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EDGE BASES OF COMPLETE UNIFORM HYPERGRAPHS

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Let $H = (V, E)$ be a finite simple hypergraph, let $V = \{v_1, v_2, \ldots, v_n\}$ be its vertex set, let $E = \{e_1, e_2, \ldots, e_m\}$ be its edge set. Let the rank of the hypergraph be k, i. e. $\max |e_i| = k$. (See e. g. [1], p. 373.) Evidently the relation $|e_i \cap e_j| \leqslant k - 1$ holds for two arbitrary edges.

Definition 1. *Let $r \leqslant k - 1$ be a non-negative integer. An edge basis of the degree r of the hypergraph H is a set of edges $B_r \subseteq E$ for which the following holds:*
1) *If $e_i, e_j \in B_r$, then $|e_i \cap e_j| \leqslant r$,*
2) *If $e_i \in E - B_r$, then there exists an $e_j \in B_r$ such that $|e_i \cap e_j| \geqslant r + 1$.*
An edge basis is called minimal [maximal] if it has a minimal [maximal] possible number of edges.

It is clear that in H there always exists a minimal and a maximal edge basis of degree r.

In the following we shall consider only complete k-uniform hypergraphs in which $|V| = n$, $|E| = \binom{n}{k}$, $0 \leqslant r \leqslant k - 1 < n$, $k \geqslant 3$. Edge bases will be denoted by $B(n, k, r)$. At first we shall introduce the known theorems on edge bases $B(n, k, r)$.

Theorem 1. $|B(n, k, r)| \leqslant \left[\frac{n-r}{k-r}\right]\binom{n}{r}\Big/\binom{k}{r}$.

The proof is in [5], p. 258.

Theorem 2. *Let* $n = (k - 1)t + s$, $0 \leqslant s < k - 1$, $M(n, k) = (k - 2) \,.\, (n^2 - s^2)/2(k - 1) + \binom{s}{2}$. *Then* $|B(n, k, 1)| \geqslant \left(\binom{n}{2} - M(n, k)\right)\Big/\binom{k}{2} = d(n, k)$.

Theorem 3. *Minimal bases $B(n, k, 1)$ with $d(n, k)$ edges exist if and only if n is divisible by $k - 1$ and for $n_1 = n/(k - 1)$ there exist a tactical system $S(2, k, n_1)$.*

The proof of both these theorems are in [2], p. 399.

Theorem 4. *Maximal bases* $B(n, 3, 1)$ *have* $\left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right]$ *edges, if* $n \not\equiv 5 \pmod 6$, $\left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] - 1$ *edges, if* $n \equiv 5 \pmod 6$.

The proof is in [6].

Definition 2. *Edge bases* $B(n, k, r)$ *in which the number of edges is equal to* $\left[\frac{n-r}{k-r}\right]\binom{n}{r}\Big/\binom{k}{r}$ *are called dense.*

The question, when dense edge bases exist is not completely solved. The following theorem concerns this topic:

Theorem 5. *Dense edge bases* $B(n, 3, 1)$. *or* $B(n, 4, 2)$ *exist if and only if* $n \equiv 0, 1, 2, 3 \pmod 6$, *or* $n \equiv 1, 2, 3, 4 \pmod 6$, *respectively.*

The proof is in [4], pp. 137 and 139.

The aim of this paper is to construct minimal edge bases $B(n, 3, 1)$ for each positive integer $n \geqslant 3$. If we apply Theorem 3, we obtain the solution easily, if n is even and there exists a tactical system $S(2, 3, n/2)$, i. e. a Steiner triple system.

It is well known that there exist Steiner triple systems for $n_1 = 6t + 1$, or $n_1 = 6t + 3$ resp. This implies $n = 12t + 2$, or $n = 12t + 6$ resp. In these two cases we construct a minimal basis $B(n, 3, 1)$, so that we construct a Steiner system S_1 of triples of the elements $1, 2, \ldots, n/2$ and a Steiner system S_2 of triples of the elements $n/2 + 1, n/2 + 2, \ldots, n$. Then $B(n, 3, 1) = S_1 \cup S_2$. This assertion is proved also in paper [2], p. 401.

We have to construct minimal bases $B(n, 3, 1)$ in the remaining cases. i. e. for $n = 12t + i$, $i = 0, 1, \ldots, 11$, $i \neq 2, 6$. In our considerations we shall use some theorems on Eulerian graphs without triangles.

1. Graphs assigned to $B(n, 3, 1)$

There exist Eulerian graphs of the order n without triangles. We shall denote them by $U(n, p)$, where p denotes the number of edges. If p is maximal at the given n, we put $p = m(n)$.

Theorem 6. *If* $n \geqslant 4$, *then*

$$m(n) = \begin{cases} n^2/4 & \text{for } n \equiv 0 \pmod 4, \\ (n-1)^2/4 + 1 & \text{for } n \equiv 1 \pmod 4, \\ n^2/4 - 1 & \text{for } n \equiv 2 \pmod 4, \\ (n-1)^2/4 & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

The proof is in [3]. This paper contains also a construction of extremal graphs $U(n, m(n))$.

A complete bipartite graph in which there are n_1 vertices of one colour in a set C_1 and n_2 vertices of the other colour in a set C_2 will be denoted by $S(n_1, n_2)$. Further let us denote by $S_{ij}(n_1, n_2)$ the graph with $n_1 + n_2 + 1$ vertices obtained from the graph $S(n_1, n_2)$ in the following way: 1) we add a further vertex z to $S(n_1, n_2)$, which will be joined with i vertices $u_1, u_2, \ldots, u_i$ of the set C_1 and with j vertices $v_1, v_2, \ldots, v_j$ of the set C_2. 2) from the graph $S(n_1, n_2)$ we delete the edges joining a vertex u_k with a vertex v_g for $k = 1, 2, \ldots, i$; $g = 1, 2, \ldots, j$.

The extremal graphs $U(n, m(n))$, important for us, have the following structure:

If $n \equiv 1 \pmod 4$, then $U(n, m(n)) = S_{1f}((n-1)/2, (n-1)/2), f = 1, 3, 5, \ldots, (n-3)/2$.

If $n \equiv 3 \pmod 4$, then $U(n, m(n))$ is either the graph $S_{1f}((n+1)/2, (n-3)/2), f = 1, 3, 5, \ldots, (n-5)/2$, or the graph $S_{e1}((n+1)/2, (n-3)/2), e = 1, 3, 5, \ldots, (n-1)/2$.

The graphs, all vertices of which have odd degrees, will be called *anti-Eulerian*. If an anti-Eulerian graph without triangles has n vertices and p edges, we denote it by $A(n, p)$. The symbol $m(n)$ denotes the maximal possible number of edges of an $A(n, p)$.

The following theorem holds for anti-Eulerian graphs $A(n, m(n))$:

Theorem 7. 1) $m(n) = n^2/4 - 1$ *for* $n \equiv 0 \pmod 4$. *The graph* $A(n, m(n))$ *is the complete bipartite graph* $S(n/2 + 1, n/2 - 1)$. 2) $m(n) = n^2/4$ *for* $n \equiv 2 \pmod 4$. *The graph* $A(n, m(n))$ *is the complete bipartite graph* $S(n/2, n/2)$.

Now let us consider the graph $G(n, 3, 1)$ which will be assigned to a basis $B(n, 3, 1)$. Let $v_i\, v_j\, v_k$ be an arbitrary edge of $B(n, 3, 1)$. Then the assigned graph $G(n, 3, 1)$ must contain the vertices v_i, v_j, v_k and the edges v_iv_j, v_iv_k, v_jv_k. Therefore the graph $G(n, 3, 1)$ will be formed by triangles which are edge disjoint. Necessarily such a graph has all vertices of even degrees, therefore it is Eulerian. Further it has the property that for an arbitrary triple of its vertices there must exist at least one edge joining two of these vertices. Otherwise we could add such a triple of vertices as an edge to the basis $B(n, 3, 1)$ which is not possible.

Let us denote by $\bar{G}(n, 3, 1)$ the complementary graph to the graph $G(n, 3, 1)$. The graph $\bar{G}(n, 3, 1)$ does not contain triangles and it is either Eulerian if n is odd, or anti-Eulerian if n is even. The graph $\bar{G}(n, 3, 1)$ can have at most $m(n)$ edges where $m(n)$ has the values from Theorems 6 and 7. This implies that the graph $G(n, 3, 1)$ has at least $\binom{n}{2} - m(n)$ edges. By q let us denote the number of edges of the basis $B(n, 3, 1)$. Thus for q the inequality

$$(1) \qquad q \geqslant \frac{\binom{n}{2} - m(n)}{3} = d_1(n),$$

must hold. Thus for each n we determine the least integer $d(n)$ for which $d(n) \geqslant d_1(n)$ holds. If we find a basis $B(n, 3, 1)$ which has $d(n)$ edges, then this basis is minimal. We shall not study the problem of the number of non-isomorphic minimal bases.

2. Lemmas

Lemma 1. *Let $n \equiv 3 \pmod 4$. Then the Eulerian graph $U(n, m(n))$ cannot be complementary to $G(n, 3, 1)$.*

Proof. As we can read in [3], the Eulerian graph $U(n, m(n))$ has the structure illustrated in Fig. 1 (where $n = 11$). The vertex set is decomposed into a vertex z and two classes C_1, C_2. We have $|C_1| = (n + 1)/2$, $|C_2| = (n - 3)/2$. The vertex z is joined with one vertex u of the set C_1 and with vertices $v_1, v_2, \ldots, v_f$ of the set C_2, where f is an odd number, or with a vertex v of the set C_2 and with vertices $u_1, u_2, \ldots, u_g$ of the set C_1, where g is an odd number. In Fig. 1 a case is illustrated with the edges zu and zv_i, $i = 1, 2, 3$. In the following it suffices to consider only this case because for the other case with the edges zv, zu_i, the consideration is the same.

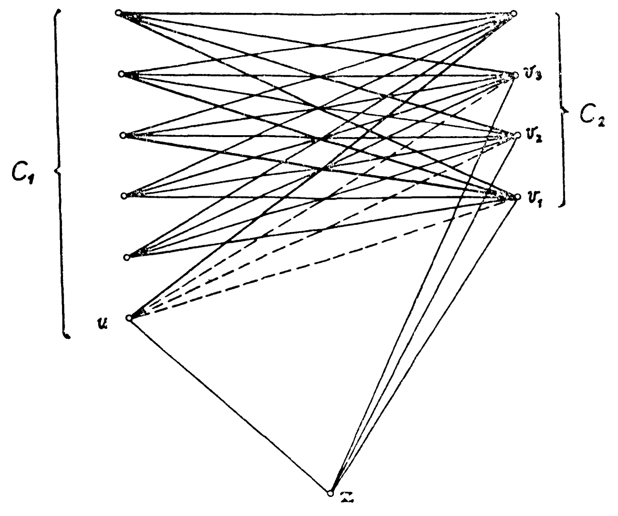

Fig. 1

Let us denote by $\overline{U}(n)$ the complementary graph to the graph $U(n, m(n))$. The graph $\overline{U}(n)$ has $\binom{n}{2} - m(n)$ edges and it is formed by the complete graphs $\langle|C_1|\rangle$, $\langle|C_2|\rangle$, further by an odd number of edges which join the vertex u with vertices of the set C_2 and finally by the edges which join the vertex z with vertices of the sets C_1, C_2. Note that in the graph $\overline{U}(n)$ the vertex z is not joined with the vertex $u \in C_1$.

We ask whether the graph $\overline{U}(n)$ can be the graph of some basis $B(n, 3, 1)$, i. e. the graph $G(n, 3, 1)$. Let us suppose that it is so. Then the graph $\overline{U}(n)$ has to be composed from edge-disjoint triangles. Each triangle incident with the vertex u must have both further vertices either in the set C_1, or in the set C_2. If one vertex lay in C_1 and another in C_2, then in the graph $\overline{U}(n)$ there would exist an edge joining these vertices and different from the edges incident with the vertex u. But this is not possible. This implies further that the number of edges joining the vertex u with the vertices of the set C_2 must be even. We obtain a contradiction because this number of edges is odd. The graph $\overline{U}(n)$ cannot be the graph $G(n, 3, 1)$ and the graph $U(n, m(n))$ cannot be the graph $\overline{G}(n, 3, 1)$.

Lemma 2. *Let* $n = 12t + 9$, *let* $p_1 = 36t^2 + 48t + 15$. *Then the graph* $U(n, p_1)$ *does not exist.*

Proof. Suppose that the graph $U(n, p_1)$ exists. We denote by D the sum of degrees of all its vertices. In the graph $U(n, p_1)$ there must exist a vertex of the degree at least $6t + 4$. Otherwise there would be

$$D \leqslant (12t + 9) \, . \, (6t + 2) = 72t^2 + 78t + 18 < 72t^2 + 96t + 30 = D.$$

The existence of a vertex of a certain degree will be a frequent consideration in the following. We shall refer to it as to *Consideration* 1.

Let x be a vertex of the maximal degree in the graph $U(n, p_1)$. Its degree is equal to $6t + 4 + c$, where $c \geqslant 0$ is an even number. Among the vertices which are joined with x there must exist at least one vertex y whose degree is exactly $6t + 4 - c$. It is easy to see that he degree of the vertex y cannot be higher. If it were so, we should have the number of vertices higher than $12t + 9$ because the degree is an even number.

If all vertices adjacent to x had degrees less than $6t + 4 - c$, i. e. at most $6t + 2 - c$, then the following relation would hold:

$$D \leqslant 6t + 4 + c + (6t + 4 + c) \, (6t + 2 - c) + (6t + 4 - c) \, . \, (6t + 4 + c) = 72t^2 + 90t + 28 - 2c^2 - c < 72t^2 + 96t + 30 = D.$$

This consideration will be referred to several times as *Consideration* 2.

Therefore we have in the graph $U(n, p_1)$ a vertex x of the degree $6t + 4 + c$ and a vertex y of the degree $6t + 4 - c$ joined by an edge. The vertices adjacent to x form a set C_1, into which we shall not include the vertex y; the vertices adjacent to y form a set C_2, into which we do not include the vertex x. We have $|C_1| = 6t + 3 + c$, $|C_2| = 6t + 3 - c$. These sets of vertices are disjoint. The remaining vertex will be denoted by z. (Fig. 2.)

As the graph $U(n, p_1)$ does not conatin triangles, its edges are

a) the edges incident with x and y whose number is

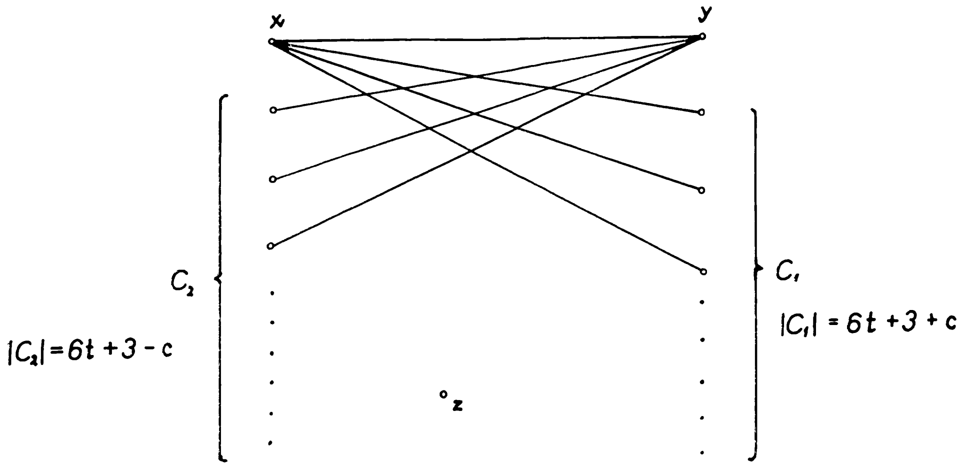

Fig. 2

$$1 + 6t + 3 + c + 6t + 3 - c = 12t + 7,$$

b) a certain subset of the edge set of the complete bipartite graph $S(C_1, |C_2)$ whose cardinality is at most

$$(6t + 3 + c) \,.\, (6t + 3 - c) = 36t^2 + 36t + 9 - c^2,$$

c) the edges incident with the vertex z. Let the vertex z be joined with e vertices of the set C_1 and f vertices of the set C_2.

Therefore the following holds:

$$(2) \qquad p_1 \quad 36t^2 + 48t + 15 - 12t + 7 + 36t^2 + 36t - 9 - c^2 - g -$$

$$+ e + f - e \,.\, f.$$

If namely $e \neq 0, f \neq 0$, then from the complete bipartite graph $S(C_1, C_2)$ we must omit $e \,.\, f$ edges in order that no triangles with the vertex z and further two vertices in the sets C_1, C_2 might be obtained. Therefore the number of edges which join vertices of the set C_1 with vertices of the set C_2 is equal to $36t^2 + 36t + 9 - c^2 - e \,. f - g$, where g is a non-negative integer.

Now let us consider a case when the equality in the relation (2) occurs. The expression $e + f - e \,. f$ is

a) equal to one if at least one of the numbers e, f is equal to one;

b) equal to zero for $e = f = 2$ or $e = f = 0$;

c) negative for any other choice of the numbers e, f.

We shall study these possibilities.

a) In this case there must be $c = 0$, $g = 2$ in order that the equality (2) may hold. Therefore the graph $U(n, p_1)$ will be obtained so that from the graph $U(n, m(n))$ we omit two edges. This follows immediately from the comparison of the structure of the graph $U(n, p_1)$ with the structure of the graph $U(n, m(n))$. But if we omit two edges from a Eulerian graph, we cannot obtain a Eulerian graph. Therefore this case is not possible.

b) In this case there must be $c = 0$, $g = 1$ in order that the relation (2) may be satisfied.

Let $e = f = 2$ and let the vertex z be joined with vertices u_1, u_2 of the set C_1 and with vertices v_1, v_2 of the set C_2. From the complete bipartite graph $S(C_1, C_2)$ we must delete 4 edges $u_i v_j$, $i, j = 1, 2$. By this the degrees of all the vertices remain even. But we must delete one more edge according to $g = 1$. By this further deleting we change the degrees of two vertices into odd ones, which is not possible.

If $e = f = 0$, then by deleting one edge the degrees of two vertices in the graph $U(n, p_1)$ are changed into odd ones. This again is not possible.

c) If $e + f - e \,. f = -1$, then $e = 2$, $f = 3$ or inversely. This implies that the vertex z is of an odd degree, which is not possible.

If $e + f - e \,. f < -1$, then the relation (2) cannot be satisfied.

Therefore we see that the graph $U(n, p_1)$ does not exist.

Lemma 3. *The graph* $A(12, 33)$ *does not exist. The graph* $A(12, 30)$ *cannot serve as complementary to* G$(12, 3, 1)$.

Proof. Suppose that there exists $A(12), 33)$. Therefore $D = 66$. In the graph a vertex of the degree at least 7 must exist according to Consideration 1. We denote by x the vertex of the maximal degree. Its degree is equal to $7 + c$, where $c \geqslant 0$ is an even number. Among the vertices which are adjacent to x there must be at least one vertex y of the degree $5 - c$, as it follows from Consideration 2. The vertices adjacent to x form a set C_1 into which we do not include y; the vertices adjacent to y form a set C_2 into which we do not include x. These sets must be disjoint in order that in the graph no triangle may be formed. (Fig. 3.) We have $|C_1| = 6 + c$, $|C_2| = 4 - c$. The remaining edges form a subset of the edge set of the complete bipartite graph $S(|C_1|, |C_2|)$.

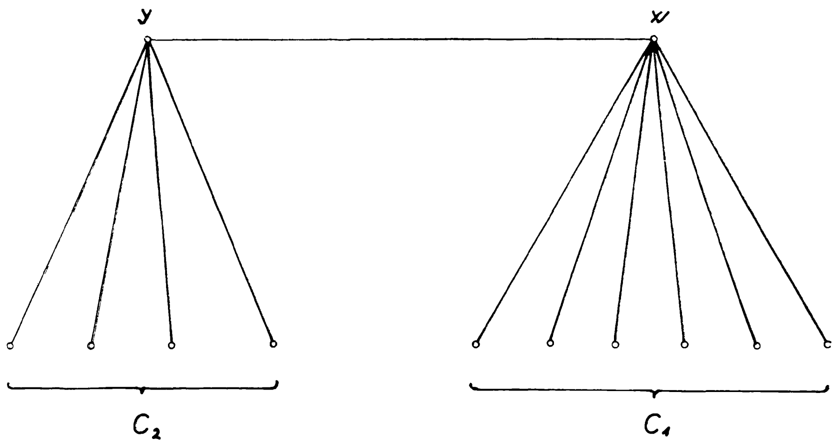

Fig. 3

Totally we can have at most $1 + 6 + c + 4 - c + (6 + c)\,.\,(4 - c) = 35 - 2c - c^2$ edges. In order that the number of edges may be equal to 33, necessarily $c = 0$. We must omit two edges from the complete bipartite graph $S(|C_1|, |C_2|)$. But by omitting any two edges we cannot obtain an anti-Eulerian graph from the anti-Eulerian graph A(12, 35) because some vertices will necessarily have even degrees. Therefore the graph $A(12, 33)$ does not exist.

This consideration on the non-existence of an anti-Eulerian graph will also be repeated in the following. (*Consideration* 3.)

Now let us suppose that there exists the graph $A(12, 30)$. In the graph there must exist a vertex x whose degree is at least 5. Therefore we can take into account the degrees 5, 7, 9.

If the highest degree of a vertex in the graph is 7, then the graph must have a structure as in Fig. 3, i. e. the vertex of the degree 7 is joined with a vertex of the degree 5 and further edges join vertices of the set C_1 with vertices of the set C_2. According to Consideration 3 a graph $A(12, 30)$ with the maximal degree 7 cannot exist.

If the highest degree is equal to, 9, the obtained graph can have at most 27 edges. (Fig. 4.)

Suppose that the highest degree is equal to 5. Then all vertices must have the same degree 5 in order that the sum of all degrees may be 60. Let us denote two adjacent vertices of the degree 5 by x and y, the sets of adjacent vertices corresponding to them by C_1, C_2, where $|C_1| = |C_2| = 4$, $C_1 \cap C_2 = \emptyset$. The remaining two vertices in the graph will be denoted by z, t.

Let us suppose that they are not joined by an edge. Then the vertex z must

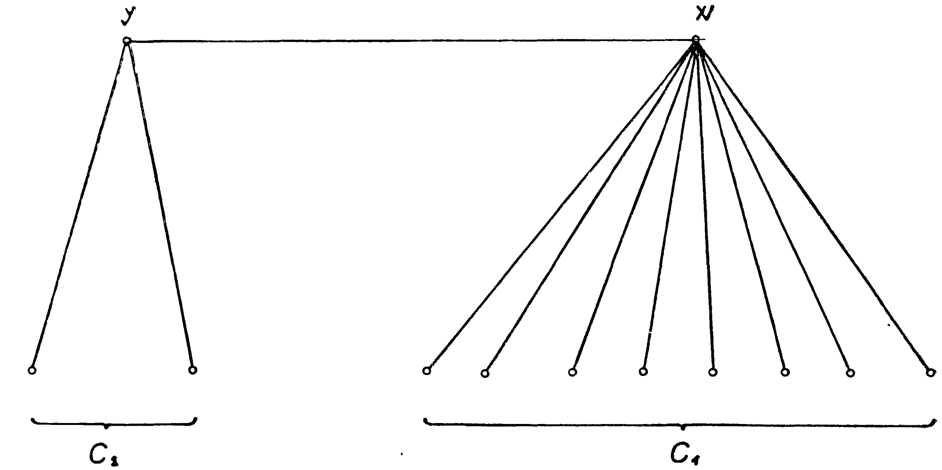

Fig. 4

be joined only with vertices of bo h sets C_1, C_2, because its degree is 5. There are two possibilities:

a) The vertex z is joined with one vertex u of the set C_i and with four vertices of the set C_j, $i \neq j$.

b) The vertex z is joined with two vertices u, v of the set C_i and with three vertices of the set C_j, $i \neq j$. (Fig. 5.)

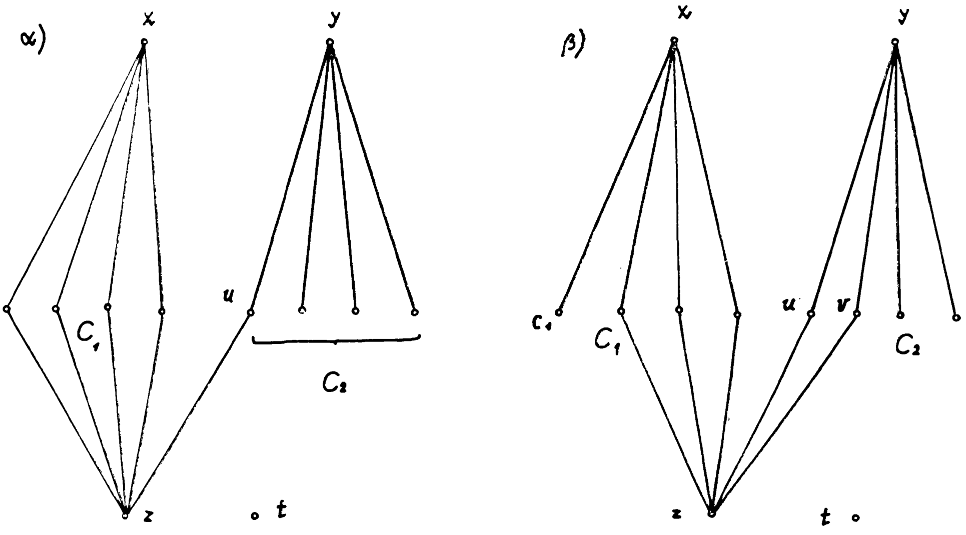

Fig. 5

In the case of a) we can join the vertex u only with the vertex t, therefore its degree is at most 3, not 5.

In the case of b) we can join the vertex u with the vertex t and with a vertex of C_1, therefore its degree is at most 4, not 5.

The assumption that the vertices z, t are not joined by an edge is nost correct.

Thus let the vertices z, t be joined by the edge zt. It is easy to see that in this case the vertex z must be joined with all vertices of the set C_i, the vertex t with all vertices of the set C_j, $i \neq j$. In the graph we have 18 edges now which are incident with the vertices x, y, z, t. We need 12 more edges which can join only vertices of the set C_1 with vertices of the set C_2. From the complete bipartite graph $S(|C_1|, |C_2|)$ it is necessary to omit four edges so that each vertex may have the degree 3. This can be done so that any two omitted edges are vertex disjoint. Thus we have found an anti-Eulerian graph without triangles with 30 edges, i. e. $A(12, 30)$.

We ask whether the graph $\bar{A}$ complementary to it with 36 edges can be the graph $G(12, 3, 1)$ composed of triangles, i. e. whether a decomposition of the graph $\bar{A}$ into edge-disjoint triangles is possible. The graph $\bar{A}$ is shown in Fig. 6.

But four edges joining vertices of the set C_1 with vertices of the set C_2 cannot be put into any triangle. It is so because between the sets of vertices C_1, C_2 there are only these four vertex-disjoint joining edges and the vertices x, y, z, t have the property that none of them is joined with vertices of both sets

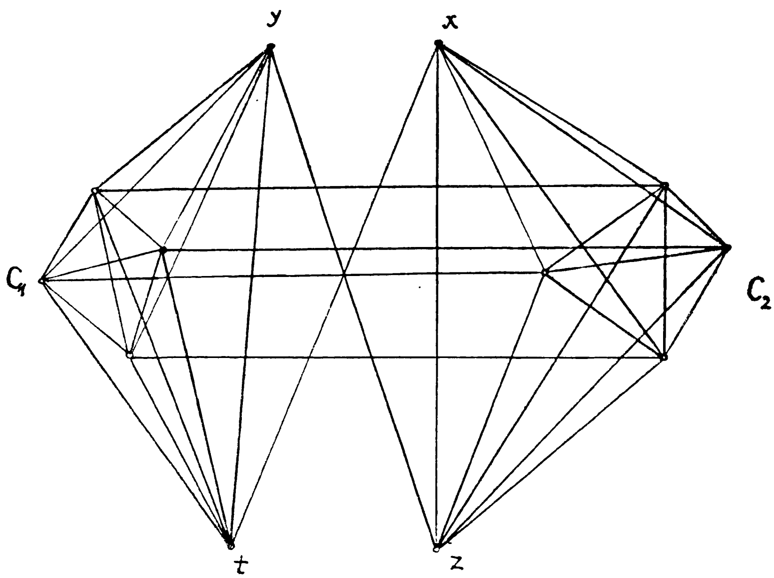

Fig. 6

C_1, C_2. Therefore the graph $A(12, 20)$ cannot be used for the construction of the graph $G(12, 3, 1)$.

Lemma 4. *The graphs* $A(12t, 36t^2 - 3)$ *and* $A(12t, 36t^2 - 6)$ *do not exist, if* $t > 1$.

Proof. Let us suppose that there exist anti-Eulerian graphs which have $36t^2 - 3$, or $36t^2 - 6$ edges. For them $D = 72t^2 - 6$, or $D = 72t^2 - 12$, respectively.

The highest degree of a vertex in the graph must be at least $6t + 1$, which we obtain from Consideration 1 with the assumtion $t > 1$.

Let the highest degree of a vertex in the graph be $6t + c$, c odd, $1 \leqslant c \leqslant 6t - 1$. The vertex of this degree will be denoted by x. Then, according to Consideration 2, with the assumption $t > 1$ among the vertices adjacent to x there must exist at least one vertex y of the degree $6t - c$ exactly. Therefore the graph has the structure as in Fig. 3 to which it is necessary to add $|C_2| = 6t - c - 1$, $|C_1| = 6t + c - 1$. No other vertices will be in the graph. For the number p of edges we have

$$p \leqslant 1 + 12t - 2 + (6t - c - 1) \,.\, (6t + c - 1) = 36t^2 - c^2.$$

For our considerations only the value $c = 1$ has a meaning, other odd values of c lead to a smaller number of edges than that which we inverstigate. According to Consideration 3 it is easy to show that the graphs $A(12t, 36t^2 - 3)$ and $A(12t, 36t^2 - 6)$ do not exist.

Lemma 5. *The graphs* $A(12t + 8, 36t^2 + 48t + 13)$ *and* $A(12t + 8, 36t^2 + 48t + 10)$ *do not exist.*

Proof. Let $n = 12t + 8$, $p_1 = 36t^2 + 48t + 13$, $p_2 = 36t^2 + 48t + 10$. Let us suppose that $A(n, p_1)$, or $A(n, p_2)$ exists. Then in such a graph according to Consideration 1 there must exist at least one vertex of the degree at least $6t + 5$. By x we denote the vertex of the highest degree $6t + 5 + c$, $c \geqslant 0$, c even. According to Condiseration 2 there must exist a vertex y of the degree $6t + 3 - c$ among the vertices adjacent to x.

The structure of the graph is therefore the same as in Fig. 3. Evidently $|C_1| = 6t + 4 + c$, $|C_2| = 6t + 2 - c$. For the number p_i of edges, $i = 1, 2$, we have:

$$p_i \leqslant 1 + 6t + 2 - c + 6t + 4 + c + (6t + 2 - c) \,.\, (6t + 4 + c) =$$
$$= 36t^2 + 48t + 15 - 2c - c^2.$$

It is easy to see that $c = 0$. We have obtained the graph $A(n, m(n))$ from which by omitting two or five edges we should obtain $A(n, p_1)$, or $A(n, p_2)$. But this is not possible. This implies that the graphs $A(n, p_1)$ and $A(n, p_2)$ do not exist.

Lemma 6. *The graph* $A(12t + 10,\ 36t^2 + 60t + 24)$ *does not exist.*
The proof of this lemma is the same as the proof of Lemma 5.

3. The construction of minimal bases $B(n, 3, 1)$

Let V be a vertex set, $|V| = n \geqslant 3$. We obtain a minimal basis $B_{\min}(n, 3, 1)$ by the following.

Construction 1. 1) We decompose the set V into two disjoint classes V_1, V_2. Let $|V_1| = n_1$ be the nearest integer to $n/2$ for which the Steiner triple system exists. Let $|V_2| = n_2 = n - n_1$.

2) We construct a maximal basis $B_{\max}(n_1, 3, 1)$ from the elements of the set V_1. It is a Steiner triple system.

Further we construct a maximal basis $B_{\max}(n_2, 3, 1)$ from the elements of the set V_2 according to Theorem 4.

3) We find all pairs of elements of the set V_2 which are not contained in the triples of the basis $B_{\max}(n_2, 3, 1)$. To each of these pairs we add one element of the set V_1. These added elements of the set V_1 must be different. Thus we obtain a set T of triples. (It may be empty.)

4) A minimal basis $B_{\min}(n, 3, 1)$ is formed by two maximal bases $B_{\max}(n_1, 3, 1)$ and $B_{\max}(n_2, 3, 1)$ and by the set T of triples obtained in 3).

Now we shall prove our main theorem.

Theorem 8. *Let* $n \geqslant 3$ *be an integer. We obtain a minimal basis* $B_{\min}(n, 3, 1)$ *by Construction* 1.

Proof. 1) Let $n = 12t$, $t \geqslant 1$.

Let $B_{\min}(12t, 3, 1)$ be a minimal basis to which the graph $G_{\min}(12t, 3, 1)$ is assigned. The graph $G_{\min}(12t, 3, 1)$ is Eulerian, the complementary graph $\bar{G}$ is anti-Eulerian without triangles, i. e. $\bar{G} = A(12t, p)$. For the number p of edges in $\bar{G}$ the following relation must hold:

$$p \leqslant m(n) = 36t^2 - 1.$$

(We used Theorem 7.)

As $\binom{12t}{2}$ is divisible by three, the number p must also be divisible by three. Therefore the number $36t^2 - 1$ comes not into account. The further values of p are $36t^2 - 3$ and $36t^2 - 6$. According to the Lemmas 3 and 4 the anti-Eulerian graphs $A(12t,\ 36t^2 - 3)$, $A(12t,\ 36t^2 - 6)$, $A(12, 33)$ do not exist. The unique graph $A(12, 30)$ cannot serve as complementary to $G(12, 3, 1)$.

For the construction of the minimal basis $B_{\min}(12t, 3, 1)$ it is necessary to use an $A(12t,\ 36t^2 - 9)$ as a complementary graph. In this case the graph

$G(12t, 3, 1)$ has $36t^2 - 6t - 9$ edges, therefore the minimal basis has at least $12t^2 - 2t + 3$ edges.

In our case we have $|V_1| = n_1 = 6t + 1$, $|V_2| = n_2 = 6t - 1$. Now we apply Construction 1. A maximal basis $B_{\max}(6t + 1, 3, 1)$, i. e. a Steiner triple system, contains $\binom{6t+1}{2}/3 = 6t^2 + t$ triples, a maximal basis $B_{\max}(6t - 1, 3, 1)$ contains $[(6t - 1)/3 \, . \, [(6t - 2)/2]] - 1 = 6t^2 - 3t - 1$ triples and $18t^2 - 9t - 3$ pairs of elements of V_2. There are four pairs of elements of V_2 not contained in the triples of the basis $B_{\max}(6t - 1, 3, 1)$.

We add four distinct elements of V_1 to these pairs and we obtain 4 triples which form the set T.

Now we form the set

$$B_{\max}(6t + 1, 3, 1) \cup B_{\max}(6t - 1, 3, 1) \cup T.$$

This is evidently a basis $B(12t, 3, 1)$ which contains $12t^2 - 2t + 3$ edges, therefore it is minimal.

2) Let $n = 12t + 1$.

Let $B_{\min}(12t + 1, 3, 1)$ be a minimal basis to which the graph $G_{\min}(12t + 1, 3, 1)$ is assigned. This graph is Eulerian, the complementary graph $\bar{G}$ to it is Eulerian without triangles, i. e. $\bar{G} = U(12t + 1, p)$. For the number p of edges in $\bar{G}$ the following relation must hold according to Theorem 6:

$$p \leqslant m(n) = 36t^2 + 1.$$

For the number q of edges in $B(12t + 1, 3, 1)$ the following relations must hold according to (1):

$$q \geqslant \left(\binom{12t+1}{2} - (36t^2 + 1)\right)/3, \; q \geqslant 12t^2 + 2t.$$

Now we apply Construction 1. We have $|V_1| = n_1 = 6t + 1$, $|V_2| = n_2 = 6t$. A $B_{\max}(6t + 1, 3, 1)$ contains $6t^2 + t$ triples, a $B_{\max}(6t^2, 3, 1)$ contains $6t^2 - 2t$ triples. There are $3t$ pairs of elements from the set V_2 which are not contained in the triples of the maximal basis $B_{\max}(6t, 3, 1)$. Thus we have $|T| = 3t$. We form the set

$$B_{\max}(6t + 1, 3, 1) \cup B_{\max}(6t, 3, 1) \cup T.$$

It is evidently a basis $B(12t + 1, 3, 1)$ with $12t^2 + 2t$ edges. Therefore it is minimal.

3) Let $n = 12t + i$, $i = 2, 3, 4, 5, 6$.

If we apply the Construction 1 we obtain minimal bases $B_{\min}(12t + i, 3, 1)$ The proof is the same as in 2)

4) Let $n = 12t + 7$.

The complementary graph to $G(n, 3, 1)$ is Eulerian without triangles. Thus $\bar{G} = U\,(12t + 7, p)$. For the number p of edges in $\bar{G}$ the relation

$$p \leqslant m(n) = 36t^2 + 36t + 9,$$

must hold. According to Lemma 1 the graph $U(12t + 7,\ 36t^2 + 36t + 9)$ cannot be complementary to $G(12t + 7, 3, 1)$. Therefore we must use a $U(12t + 7, 36t^2 + 36t + 6)$. For the number q of edges in a basis $B(12t + 7, 3, 1)$ we obtain the relation:

$$q \geqslant \left(\binom{12t + 7}{2} - 36t^2 - 36t - 6\right)/3 = 12t^2 + 14t + 5.$$

If we apply Construction 1 for $n_1 = 6t + 3$, $n_2 = 6t + 4$, we obtain a basis with $12t^2 + 14t + 5$ edges, i. e. a minimal basis. We find the numbers of triples in the bases $B_{\max}\,(6t + 3, 3, 1)$ and $B_{\max}(6t + 4, 3, 1)$ and in the set T in the same way as in the previous cases.

5) Let $n = 12t + 11$.
According to Lemma 1 the graph $U(n, m(n))$ cannot be complementary to $G(12t + 11, 3, 1)$. The further procedure of the proof in this case is the same as in the case 4).

6) Let $n = 12t + 8$.
The graph $\bar{G}(n, 3, 1)$ is anti-Eulerian without triangles. Thus

$$\bar{G}(n, 3, 1) = A(n, p),$$

where $p \leqslant m(n) = 36t^2 + 48t + 15$.
As the difference $\binom{n}{2} - p$ must be divisible by three, the number of edges $p \leqslant 36t^2 + 48t + 13$ comes into account. According to Lemma 5 the graphs $A(12t + 8,\ 36t^2 + 48t + 13)$ and $A(12t + 8,\ 36t^2 + 48t + 10)$ do not exist. Therefore $p \leqslant 36t^2 + 48t + 7$ and $q \geqslant 12t^2 + 14t + 7$.

By Construction 1 we obtain a basis $B(12t + 8, 3, 1)$ with $12t^2 + 14t + 7$ edges. Therefore this basis is minimal.

7) Let $n = 12t + 9$. The graph $\bar{G}$ is Eulerian without trangles, thus $\bar{G} = U(12t + 9, p)$, where $p \leqslant m(n) = 36t^2 + 48t + 17$ according to Theorem 6. But as the difference $\binom{n}{2} - m(n)$ is not divisible by three, we have $p \leqslant 36t^2 + 48t + 15$.

According to Lemma 2 the graph $U(12t + 9,\ 36t^2 + 48t + 15)$ does not exist. Thus $p \leqslant 36t^2 + 48t + 12$ and

$$q \geqslant \left(\binom{n}{2} - 36t^2 + 48t \quad 12\right) 3 = 12t^2 + 18t + 8.$$

By Construction 1 we obtain a basis $B(12t + 9, 3, 1)$ with $12t^2 + 48t + 8$ edges, which is therefore minimal.

8) Let $n = 12t + 10$. The graph $\bar{G}$ is anti-Eulerian without triangles, thus $\bar{G} = A(12t + 10, p)$, $p \leqslant m(n) = 36t^2 + 60t + 25$ according to Theorem 7. But as the difference $\binom{n}{2} - m(n)$ is not divisible by three, we have $p \leqslant$ $\leqslant 36t^2 + 60t + 24$.

According to Lemma 6 the graph $A(12t + 10, 36t^2 + 60t + 24)$ does not exist. Thus $p \leqslant 36t^2 + 60t + 21$, $q \geqslant 12t^2 + 18t + 8$. By Construction 1 we obtain a minimal basis $B(n, 3, 1)$ with $12t^2 + 18t + 8$ edges, which is therefore minimal.

By this we have exhausted all cases and we have proved the main theorm.

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