Matematický časopis

Ol'ga Vavrová

A Tote on the Completeness of L_q

Matematický časopis, Vol. 23 (1973), No. 3, 267--269

Persistent URL: http://dml.cz/dmlcz/126890

Matematický časopis 23 (1973), No. 3

A NOTE ON THE COMPLETENESS OF L_q

OĽGA VAVROVÁ, Košice

There is a connection between the completeness of L_q and the completeness of the metric space of all sets of finite measure (see [1]). It has been shown in [2] that the completeness of the measure space can be formulated and proved by means of some properties of the families of sets of "small measure". We use a similar method in the present paper to prove a generalization of an L_q-completeness theorem.

First we introduce a sequence $\{G_n\}_{n=0}^{\infty}$ of sets of extended real valued measurable functions defined on a set S and satisfying some axioms. An example of such a sequence is the following. Let $(S, \sum, \mu)$ be a finite measurable algebra, $G_0 = \{f\text{-measurable}, \int_S |f|^q \mathrm{d}\mu < \infty$, $G_n = \{f, f \in G_0, \int_S |f|^q \mathrm{d}\mu < 2^{-n}\}$.

The operations $f + g$, αf etc. are defined as usually, only we put $\infty + (-\infty) = (-\infty) + (\infty) = 0$, $0 \, . \, \infty = 0$. Hence we list the axioms:

I. If $f \in G_n$, then $|f| \in G_n$, $n = 0, 1, 2, \ldots$

II. If $f \in G_m$, g is a measurable function such that $|g| \leqq f$ on S, then also $g \in G_n$.

III. If $f \in G_{n+1}$, $g \in G_{n+1}$, then $f + g \in G_n$, $f + g \in G_0$ for $f, g \in G_0$.

IV. If $f_n \in G_0$, $n = 1, 2, 3, \ldots, f_n \nearrow f$, $f_{n+1} - f_n \in G_n$, then also $f \in G_0$ ($f_n \nearrow f$ if $f_n(x) \leqq f_{n+1}(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in S$).

V. If $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of real valued constant functions and $\lim_{n\to\infty} \alpha_n = 0$, then to any n there is m such that the constant function $f(x) = \alpha_m$, $x \in S$, belongs to G_n.

VI. For every real nonzero constant λ and positive integer n there exists an index m such that $f \in G_m$, implies $\lambda f \in G_n$ $((\lambda f)\,(x) = \lambda f(x)$ for every $x \in S)$.

VII. If $f_n \to f$ (i. e. for every $x \in S$ $\lim_{n\to\infty} f_n(x) = f(x)$), $f_n \in G_{k+1}$ for $n = 0, 1, 2, \ldots$, then $f \in G_k$.

VIII. If $f \in G_0$, $M = \{x : |f(x)| < \infty\}$ and g measurable, $g \cdot \chi_M \in G_i$, then $g \in G_i$.

Theorem. *Let* $g \geqq 1$, $A = \{f \in G_0, \ |f|^q \in G_0\}$, $U_n = \{(f, g) : |f - g|^q \in G_n\}$ $(n = 0, 1, 2, \dots)$ and $\mathscr{B} = \{U_n\}_{n=0}^{\infty}$. *Then* $(A, \mathscr{B})$ *is a complete uniform pseudometrizable space. Furthermore, there is a translation invariant pseudometric* d *on* A *such that* d *and* $\mathscr{B}$ *generate the same uniformity on* A, *and* $\lambda \in \mathscr{R}$, $\{f_n\}_{n=1}^{\infty}$ *in* A, $d(f_n, 0) \to 0$ *imply* $d(\lambda f_n, 0) \to 0$.

Proof. Let $q > 1$.

We prove the completeness of $(A, \mathscr{B})$. The base $\mathscr{B}$ of A is countable. Hence A is complete if every Cauchy sequence is convergent (see [3]). Let $f_n \xrightarrow{q} f$ denote the convergence in $(A, \mathscr{B})$. It means: $f \in A$ and to every k there exists N_0 such that $(f_n, f) \in U_k$ for $n \geqq N_0$. A sequence $\{f_n\}_{n=1}^{\infty}$ is Cauchy in $(A, \mathscr{B})$ if for each k there exists N such that $(f_n, f_m) \in U_k$ for $n, m \geqq N$.

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(A, \mathscr{B})$ and let $i \geqq 1$ be given. By V there is $\lambda > 0$ such that

(1) $$\frac{1}{p} \lambda^{p-1} \in G_{i+1}, \quad \text{where } p = \frac{q}{q-1}.$$

By VI there is m_i such that

(2) $$(\lambda q)^{-1} G_{m_i} \subset G_{i+1}.$$

Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy, there exists k_i' such that

(3) $$(f_n, f_m) \in U_{m_i} \text{ for all } n, m \geqq k_i'.$$

From (2) and (3) it follows that

(4) $$(\lambda q)^{-1} |f_n - f_m|^q \in G_{i+1} \text{ for all } n, m \geqq k_i'.$$

The inequality

$$a \cdot b \leqq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b \geqq 0)$$

implies ($a = \lambda$, $b = |f_n(x) - f_m(x)|$, $x \in S$):

(5) $$|f_n - f_m| \leqq (\lambda q)^{-1} |f_n - f_m|^q + \frac{1}{p} \lambda^{p-1} \quad (n, m \geqq k_i').$$

But (1), (4), (5), III and II imply

(6) $$f_n - f_m \in G_i \text{ for all } n, m \geqq k_i'.$$

Let $\{k_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of integers such that $k_i \geqq k_i'$ (for example, $k_i = \max\{k_1', \dots, k_i'\} + 1$). Then (6) implies $f_{k_{i+1}} - f_{k_i} \in G_i$ $(i \geqq 1)$, since $k_{i+1} > k_i \geqq k_i'$.

Put now $h_0 = |f_{k_1}|$, $h_i = |f_{k_{i+1}} - f_{k_i}|$, $i = 1, 2, \ldots$. Then $\sum_{i=0}^{n} h_i \nearrow \sum_{i=0}^{\infty} h_i$, $h_n = \sum_{i=0}^{n} h_i - \sum_{i=0}^{n-1} h_i \in G_n$, hence $\sum_{i=0}^{\infty} h_i \in G_0$ according to IV. Finally define

$$f(x) = f_{k_1}(x) + \sum_{i=1}^{\infty} (f_{k_{i+1}}(x) - f_{k_i}(x)),$$

if $\sum_{i=0}^{\infty} h_i(x)$ converges and

$$f(x) = 0$$

in the opposite case. Then f is a measurable function, for which $|f| \leqq \sum_{i=0}^{\infty} h_i \in G_0$, hence $f \in G_0$ according to II. Put $M = \{x: \sum_{i=0}^{\infty} h_i(x) < \infty\}$. Evidently $f_{k_i} \cdot \chi_M \to \to f \cdot \chi_M$. According to VII and to VIII $f_{k_i} \xrightarrow{q} f$. Now it is not difficult to prove that $|f|^q \in G_0$ and also $f_n \xrightarrow{q} f$.

The base $\mathscr{B}$ gives on A a base of neighbourhoods of 0, which form a topology on A; the discrete product of these neighbourhoods forms a topology on $A \times \times A = \{(x, y) : x \in A, y \in A\}$. Since the function $f(x, y) = x + y$ from $A \times A$ into A is a continuous function (III) and the function $g(\alpha, x) = \alpha x$ from $\mathscr{R} \times A = \{(\alpha, x), \alpha\text{-real number}, x \in A\}$ into A is a continuous function too, A is a linear topological space. One can easily define on A a translation invariant pseudometric d, such that d generates the same uniformity on A as $\mathscr{B}$, and the following holds true: for every sequence $\{f_n\}_{n=0}^{\infty}$ of elements of A, if $d(f_n, 0) \to 0$, then $d(\lambda f_n, 0) \to 0$ for every real number λ ([4,5]).

In a case $q = 1$ the proof is simple.

Let us remark that the space A needs not be separated.

REFERENCES

[1] ALFSEN, E. M.: Order Theoretic Foundations of Integration. Math. Ann. *149*, 1963, 419—461.

[2] RIEČAN, B.: Abstract formulation of some theorems of measure theory II. Mat. časop. *19*, 1969, 138—144.

[3] KELLEY, J. L.: General Topology. New York 1955.

[4] KOETHE, G.: Topologische lineare Räume. Springer 1966.

[5] BOURBAKI, N.: Espaces vectoriels topologiques. Paris 1953.

Received March 28, 1972

I. katedra matematiky
Prírodovedeckej fakulty Univerzity P. J. Šafárika
Košice