \mathbf{x}(s))}{\partial s}\, \mathrm{d}s\,.$$

From here on the proof is analogous to that of Theorem 1.

Remark 5. Theorem 2 will also hold for solutions of (26) if for condition (8) we substitute (27) with F defined by the relation (7).

Theorem 13. *Suppose that* $G(\mathbf{y}) = G(y_1, \dots, y_n)$ *satisfies the conditions of Theorem* 11 *and that the assumptions of Theorem* 3 *are valid. Then every solution of* (26) *satisfying the condition* (27) *and its first derivative are bounded on* $\langle t_0, \infty)$.

Proof. That the solution itself is bounded is evident from Remark 5. From (28) we get

$$G(\mathbf{x}'(t)) \leqq G(\mathbf{x}'(t_0)) + F(t_0, \mathbf{x}(t_0))\,,$$

which means that $G(\mathbf{x}'(t))$ is a bounded function of t and therefore $\|\mathbf{x}'(t)\|$ is also bounded.

The proof that a solution satisfying (27) can be extended to $\langle t_0, \infty)$ is analogous to the corresponding part of the proof of Theorem 11. This completes the proof.

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$$x_i'' + a_i(t) \sum_{k=1}^{n} b_{i,k}(t) x_k' + a_i(t) \frac{\partial F}{\partial x_i} = 0\,,$$

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