Jorge J. Betancor; Isabel Marrero

Convolution operators on Kucera-type spaces for the Hankel transformation

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 1, 47–72

Persistent URL: http://dml.cz/dmlcz/127338

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

CONVOLUTION OPERATORS ON KUCERA-TYPE SPACES FOR THE HANKEL TRANSFORMATION

J. J. BETANCOR and I. MARRERO, La Laguna

(Received June 2, 1994)

1. INTRODUCTION AND PRELIMINARIES

The Hankel integral transformation, defined by

$$(\mathfrak{H}_\mu\varphi)(x) = \int_0^\infty (xt)^{1/2} J_\mu(xt)\varphi(t)\,\mathrm{d}t \quad (\mu \geqslant -\tfrac{1}{2}),$$

where, as usual, J_μ denotes the Bessel function of the first kind and order μ, was studied on distribution spaces by A. H. Zemanian [14], [15]. Given $\mu \in \mathbb{R}$, this author [15, Chapter 5] introduced the space $\mathscr{H}_\mu$ of all those smooth, complex-valued functions $\varphi = \varphi(x)$, $x \in I =]0,\infty[$, such that

$$\gamma^\mu_{m,k}(\varphi) = \sup_{x\in I} |x^m (x^{-1}D)^k x^{-\mu-1/2}\varphi(x)| < \infty$$

for every $m, k \in \mathbb{N}$. When endowed with the topology generated by the family of seminorms $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbb{N}}$, $\mathscr{H}_\mu$ becomes a Fréchet space where $\mathfrak{H}_\mu$ is an automorphism provided that $\mu \geqslant -\frac{1}{2}$ [15, Theorem 5.4-1]. The generalized Hankel transformation $\mathfrak{H}'_\mu$ is then defined on $\mathscr{H}'_\mu$, the dual space of $\mathscr{H}_\mu$, as the transpose of $\mathfrak{H}_\mu$.

Also, A. H. Zemanian [14] defined the space $\mathscr{B}_{\mu,a}$ ($\mu \in \mathbb{R}$, $a > 0$), consisting of all those smooth, complex-valued fuctions $\varphi = \varphi(x)$, $x \in I$, such that $\varphi(x) = 0$ if $x \geqslant a$, for which the quantities

$$\gamma^\mu_k(\varphi) = \sup_{x\in I} |(x^{-1}D)^k x^{-\mu-1/2}\varphi(x)| \quad (k \in \mathbb{N})$$

are finite. Equipped with the topology associated to the system of seminorms $\{\gamma^\mu_k\}_{k\in\mathbb{N}}$, each $\mathscr{B}_{\mu,a}$ is a Fréchet space. The inductive limit $\mathscr{B}_\mu$ of the family $\{\mathscr{B}_{\mu,a}\}_{a>0}$ is a dense subspace of $\mathscr{H}_\mu$.

The theory of the Hankel convolution on $\mathscr{H}_\mu$, $\mathscr{B}_\mu$, and their duals has been developed by the authors in a series of papers [2], [3], [4], [10].

In a previous work [5], we introduced a chain $\{\mathscr{H}_\mu^p\}_{p\in\mathbb{Z}}$ of Hilbert spaces where the Hankel transformation is an automorphism. For every $p \in \mathbb{N}$, $\mathscr{H}_\mu^p$ is the space of all those $\varphi \in L^2(I)$ such that the distributions $T_{\mu,j}\varphi$ $(0 \leqslant j \leqslant p)$ are regular and satisfy

$$\|\varphi\|_{\mu,p} = \Big\{ \sum_{i+j=0}^{p} \int_0^\infty |x^i T_{\mu,j}\varphi(x)|^2 \, \mathrm{d}x \Big\}^{1/2} < \infty,$$

where $T_{\mu,0}$ is the identity operator and $T_{\mu,j}$ denotes the operator $N_{\mu+j-1}\dots N_\mu$ $(j \in \mathbb{N},\ j \geqslant 1)$, with $N_\mu = x^{\mu+1/2}Dx^{-\mu-1/2}$. The space $\mathscr{H}_\mu^p$ and its dual $\mathscr{H}_\mu^{-p}$ $(p \in \mathbb{N})$ are Hilbert, the norm $\|\cdot\|_{\mu,p}$ being induced by the inner product

$$[\varphi,\psi]_{\mu,p} = \sum_{i+j=0}^{p} \int_0^\infty x^{2i} T_{\mu,j}\varphi(x) T_{\mu,j}\bar{\psi}(x) \, \mathrm{d}x \quad (\varphi,\psi \in \mathscr{H}_\mu^p).$$

Moreover, $\operatorname{proj\,lim}_{p\to\infty} \mathscr{H}_\mu^p = \mathscr{H}_\mu$ and $\operatorname{ind\,lim}_{p\to\infty} \mathscr{H}_\mu^{-p} = \mathscr{H}_\mu'$ [5, Proposition 2.15]. The study of the multipliers and Hankel convolution operators on the spaces $\mathscr{H}_\mu^p$ $(p \in \mathbb{Z})$ was initiated by the authors in [6]. In this paper we complete our investigation about multipliers (Sections 2, 5) and Hankel convolution operators (Section 4) of the spaces $\mathscr{H}_\mu^p$ $(p \in \mathbb{Z})$. As a consequence, the space $\mathscr{O}$ of multipliers, respectively $\mathscr{O}'_{\mu,\sharp}$ of Hankel convolution operators, of both $\mathscr{H}_\mu$ and $\mathscr{H}_\mu'$ are expressed as projective-inductive limits of Hilbert spaces. We also examine the joint continuity of the product, respectively the Hankel convolution, from $\mathscr{O} \times \mathscr{H}_\mu'$, respectively $\mathscr{O}'_{\mu,\sharp} \times \mathscr{H}_\mu'$, into $\mathscr{H}_\mu'$ (Section 6). In Section 3 we deal with some auxiliary machinery, mainly the behavior of the Hankel translation operator on $\mathscr{H}_\mu^p$ $(p \in \mathbb{Z})$. Our work is motivated by the study developed in [7], [8], [9] and [11] for the Fourier transformation.

Throughout this paper μ will represent a real number not less than $-\frac{1}{2}$. Also, the letter C will always stand for a positive constant (not necessarily the same in each occurrence).

2. Hankel multiplication distributions

Let $p,q \in \mathbb{N}$. In [6], the authors introduced the space $\mathscr{O}_{p,q}$ of multipliers from $\mathscr{H}_\mu^p$ into $\mathscr{H}_\mu^q$, that is, of all those functions $\theta\colon I \longrightarrow \mathbb{C}$ such that $\theta\varphi \in \mathscr{H}_\mu^q$ for each $\varphi \in \mathscr{H}_\mu^p$ and the mapping $\varphi \longmapsto \theta\varphi$ is continuous. Of course, $\mathscr{O}_{p,q}$ acts as a space of multipliers from $\mathscr{H}_\mu^{-q}$ into $\mathscr{H}_\mu^{-p}$ by transposition.

The main properties of $\mathscr{O}_{p,q}$ were also investigated in [6]. In particular, this space was shown to be Banach under the norm

$$\|\theta\|_{p,q} = \sup\{\|\theta\varphi\|_{\mu,q} \colon \varphi \in \mathscr{H}_\mu^p, \|\varphi\|_{\mu,p} \leqslant 1\} \quad (\theta \in \mathscr{O}_{p,q}).$$

For our purposes it will be convenient to formulate the following alternate description of $\mathcal{O}_{p,q}$.

Proposition 2.1. *Let $p, q \in \mathbb{N}$, $\theta\colon I \longrightarrow \mathbb{C}$. Then $\theta \in \mathcal{O}_{p,q}$ if, and only if, $\theta\varphi \in \mathcal{H}_\mu^q$ for each $\varphi \in \mathcal{H}_\mu$ and*

$$\|\theta\varphi\|_{\mu,q} \leqslant C\|\varphi\|_{\mu,p} \quad (\varphi \in \mathcal{H}_\mu). \tag{2.1}$$

Proof. The condition is plainly necessary. To show that it also suffices, let $\varphi \in \mathcal{H}_\mu^p$. There exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ in $\mathcal{H}_\mu$ which converges to φ in the topology of $\mathcal{H}_\mu^p$ [5, Proposition 2.12]. A subsequence $\{\varphi_{j_k}\}_{k\in\mathbb{N}}$ of $\{\varphi_j\}_{j\in\mathbb{N}}$ converges a.e. to φ. Therefore, $\{\theta\varphi_{j_k}\}_{k\in\mathbb{N}}$ converges a.e. to $\theta\varphi$. On the other hand, since $\{\varphi_j\}_{j\in\mathbb{N}}$ is Cauchy in $\mathcal{H}_\mu^p$, so is $\{\theta\varphi_j\}_{j\in\mathbb{N}}$ in $\mathcal{H}_\mu^q$, by (2.1), and hence it converges to some $\psi \in \mathcal{H}_\mu^q$ in $\mathcal{H}_\mu^q$. In particular, $\{\theta\varphi_{j_k}\}_{k\in\mathbb{N}}$ converges to ψ in $\mathcal{H}_\mu^q$. Then ψ is the limit a.e. of some subsequence of $\{\theta\varphi_{j_k}\}_{k\in\mathbb{N}}$. By uniqueness, $\theta\varphi = \psi$ a.e. Furthermore,

$$\|\theta\varphi\|_{\mu,q} = \lim_{j\to\infty} \|\theta\varphi_j\|_{\mu,q} \leqslant C \lim_{j\to\infty} \|\varphi_j\|_{\mu,p} = C\|\varphi\|_{\mu,p}.$$

This completes the proof. □

Let $p, q \in \mathbb{N}$. It is known that $x^{\mu+1/2}\mathcal{O}_{p,q} \subset \mathcal{H}_\mu'$, at least when $q > \mu + 1$ [6, Proposition 7 and 5, Proposition 2.15]. We want to describe $x^{\mu+1/2}\mathcal{O}_{p,q}$ as a subspace of $\mathcal{H}_\mu'$.

Lemma 2.2. *Let $p, q \in \mathbb{N}$ and $T \in \mathcal{H}_\mu'$. Then $T \in x^{\mu+1/2}\mathcal{O}_{p,q}$ if, and only if, $x^{-\mu-1/2}\varphi(x)T(x) \in \mathcal{H}_\mu^q$ for each $\varphi \in \mathcal{H}_\mu$, with*

$$\|x^{-\mu-1/2}\varphi(x)T(x)\|_{\mu,q} \leqslant C\|\varphi\|_{\mu,p} \quad (\varphi \in \mathcal{H}_\mu). \tag{2.2}$$

Proof. The necessity is clear. By Proposition 2.1, for the sufficiency it is enough to show that T is a function.

Choose $\beta_1 \in C^\infty(I)$ such that $0 \leqslant \beta_1(x) \leqslant 1$ $(x \in I)$, $\beta_1(x) = 1$ $(0 < x \leqslant 1)$, and $\beta_1(x) = 0$ $(x \geqslant 2)$, and define $\beta_n(x) = \beta_1(x/n)$ $(n \in \mathbb{N},\ n \geqslant 2,\ x \in I)$. Also, set $g_n(x) = \beta_n(x)T(x)$ $(n \in \mathbb{N},\ n \geqslant 1,\ x \in I)$ and $g(x) = g_n(x)$ $(n - 1 < x \leqslant n,\ n \in \mathbb{N},\ n \geqslant 1)$.

If $n \in \mathbb{N}$, $n \geqslant 1$, and $\varphi \in \mathcal{B}_\mu$, with $\varphi(x) = 0$ $(x \geqslant n)$, then

$$\begin{aligned}\int_0^n g_{n+1}(x)\varphi(x)\,\mathrm{d}x &= \int_0^\infty g_{n+1}(x)\varphi(x)\,\mathrm{d}x = \langle g_{n+1}, \varphi\rangle = \langle \beta_{n+1}T, \varphi\rangle \\ &= \langle T, \beta_{n+1}\varphi\rangle = \langle T, \varphi\rangle = \langle T, \beta_n\varphi\rangle \\ &= \langle \beta_n T, \varphi\rangle = \langle g_n, \varphi\rangle = \int_0^\infty g_n(x)\varphi(x)\,\mathrm{d}x = \int_0^n g_n(x)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Consequently $g_{n+1}(x) = g_n(x)$ a.e. in $]0, n]$, whence $g(x) = g_n(x)$ a.e. in $]0, n]$ ($n \in \mathbb{N}$, $n \geqslant 1$), and $\langle g, \varphi \rangle = \langle T, \varphi \rangle$ whenever $\varphi \in \mathscr{B}_\mu$. The space $\mathscr{B}_\mu$ being dense in $\mathscr{H}_\mu$, to complete the proof it suffices to show that g defines a distribution in $\mathscr{H}_\mu'$.

Let $\varphi \in \mathscr{H}_\mu$, and choose $r \in \mathbb{N}$, $r > \mu + 1$. Then $(1 + x^2)^r \varphi(x) \beta_n(x) \in \mathscr{H}_\mu$ for all $n \in \mathbb{N}$, $n \geqslant 1$. By (2.2) and [5, Lemma 2.6], we may write

$$
\begin{aligned}
|\langle g, \varphi \rangle| &\leqslant \int_0^\infty |g(x)\varphi(x)| \, \mathrm{d}x \\
&= \lim_{N \to \infty} \int_0^N |g_N(x)\varphi(x)| \, \mathrm{d}x \leqslant \lim_{N \to \infty} \int_0^\infty |g_N(x)\varphi(x)| \, \mathrm{d}x \\
&\leqslant \left\{ \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^r} \, \mathrm{d}x \right\}^{1/2} \lim_{N \to \infty} \left\{ \int_0^\infty |x^{-\mu-1/2}(1+x^2)^r \varphi(x) g_N(x)|^2 \, \mathrm{d}x \right\}^{1/2} \\
&\leqslant C \lim_{N \to \infty} \|x^{-\mu-1/2}(1+x^2)^r \varphi(x) g_N(x)\|_{\mu,q} \\
&\leqslant C \lim_{N \to \infty} \|(1+x^2)^r \varphi(x) \beta_N(x)\|_{\mu,p} \\
&= C \, \|(1+x^2)^r \varphi(x)\|_{\mu,p}.
\end{aligned}
$$

Since $(1 + x^2)^r \varphi(x) \in \mathscr{H}_\mu$ and $\mathscr{H}_\mu^p$ continuously contains $\mathscr{H}_\mu$ [5, Proposition 2.15], Lemma 2.2 is proved. □

Motivated by [11], given $p, q \in \mathbb{N}$ we introduce the space

$$
\begin{gathered}
\mathscr{M}\mathscr{D}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q) = \{F \in \mathscr{H}_\mu' \mid \exists L_F \in \mathscr{L}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q) \\
\text{with } x^{-\mu-1/2}\varphi(x)F(x) = (L_F\varphi)(x) \ (\varphi \in \mathscr{H}_\mu)\}
\end{gathered}
$$

of the so-called *multiplication distributions* from $\mathscr{H}_\mu^p$ into $\mathscr{H}_\mu^q$. Here, and in what follows, $\mathscr{L}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ denotes the space of all continuous linear mappings from $\mathscr{H}_\mu^p$ into $\mathscr{H}_\mu^q$.

Proposition 2.3. *For each $p, q \in \mathbb{N}$, the identity $\mathscr{M}\mathscr{D}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q) = x^{\mu+1/2}\mathscr{O}_{p,q}$ holds.*

P r o o f. It suffices to apply Lemma 2.2. □

3. THE HANKEL TRANSLATION OPERATOR ON $\mathscr{H}_\mu^p$ ($p \in \mathbb{Z}$)

For each $y \in I$, a Hankel translation operator τ_y has been defined on $\mathscr{H}_\mu$ [2, 10]. Before investigating the behavior of that operator on the spaces $\mathscr{H}_\mu^p$ ($p \in \mathbb{N}$) we must prove the following auxiliary result.

Lemma 3.1. *For every $p \in \mathbb{N}$ and $y \in I$, the function $g_{\mu,y}(x) = y^{\mu+1/2}(xy)^{-\mu} \times J_\mu(xy)$ ($x \in I$) lies in $\mathscr{O}_{p,p}$.*

Proof. Fix $p \in \mathbb{N}$ and $y \in I$. If $(x^{-1}D_x)^i\theta(x) \in L^\infty(I)$ $(0 \leqslant i \leqslant p)$, then $\theta \in \mathscr{O}_{p,p}$ [6, Proposition 4]. Now, for $0 \leqslant i \leqslant p$ we have

$$(x^{-1}D_x)^i g_{\mu,y}(x) = (-1)^i y^{2i+\mu+1/2}(xy)^{-\mu-i}J_{\mu+i}(xy) \ (x \in I).$$

Since the function $z^{-\mu-i}J_{\mu+i}(z)$ is bounded on I, this completes the proof. □

At this point, given $p \in \mathbb{N}$ and $y \in I$, we are in a position to define the Hankel translation operator τ_y on $\mathscr{H}_\mu^p$ by the formula

$$(\tau_y\varphi)(x) = \mathfrak{H}_\mu(g_{\mu,y}\mathfrak{H}_\mu\varphi)(x) \quad (\varphi \in \mathscr{H}_\mu^p,\ x \in I),$$

where $g_{\mu,y}$ is the function defined in Lemma 3.1.

Note that the operator τ_y ($y \in I$) reduces to the usual Hankel translation operator when restricted to $\mathscr{H}_\mu$ [2, Equation (3.1)].

Proposition 3.2. *Let $p \in \mathbb{N}$. The operator τ_y ($y \in I$) is well-defined and continuous from $\mathscr{H}_\mu^p$ into itself. Moreover, if $p \geqslant 1$ then the identity*

$$(\tau_y\varphi)(x) = (\tau_x\varphi)(y) \quad (x, y \in I) \tag{3.1}$$

holds for every $\varphi \in \mathscr{H}_\mu^p$.

Proof. Fix $p \in \mathbb{N}$ and $y \in I$. That τ_y is a continuous endomorphism of $\mathscr{H}_\mu^p$ follows from Lemma 3.1 and from the fact that $\mathfrak{H}_\mu$ is an automorphism of $\mathscr{H}_\mu^p$ [5, Theorem 2.2].

Equation (3.1) holds when $\varphi \in \mathscr{H}_\mu$, and $\mathscr{H}_\mu$ is dense in $\mathscr{H}_\mu^p$ [5, Proposition 2.12]. Since convergence in $\mathscr{H}_\mu^p$, $p \geqslant 1$, is stronger than pointwise convergence [5, Lemma 2.14], necessarily (3.1) also holds when $\varphi \in \mathscr{H}_\mu^p$ and $p \geqslant 1$. □

For $y \in I$ and $q \in \mathbb{N}$, the translation operator τ_y is defined on $\mathscr{H}_\mu^{-q}$ by transposition. Then, by Proposition 3.2, τ_y is a continuous linear mapping from $\mathscr{H}_\mu^{-q}$ into itself.

The functional δ_μ introduced in Proposition 3.3 below will be very useful later.

Proposition 3.3. *Let $p \in \mathbb{N}$, $p > \mu + 3/2$. The limit $\lim_{x \to 0+} x^{-\mu-1/2}\varphi(x)$ exists for all $\varphi \in \mathscr{H}_\mu^p$. The linear functional*

$$\langle \delta_\mu, \varphi \rangle = \lim_{x \to 0+} c_\mu x^{-\mu-1/2}\varphi(x) \quad (\varphi \in \mathscr{H}_\mu^p), \tag{3.2}$$

where $c_\mu = 2^\mu \Gamma(\mu + 1)$, is continuous. Furthermore,

$$\langle \delta_\mu, \tau_y \varphi \rangle = \varphi(y) \quad (y \in I,\ \varphi \in \mathscr{H}_\mu^p). \tag{3.3}$$

Proof. Fix $p \in \mathbb{N}$, $p > \mu + 3/2$, and let $\varphi \in \mathscr{H}_\mu^p$. It is known that

$$x^{-\mu-1/2}\varphi(x) = \int_0^\infty y^{\mu+1/2}(\mathfrak{H}_\mu \varphi)(y)(xy)^{-\mu} J_\mu(xy)\, dy$$

for $x \in I$ (see the proof of [5, Lemma 2.14]).

The integrand above lies in $L^1(I)$. Certainly, the function $z^{-\mu}J_\mu(z)$ $(z \in I)$ is bounded. Moreover,

$$|y^{\mu+1/2}(\mathfrak{H}_\mu\varphi)(y)| = \frac{y^{\mu+1/2}}{(1+y)^{p-1}}|(1+y)^{p-1}(\mathfrak{H}_\mu\varphi)(y)| \leqslant \sum_{j=0}^{p-1} |y^j(\mathfrak{H}_\mu\varphi)(y)| \ (y \in I),$$

with $y^j(\mathfrak{H}_\mu\varphi)(y) \in L^1(I)$ $(0 \leqslant j \leqslant p-1)$; see the proof of [5, Lemma 2.14].

Now, by dominated convergence,

$$\lim_{x \to 0+} x^{-\mu-1/2}\varphi(x) = c_\mu^{-1} \int_0^\infty y^{\mu+1/2}(\mathfrak{H}_\mu\varphi)(y)\, dy.$$

The linear functional δ_μ defined by (3.2) satisfies

$$|\langle \delta_\mu, \varphi \rangle| \leqslant \int_0^\infty |y^{\mu+1/2}(\mathfrak{H}_\mu\varphi)(y)|\, dy \leqslant \sum_{j=0}^{p-1} \int_0^\infty |y^j(\mathfrak{H}_\mu\varphi)(y)|\, dy \leqslant C\|\varphi\|_{\mu,p}$$

for every $\varphi \in \mathscr{H}_\mu^p$ (proof of [5, Lemma 2.14]), hence it is continuous.

Finally, let us prove (3.3). For fixed $\varphi \in \mathscr{H}_\mu^p$ and $x \in I$, there holds

$$\mathfrak{H}_\mu(\varphi(y) - c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y))(t) = (1 - c_\mu(xt)^{-\mu}J_\mu(xt))(\mathfrak{H}_\mu\varphi)(t) \ (t \in I). \tag{3.4}$$

As $(\mathfrak{H}_\mu\varphi)(t)$ $(t \in I)$ is bounded [5, Theorem 2.2 and Lemma 2.14], and since $\lim_{t \to 0+}(1 - c_\mu(xt)^{-\mu}J_\mu(xt)) = 0$, it follows from (3.4) that

$$\lim_{t \to 0+} \mathfrak{H}_\mu(\varphi(y) - c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y))(t) = 0. \tag{3.5}$$

The difference $1 - c_\mu (xt)^{-\mu} J_\mu(xt)$ being bounded for all $t \in I$, (3.4) also implies

$$|\mathfrak{H}_\mu(\varphi(y) - c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y))(t)| \leqslant C|(\mathfrak{H}_\mu\varphi)(t)|, \tag{3.6}$$

with $(\mathfrak{H}_\mu\varphi)(t) \in L^1(I)$ and $C > 0$ not depending on $t \in I$.

On the other hand, we have

$$\begin{aligned}\varphi(y) &- c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y)\\ &= \int_0^\infty \mathfrak{H}_\mu(\varphi(y) - c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y))(t)(yt)^{1/2}J_\mu(yt)\,\mathrm{d}t \quad (y \in I),\end{aligned} \tag{3.7}$$

because $\varphi(y) - c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y) \in L^1(I)$.

Formulas (3.5) and (3.7), along with the dominated convergence theorem (which applies by virtue of (3.6)), yield

$$\varphi(y) = \lim_{x\to 0+} c_\mu x^{-\mu-1/2}(\tau_x\varphi)(y) \quad (\varphi \in \mathscr{H}_\mu^p, y \in I).$$

In view of Proposition 3.2, this establishes (3.3). □

4. Hankel convolution operators on $\mathscr{H}_\mu^p$ ($p \in \mathbb{Z}$)

Let $p, q \in \mathbb{N}$. In [6], the spaces of convolution operators

$$\mathcal{O}_{p,q}^\sharp = \{T \in \mathscr{H}_\mu' \colon y^{-\mu-1/2}(\mathfrak{H}_\mu' T)(y) \in \mathcal{O}_{p,q}\}$$

were defined and endowed with the norm

$$|T|_{p,q}^\sharp = \|y^{-\mu-1/2}(\mathfrak{H}_\mu' T)(y)\|_{p,q} \quad (T \in \mathcal{O}_{p,q}^\sharp).$$

By topologizing the space $x^{\mu+1/2}\mathcal{O}_{p,q}$ so as to make it isometric to $\mathcal{O}_{p,q}$, we have the following.

Proposition 4.1. *The generalized Hankel transformation is an isomorphism from $\mathcal{O}_{p,q}^\sharp$ onto $x^{\mu+1/2}\mathcal{O}_{p,q}$.*

Taking into account Lemma 2.2, another description of the spaces $\mathcal{O}_{p,q}^\sharp$ may be given. To this end, we recall that the Hankel convolution of $T \in \mathscr{H}_\mu'$ and $\varphi \in \mathscr{H}_\mu$ is the function $(T\sharp\varphi)(x) = \langle T, \tau_x\varphi\rangle$ $(x \in I)$ [10, Definition 3.1]. For all $T \in \mathscr{H}_\mu'$ and $\varphi \in \mathscr{H}_\mu$, the exchange formula

$$\mathfrak{H}_\mu'(T\sharp\varphi)(y) = y^{-\mu-1/2}(\mathfrak{H}_\mu\varphi)(y)(\mathfrak{H}_\mu' T)(y) \quad (y \in I)$$

holds [10, Proposition 3.5].

Proposition 4.2. *For each* $p, q \in \mathbb{N}$, $\mathcal{O}^{\sharp}_{p,q}$ *is the space of all those* $T \in \mathcal{H}'_\mu$ *such that* $T\sharp\varphi \in \mathcal{H}^q_\mu$ *for all* $\varphi \in \mathcal{H}_\mu$ *and*

$$\|T\sharp\varphi\|_{\mu,q} \leqslant C\|\varphi\|_{\mu,p} \quad (\varphi \in \mathcal{H}_\mu).$$

Proof. Let $T \in \mathcal{H}'_\mu$. By Lemma 2.2 and [5, Theorem 2.2], $y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y) \in \mathcal{O}_{p,q}$ if, and only if, $y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)(\mathfrak{H}_\mu\varphi)(y) = \mathfrak{H}_\mu(T\sharp\varphi)(y) \in \mathcal{H}^q_\mu$ for each $\varphi \in \mathcal{H}_\mu$ and

$$\|T\sharp\varphi\|_{\mu,q} = \|\mathfrak{H}_\mu(T\sharp\varphi)\|_{\mu,q} \leqslant C\|\mathfrak{H}_\mu\varphi\|_{\mu,p} = C\|\varphi\|_{\mu,p} \quad (\varphi \in \mathcal{H}_\mu).$$

□

Fix $p, q \in \mathbb{N}$. Since $\mathcal{H}_\mu$ is dense in $\mathcal{H}^p_\mu$ [5, Proposition 2.12], for each $T \in \mathcal{O}^{\sharp}_{p,q}$ the continuous mapping

$$\begin{aligned} \sharp\colon (\mathcal{H}_\mu, \|\cdot\|_{\mu,p}) &\longrightarrow (\mathcal{H}^q_\mu, \|\cdot\|_{\mu,q}) \\ \varphi &\longmapsto T\sharp\varphi \end{aligned}$$

admits a unique extension up to $\mathcal{H}^p_\mu$ preserving the norm, which we keep denoting by the same symbol $\sharp$. Hence, $\mathcal{O}^{\sharp}_{p,q}$ may be regarded as a subspace of $\mathcal{L}(\mathcal{H}^p_\mu, \mathcal{H}^q_\mu)$. The norm of $\mathcal{L}(\mathcal{H}^p_\mu, \mathcal{H}^q_\mu)$ restricted to $\mathcal{O}^{\sharp}_{p,q}$ will be represented by $\|\cdot\|^{\sharp}_{p,q}$.

The following exchange formula will be of great utility later.

Proposition 4.3. *Let* $p, q \in \mathbb{N}$. *For each* $T \in \mathcal{O}^{\sharp}_{p,q}$ *and* $\varphi \in \mathcal{H}^p_\mu$, *the identity*

$$\mathfrak{H}_\mu(T\sharp\varphi)(y) = y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)(\mathfrak{H}_\mu\varphi)(y) \quad (y \in I) \tag{4.1}$$

holds. Furthermore, the norms $\|\cdot\|^{\sharp}_{p,q}$ *and* $|\cdot|^{\sharp}_{p,q}$ *coincide on* $\mathcal{O}^{\sharp}_{p,q}$.

Proof. If $T \in \mathcal{O}^{\sharp}_{p,q}$ then $y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y) \in \mathcal{O}_{p,q}$. Consequently, both the left- and the right-hand sides of (4.1) define continuous linear mappings from $\mathcal{H}^p_\mu$ into $\mathcal{H}^q_\mu$. Since (4.1) holds if $\varphi \in \mathcal{H}_\mu$ with $\mathcal{H}_\mu$ dense in $\mathcal{H}^p_\mu$ [5, Proposition 2.12], necessarily it also holds for all $\varphi \in \mathcal{H}^p_\mu$.

Now, the fact that $\mathfrak{H}_\mu$ is an isometry of $\mathcal{H}^q_\mu$ [5, Theorem 2.2], along with (4.1), implies

$$\begin{aligned} \|T\|^{\sharp}_{p,q} &= \sup\{\|T\sharp\varphi\|_{\mu,q} : \varphi \in \mathcal{H}^p_\mu, \|\varphi\|_{\mu,p} \leqslant 1\} \\ &= \sup\{\|\mathfrak{H}_\mu(T\sharp\varphi)\|_{\mu,q} : \varphi \in \mathcal{H}^p_\mu, \|\varphi\|_{\mu,p} \leqslant 1\} \\ &= \sup\{\|y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)\varphi(y)\|_{\mu,q} : \varphi \in \mathcal{H}^p_\mu, \|\varphi\|_{\mu,p} \leqslant 1\} \\ &= \|y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)\|_{p,q} = |T|^{\sharp}_{p,q} \quad (T \in \mathcal{O}^{\sharp}_{p,q}). \end{aligned}$$

□

Let $p, q \in \mathbb{N}$. The Hankel convolution of $T \in \mathcal{O}_{p,q}^{\sharp}$ and $u \in \mathcal{H}_\mu^{-q}$ is the functional $T\sharp u \in \mathcal{H}_\mu^{-p}$, given by

$$\langle T\sharp u, \varphi\rangle = \langle u, T\sharp\varphi\rangle \quad (\varphi \in \mathcal{H}_\mu^p).$$

Note that, for a fixed $T \in \mathcal{O}_{p,q}^{\sharp}$, the mapping $u \longmapsto T\sharp u$ from $\mathcal{H}_\mu^{-q}$ into $\mathcal{H}_\mu^{-p}$ is the transpose of the mapping $\varphi \longmapsto T\sharp\varphi$ from $\mathcal{H}_\mu^p$ into $\mathcal{H}_\mu^q$.

Note also that if $T \in \mathcal{O}_{p,q}^{\sharp}$ and $\psi \in \mathcal{H}_\mu$ then [5, Propositions 2.4 and 2.15] provide us with two (coincident) definitions of the functional $T\sharp\psi \in \mathcal{H}_\mu^{-p}$. In fact, on the one hand, since $\psi \in \mathcal{H}_\mu \subset \mathcal{H}_\mu^p$, the function $v_1(x) = \langle T, \tau_x\psi\rangle$ $(x \in I)$ satisfies $v_1 \in \mathcal{H}_\mu^q \subset \mathcal{H}_\mu^{-p}$. On the other hand, as $\psi \in \mathcal{H}_\mu^{-q}$, we may consider the functional $v_2 \in \mathcal{H}_\mu^{-p}$ defined by $\langle v_2, \varphi\rangle = \langle \psi, T\sharp\varphi\rangle$ $(\varphi \in \mathcal{H}_\mu^p)$. Now, $\mathcal{H}_\mu^{-p} \subset \mathcal{H}_\mu'$, and $\langle v_1, \varphi\rangle = \langle v_2, \varphi\rangle$ whenever $\varphi \in \mathcal{H}_\mu$ [10, Proposition 3.5]. Since $\mathcal{H}_\mu$ is dense in $\mathcal{H}_\mu^p$ [5, Proposition 2.12], v_1 and v_2 define the same functional in $\mathcal{H}_\mu^{-p}$.

Corollary 4.4. *Let $p, q \in \mathbb{N}$. Given $T \in \mathcal{O}_{p,q}^{\sharp}$ and $u \in \mathcal{H}_\mu^{-q}$, the identity*

$$\mathfrak{H}_\mu'(T\sharp u)(y) = y^{-\mu-1/2}(\mathfrak{H}_\mu' T)(y)(\mathfrak{H}_\mu' u)(y)$$

holds.

Proof. By Proposition 4.3, for all $\varphi \in \mathcal{H}_\mu^p$ we have

$$\begin{aligned}\left\langle \mathfrak{H}_\mu'(T\sharp u), \mathfrak{H}_\mu\varphi\right\rangle &= \langle T\sharp u, \varphi\rangle = \langle u, T\sharp\varphi\rangle = \left\langle \mathfrak{H}_\mu' u, \mathfrak{H}_\mu(T\sharp\varphi)\right\rangle \\ &= \left\langle (\mathfrak{H}_\mu' u)(y), y^{-\mu-1/2}(\mathfrak{H}_\mu' T)(y)(\mathfrak{H}_\mu\varphi)(y)\right\rangle \\ &= \left\langle y^{-\mu-1/2}(\mathfrak{H}_\mu' T)(y)(\mathfrak{H}_\mu' u)(y), (\mathfrak{H}_\mu\varphi)(y)\right\rangle.\end{aligned}$$

□

Next we aim to characterize $\mathcal{O}_{p,q}^{\sharp}$ $(p, q \in \mathbb{N},\ q > \mu + 3/2)$ as the space of all those continuous linear operators from $\mathcal{H}_\mu^p$ into $\mathcal{H}_\mu^q$ (respectively, $\mathcal{H}_\mu^{-q}$ into $\mathcal{H}_\mu^{-p}$) commuting with Hankel translations (Proposition 4.6).

We adopt from [11] the notation

$$m(\mathcal{A}, \mathcal{B}) = \{T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \mid \tau_y T = T\tau_y \quad (y \in I)\}.$$

Here $\mathcal{A}$ and $\mathcal{B}$ denote suitable linear topological spaces where the Hankel translation is defined, and $\mathcal{L}(\mathcal{A}, \mathcal{B})$ represents the space of continuous linear mappings from $\mathcal{A}$ into $\mathcal{B}$. Recall that the spaces $\mathcal{M}\mathcal{D}(\mathcal{H}_\mu^p, \mathcal{H}_\mu^q)$ $(p, q \in \mathbb{N})$ were defined at the end of Section 2 above.

Lemma 4.5. *Assume $p, q \in \mathbb{N}$, with $q > \mu + 3/2$. To every $F \in \mathscr{MD}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ there corresponds a unique $T_F \in m(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ such that $T_F\varphi = (\mathfrak{H}'_\mu F)\sharp\varphi$ ($\varphi \in \mathscr{H}_\mu$). The mapping $\mathbb{T}\colon F \longrightarrow T_F$ is an isomorphism from $\mathscr{MD}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ onto $m(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$.*

Proof. Given $F \in \mathscr{MD}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$, we define $T_F\colon \mathscr{H}_\mu^p \longrightarrow \mathscr{H}_\mu^q$ by $T_F\varphi = \mathfrak{H}_\mu(L_F\mathfrak{H}_\mu\varphi)$, where L_F satisfies $x^{-\mu-1/2}\varphi(x)F(x) = (L_F\varphi)(x)$ ($\varphi \in \mathscr{H}_\mu$); then, $T_F \in \mathscr{L}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$. Fix $y \in I$ and $\varphi \in \mathscr{H}_\mu$, and let $g_{\mu,y}$ be the function considered in Lemma 3.1. There holds

$$g_{\mu,y}(x)(L_F\varphi)(x) = x^{-\mu-1/2}g_{\mu,y}(x)\varphi(x)F(x) = L_F(g_{\mu,y}\varphi)(x).$$

Hence, we may write:

$$\begin{aligned}\tau_y(T_F\varphi) &= \mathfrak{H}_\mu(g_{\mu,y}\mathfrak{H}_\mu(T_F\varphi)) = \mathfrak{H}_\mu(g_{\mu,y}L_F(\mathfrak{H}_\mu\varphi))\\ &= \mathfrak{H}_\mu(L_F(g_{\mu,y}\mathfrak{H}_\mu\varphi)) = \mathfrak{H}_\mu(L_F(\mathfrak{H}_\mu\tau_y\varphi)) = T_F(\tau_y\varphi).\end{aligned}$$

Since τ_yT_F and $T_F\tau_y$ both lie in $\mathscr{L}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$, and since $\mathscr{H}_\mu$ is dense in $\mathscr{H}_\mu^p$ [5, Proposition 2.12], we conclude that $T_F \in m(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$.

Conversely, assume that $T \in m(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ and define the linear functional

$$\langle H, \varphi\rangle = \langle\delta_\mu, T\varphi\rangle \quad (\varphi \in \mathscr{H}_\mu),$$

where δ_μ is as in Proposition 3.3. Then $H \in \mathscr{H}'_\mu$, because $\delta_\mu T$ is a bounded linear functional on $\mathscr{H}_\mu^p$ and the inclusion of $\mathscr{H}_\mu$ into $\mathscr{H}_\mu^p$ is continuous.

Fix $\varphi \in \mathscr{H}_\mu$. Using Proposition 3.3 we obtain

$$(T\varphi)(y) = \langle\delta_\mu, \tau_yT\varphi\rangle = \langle\delta_\mu, T\tau_y\varphi\rangle = \langle H, \tau_y\varphi\rangle = (H\sharp\varphi)(y) \quad (y \in I), \tag{4.2}$$

or, in other words, $T(\mathfrak{H}_\mu\varphi) = H\sharp(\mathfrak{H}_\mu\varphi)$. Let $F = \mathfrak{H}'_\mu H \in \mathscr{H}'_\mu$; then

$$\mathfrak{H}_\mu(T\mathfrak{H}_\mu\varphi)(y) = \mathfrak{H}'_\mu((\mathfrak{H}'_\mu F)\sharp(\mathfrak{H}_\mu\varphi))(y) = y^{-\mu-1/2}\varphi(y)F(y). \tag{4.3}$$

Since $\mathfrak{H}_\mu T\mathfrak{H}_\mu = L \in \mathscr{L}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$, we find that $F \in \mathscr{MD}(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$.

Now the proof may be completed as follows. That the mapping $\mathbb{T}$ is well defined is a consequence of the first paragraph above, along with (4.2) and (4.3). The preceding paragraph shows that $\mathbb{T}$ is onto. Finally, the remaining assertions of Lemma 4.5 are obvious. □

Proposition 4.6. *Let $p, q \in \mathbb{N}$, $q > \mu + 3/2$. A linear map $L\colon \mathscr{H}_\mu^p \longrightarrow \mathscr{H}_\mu^q$ (respectively, $L\colon \mathscr{H}_\mu^{-q} \longrightarrow \mathscr{H}_\mu^{-p}$) lies in $m(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ (respectively, $m(\mathscr{H}_\mu^{-q}, \mathscr{H}_\mu^{-p})$)*

if, and only if, there exists a unique $T \in \mathscr{O}^{\sharp}_{p,q}$ such that $Lu = T\sharp u$ for all $u \in \mathscr{H}^p_\mu$ (respectively, $u \in \mathscr{H}^{-q}_\mu$).

Proof. Assume $L \in m(\mathscr{H}^p_\mu, \mathscr{H}^q_\mu)$. By Lemma 4.5 and Propositions 2.3 and 4.1, there exists $T \in \mathscr{O}^{\sharp}_{p,q}$ such that $L\varphi = T\sharp\varphi$ if $\varphi \in \mathscr{H}_\mu$. Since $\mathscr{H}_\mu$ is dense in $\mathscr{H}^p_\mu$ [5, Proposition 2.12], by continuity $L\varphi = T\sharp\varphi$ whenever $\varphi \in \mathscr{H}^p_\mu$.

If $T \in \mathscr{O}^{\sharp}_{p,q}$ and $L\varphi = T\sharp\varphi$ for all $\varphi \in \mathscr{H}^p_\mu$, then $L\varphi = T\sharp\varphi$ for all $\varphi \in \mathscr{H}_\mu$. From Lemma 4.5 and Propositions 2.3 and 4.1, we conclude that $L \in m(\mathscr{H}^p_\mu, \mathscr{H}^q_\mu)$.

Now suppose $L \in m(\mathscr{H}^{-q}_\mu, \mathscr{H}^{-p}_\mu)$. Then is adjoint L^* lies in $m(\mathscr{H}^p_\mu, \mathscr{H}^q_\mu)$; in fact, for all $y \in I$, $\varphi \in \mathscr{H}^p_\mu$ and $u \in \mathscr{H}^{-q}_\mu$, we have

$$\begin{aligned}\langle u, \tau_y L^*\varphi\rangle &= \langle \tau_y u, L^*\varphi\rangle = \langle L\tau_y u, \varphi\rangle = \langle \tau_y Lu, \varphi\rangle = \langle Lu, \tau_y\varphi\rangle \\ &= \langle u, L^*\tau_y\varphi\rangle .\end{aligned}$$

As just proved, there exists $T \in \mathscr{O}^{\sharp}_{p,q}$ satisfying $L^*\varphi = T\sharp\varphi$ ($\varphi \in \mathscr{H}^p_\mu$). The identities

$$\langle Lu, \varphi\rangle = \langle u, L^*\varphi\rangle = \langle u, T\sharp\varphi\rangle = \langle T\sharp u, \varphi\rangle \quad (u \in \mathscr{H}^{-q}_\mu, \varphi \in \mathscr{H}^p_\mu)$$

show that $Lu = T\sharp u$ ($u \in \mathscr{H}^{-q}_\mu$).

Conversely, if $T \in \mathscr{O}^{\sharp}_{p,q}$ and $Lu = T\sharp u$ for all $u \in \mathscr{H}^{-q}_\mu$ then $L \in \mathscr{L}(\mathscr{H}^{-q}_\mu, \mathscr{H}^{-p}_\mu)$. Since $\tau_y(T\sharp\varphi) = T\sharp(\tau_y\varphi)$ for all $y \in I$ and $\varphi \in \mathscr{H}^p_\mu$, we get

$$\begin{aligned}\langle L\tau_y u, \varphi\rangle &= \langle T\sharp\tau_y u, \varphi\rangle = \langle \tau_y u, T\sharp\varphi\rangle = \langle u, \tau_y(T\sharp\varphi)\rangle \\ &= \langle u, T\sharp(\tau_y\varphi)\rangle = \langle T\sharp u, \tau_y\varphi\rangle = \langle \tau_y Lu, \varphi\rangle \quad (y \in I, \varphi \in \mathscr{H}^p_\mu).\end{aligned}$$

That is, $L \in m(\mathscr{H}^{-q}_\mu, \mathscr{H}^{-p}_\mu)$.

To prove uniqueness, assume that $T \in \mathscr{O}^{\sharp}_{p,q}$ and $T\sharp u = 0$, for all $u \in \mathscr{H}^p_\mu$ (respectively, $u \in \mathscr{H}^{-q}_\mu$). Then, by Proposition 4.3 (respectively, Corollary 4.4), $0 = x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x) \in \mathscr{O}_{p,q}$, whence $T = 0$. This completes the proof. □

As $\|\varphi\|_{\mu,p} \leqslant \|\varphi\|_{\mu,q}$ whenever $p, q \in \mathbb{N}, p \leqslant q$, and $\varphi \in \mathscr{H}^q_\mu$, the identity mappings

$$\mathscr{O}^{\sharp}_{p,q+1} \hookrightarrow \mathscr{O}^{\sharp}_{p,q} \hookrightarrow \mathscr{O}^{\sharp}_{p+1,q} \quad (p, q \in \mathbb{N})$$

are continuous. This fact allows us to consider the limits

$$\mathscr{O}^{\sharp}_q = \operatorname*{ind\,lim}_{p\to\infty} \mathscr{O}^{\sharp}_{p,q} \quad (q \in \mathbb{N}) \quad \text{and} \quad \mathscr{O}'_{\mu,\sharp} = \operatorname*{proj\,lim}_{q\to\infty} \mathscr{O}^{\sharp}_q .$$

Here $\mathscr{O}'_{\mu,\sharp}$ is the space of convolution operators of $\mathscr{H}_\mu$ and $\mathscr{H}'_\mu$ [6, Proposition 19].

In the following Proposition 4.7 we characterize $\mathscr{O}^{\sharp}_q$ ($q \in \mathbb{N}$) as the space of convolution operators from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$ and from $\mathscr{H}^{-q}_\mu$ into $\mathscr{H}'_\mu$.

Proposition 4.7. *Let $q \in \mathbb{N}$ and $T \in \mathscr{H}'_\mu$. The following are equivalent.*

(i) $T \in \mathscr{O}^\sharp_q$.

(ii) *The mapping $\varphi \longmapsto T\sharp\varphi$ is continuous from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$.*

(iii) *The mapping $u \longmapsto T\sharp u$ is continuous from $\mathscr{H}^{-q}_\mu$ into $\mathscr{H}'_\mu$ when $\mathscr{H}'_\mu$ is endowed with either its weak* or its strong topology.*

Proof. Statements (ii) and (iii) are equivalent by transposition. We shall establish the equivalence between (i) and (ii).

Assume $T \in \mathscr{O}^\sharp_q$, and let $p \in \mathbb{N}$ be such that $T \in \mathscr{O}^\sharp_{p,q}$. Then $L\varphi = T\sharp\varphi$ is continuous from $\mathscr{H}^p_\mu$ into $\mathscr{H}^q_\mu$. Since the embedding $\mathscr{H}_\mu \hookrightarrow \mathscr{H}^p_\mu$ is continuous [5, Proposition 2.15], L is continuous from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$.

Conversely, let $T \in \mathscr{H}'_\mu$ be such that $L\varphi = T\sharp\varphi$ is continuous from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$. By [6, Lemma 1], there exists $p \in \mathbb{N}$ satisfying $\|T\sharp\varphi\|_{\mu,q} \leqslant C\|\varphi\|_{\mu,p}$ $(\varphi \in \mathscr{H}_\mu)$. But this means that $T \in \mathscr{O}^\sharp_{p,q} \subset \mathscr{O}^\sharp_q$, and completes the proof. □

Our next objective is to characterize $\mathscr{O}^\sharp_q$ $(q \in \mathbb{N},\ q > \mu + 3/2)$ as the space of continuous linear operators from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$ (respectively, $\mathscr{H}^{-q}_\mu$ into $\mathscr{H}'_\mu$) that commute with Hankel translations.

Proposition 4.8. *Let $q \in \mathbb{N}$, $q > \mu + 3/2$. A linear map $L\colon \mathscr{H}_\mu \longrightarrow \mathscr{H}^q_\mu$ lies in $m(\mathscr{H}_\mu, \mathscr{H}^q_\mu)$ (respectively, $m(\mathscr{H}^{-q}_\mu, \mathscr{H}'_\mu)$, where $\mathscr{H}'_\mu$ is endowed with either its weak* or its strong topology) if, and only if, there exists a unique $T \in \mathscr{O}^\sharp_q$ such that $Lu = T\sharp u$ for all $u \in \mathscr{H}_\mu$ (respectively, $u \in \mathscr{H}^{-q}_\mu$).*

Proof. By Propositions 4.6 and 4.7, it is apparent that the mapping $L\varphi = T\sharp\varphi$ $(\varphi \in \mathscr{H}_\mu)$, with $T \in \mathscr{O}^\sharp_q$, is continuous from $\mathscr{H}_\mu$ into $\mathscr{H}^q_\mu$ and commutes with translations.

Conversely, if $L \in m(\mathscr{H}_\mu, \mathscr{H}^q_\mu)$ then the mapping $\varphi \longmapsto \langle \delta_\mu, L\varphi\rangle$ $(\varphi \in \mathscr{H}_\mu)$ defines a linear functional $T \in \mathscr{H}'_\mu$ satisfying

$$(L\varphi)(x) = \langle \delta_\mu, \tau_x L\varphi\rangle = \langle \delta_\mu, L\tau_x\varphi\rangle = \langle T, \tau_x\varphi\rangle = (T\sharp\varphi)(x)$$

for every $\varphi \in \mathscr{H}_\mu$ and $x \in I$ (Proposition 3.3). From Proposition 4.7 we conclude that $T \in \mathscr{O}^\sharp_q$.

To prove uniqueness it suffices to argue as in the proof of Proposition 4.6.

The respective part follows easily by transposition. □

At this point we aim to describe $\mathscr{O}^\sharp_q$ $(q \in \mathbb{N})$ as an inductive limit of Hilbert spaces (Proposition 4.10). For every $k \in \mathbb{N}$ we define the operator $(1 - S_\mu)^{-k}$ from $\mathscr{H}_\mu$ into itself by the formula $(1 - S_\mu)^{-k}\varphi = \mathfrak{H}_\mu((1 + x^2)^{-k}(\mathfrak{H}_\mu\varphi)(x))$ $(\varphi \in \mathscr{H}_\mu)$, and from $\mathscr{H}'_\mu$ into itself by transposition. These mappings are injective.

Given $k, q \in \mathbb{N}$, put

$$(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q} = \{(1 - S_\mu)^{-k} u \in \mathscr{H}_\mu' : u \in \mathscr{H}_\mu^{-q}\},$$
$$(1 - S_\mu)^{k} \mathscr{H}_\mu^{q} = \{T \in \mathscr{H}_\mu' : (1 - S_\mu)^{-k} T \in \mathscr{H}_\mu^{q}\},$$

and endow those spaces with the topologies that make the operator $(1 - S_\mu)^{-k}$ an isometry. It is apparent that for each k, $q \in \mathbb{N}$, the Hilbert spaces $(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q}$ and $(1 - S_\mu)^{k} \mathscr{H}_\mu^{q}$ constitute a dual system with respect to the bilinear form

$$\begin{aligned}(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q} \times (1 - S_\mu)^{k} \mathscr{H}_\mu^{q} &\longrightarrow \mathbb{C}\\ (u, v) \quad &\longmapsto \langle u, v\rangle = \left\langle (1 - S_\mu)^{k} u, (1 - S_\mu)^{-k} v\right\rangle.\end{aligned}$$

Lemma 4.9. *For every $q \in \mathbb{N}$, the following holds.*
(i) *$\mathscr{H}_\mu$ is dense in $(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q}$ ($k \in \mathbb{N}$).*
(ii) *$\mathscr{H}_\mu^{q} \subset (1 - S_\mu)\mathscr{H}_\mu^{q} \subset (1 - S_\mu)^2 \mathscr{H}_\mu^{q} \subset \ldots$, with continuous embedding.*
(iii) *$\mathscr{H}_\mu^{-q} \supset (1 - S_\mu)^{-1} \mathscr{H}_\mu^{-q} \supset (1 - S_\mu)^{-2} \mathscr{H}_\mu^{-q} \supset \ldots$, with continuous embedding.*
(iv) *$\operatorname{ind}\lim_{k\to\infty} (1 - S_\mu)^{k} \mathscr{H}_\mu^{q}$ is the strong dual of $\operatorname{proj}\lim_{k\to\infty} (1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q}$.*

Proof. Since $(1 - S_\mu)^{-k}$ defines an isometric isomorphism between $\mathscr{H}_\mu^{-q}$ and $(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q}$, and since $\mathscr{H}_\mu = (1 - S_\mu)^{-k} \mathscr{H}_\mu$ is dense in $\mathscr{H}_\mu^{-q}$ [5, Proposition 2.12], part (i) follows.

Next we observe that, for all $k \in \mathbb{N}$, $(1 - S_\mu)^{-k} \mathscr{H}_\mu^{q} \subset \mathscr{H}_\mu^{q}$ as well as $(1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q} \subset \mathscr{H}_\mu^{-q}$ [6, Corollary 1 and 5, Theorem 2.2], and that

$$\begin{aligned}(1 - S_\mu)^{-n-m} T &= \mathfrak{H}_\mu'((1 + x^2)^{-n} \mathfrak{H}_\mu' \mathfrak{H}_\mu' (1 + x^2)^{-m} (\mathfrak{H}_\mu' T)(x))\\ &= (1 - S_\mu)^{-n} (1 - S_\mu)^{-m} T \quad (n, m \in \mathbb{N},\ T \in \mathscr{H}_\mu').\end{aligned}$$

Fix $k \in \mathbb{N}$, and let $T \in (1 - S_\mu)^{k} \mathscr{H}_\mu^{q}$, so that $T \in \mathscr{H}_\mu'$ and $(1 - S_\mu)^{-k} T = \varphi \in \mathscr{H}_\mu^{q}$. Then, as just observed, $(1 - S_\mu)^{-k-1} T = (1 - S_\mu)^{-1} \varphi \in \mathscr{H}_\mu^{q}$, whence $T \in (1 - S_\mu)^{k+1} \mathscr{H}_\mu^{q}$. Moreover,

$$\begin{aligned}\|(1 - S_\mu)^{-1} \varphi\|_{\mu,q} &= \|\mathfrak{H}_\mu((1 + x^2)^{-1} \mathfrak{H}_\mu \varphi(x))\|_{\mu,q}\\ &= \|(1 + x^2)^{-1} \mathfrak{H}_\mu \varphi(x)\|_{\mu,q} \leqslant \|(1 + x^2)^{-1}\|_{q,q} \|\varphi\|_{\mu,q}.\end{aligned}$$

This proves (ii).

Similarly, if $T = (1 - S_\mu)^{-k-1} u \in \mathscr{H}_\mu'$, where $u \in \mathscr{H}_\mu^{-q}$, then

$$T = (1 - S_\mu)^{-k} (1 - S_\mu)^{-1} u \in (1 - S_\mu)^{-k} \mathscr{H}_\mu^{-q},$$

with

$$\begin{aligned}\|(1 - S_\mu)^{-1} u\|_{\mu,-q} &= \|\mathfrak{H}_\mu'((1 + x^2)^{-1} \mathfrak{H}_\mu' u(x))\|_{\mu,-q}\\ &= \|(1 + x^2)^{-1} \mathfrak{H}_\mu' u(x)\|_{\mu,-q} \leqslant \|(1 + x^2)^{-1}\|_{q,q} \|u\|_{\mu,-q}.\end{aligned}$$

Thus, (iii) is proved.

The limit $\operatorname*{proj\,lim}_{k\to\infty}(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ is reduced, because of (i). Hence, its dual with the Mackey topology may be identified with $\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q$ [12, IV-4.4]. Since $\operatorname*{proj\,lim}_{k\to\infty}(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ is semireflexive [12, IV-5.8], the strong dual topology and the Mackey topology coincide on $\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q$ [12, IV-5.5]. This establishes (iv). □

Proposition 4.10. *For all $q\in\mathbb{N}$, the identity $\mathcal{O}_q^\sharp=\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q$ holds.*

Proof. Let $p,q\in\mathbb{N}$, let $T\in\mathcal{O}_{p,q}^\sharp$, and choose $k\in\mathbb{N}$ such that $x^{\mu+1/2}(1+x^2)^{-k}\in\mathscr{H}_\mu^p$. Then, $\varphi=\mathfrak{H}_\mu(x^{\mu+1/2}(1+x^2)^{-k})\in\mathscr{H}_\mu^p$ [5, Theorem 2.2] and $f=T\sharp\varphi\in\mathscr{H}_\mu^q$. From Proposition 4.3 we obtain $(\mathfrak{H}_\mu f)(x)=(1+x^2)^{-k}(\mathfrak{H}'_\mu T)(x)\in\mathscr{H}_\mu^q$; consequently $f=\mathfrak{H}_\mu((1+x^2)^{-k}(\mathfrak{H}'_\mu T)(x))=(1-S_\mu)^{-k}T\in\mathscr{H}_\mu^q$ [5, Theorem 2.2], and hence $T\in(1-S_\mu)^k\mathscr{H}_\mu^q$.

Since the mappings

$$\begin{aligned}\mathcal{O}_{p,q}^\sharp &\longrightarrow \mathscr{H}_\mu^q \longrightarrow (1-S_\mu)^k\mathscr{H}_\mu^q,\\ T &\longmapsto T\sharp\varphi\longmapsto(1-S_\mu)^k(T\sharp\varphi)=T\end{aligned}$$

are continuous, so are the embeddings

$$\mathcal{O}_{p,q}^\sharp\hookrightarrow(1-S_\mu)^k\mathscr{H}_\mu^q\hookrightarrow\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q,$$

and from the arbitrariness of $p\in\mathbb{N}$ we infer the continuity of

$$\mathcal{O}_q^\sharp\hookrightarrow\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q.$$

Conversely, given $k\in\mathbb{N}$ and $\varphi\in\mathscr{H}_\mu$, let $\psi=(1-S_\mu)^k\varphi\in\mathscr{H}_\mu$. If $q\in\mathbb{N}$ and $\varphi\in\mathscr{H}_\mu^q$, then

$$\begin{aligned}((1-S_\mu)^kf)\sharp\varphi(x)&=\langle(1-S_\mu)^kf,\tau_x\varphi\rangle=\langle f,(1-S_\mu)^k\tau_x\varphi\rangle\\&=\langle f,\tau_x(1-S_\mu)^k\varphi\rangle=(f\sharp(1-S_\mu)^k\varphi)(x)=(f\sharp\psi)(x)\quad(x\in I)\end{aligned}$$

[10, Proposition 2.1]. Thus [5, (proof of) Theorem 2.2],

$$
\begin{aligned}
\|((1-S_\mu)^k f)\natural\varphi\|_{\mu,q} = \|f\natural\psi\|_{\mu,q} &\leqslant \sum_{n+m=0}^{q} \|x^n T_{\mu,m}(f\natural\psi)\|_2 \\
&= \sum_{n+m=0}^{q} \|x^m T_{\mu,n}(x^{-\mu-1/2}(\mathfrak{H}_\mu f)(x)(\mathfrak{H}_\mu\psi)(x))\|_2 \\
&= \sum_{n+m=0}^{q} \|x^{n+m+\mu+1/2}(x^{-1}D)^n(x^{-\mu-1/2}(\mathfrak{H}_\mu f)(x)x^{-\mu-1/2}(\mathfrak{H}_\mu\psi)(x))\|_2 \\
&\leqslant \sum_{n+m=0}^{q}\sum_{j=0}^{n}\binom{n}{j} \|x^{n+m+\mu+1/2}(x^{-1}D)^j(x^{-\mu-1/2}(\mathfrak{H}_\mu f)(x)) \\
&\qquad \times (x^{-1}D)^{n-j}(x^{-\mu-1/2}(\mathfrak{H}_\mu\psi)(x))\|_2 \\
&\leqslant \sum_{n+m=0}^{q}\sum_{j=0}^{n}\binom{n}{j} \|x^{n-j}(x^{-1}D)^{n-j}x^{-\mu-1/2}(\mathfrak{H}_\mu\psi)(x)\|_\infty \\
&\qquad \times \|x^{m+j+\mu+1/2}(x^{-1}D)^j x^{-\mu-1/2}(\mathfrak{H}_\mu f)(x)\|_2 \\
&= \sum_{n+m=0}^{q}\sum_{j=0}^{n}\binom{n}{j} \|x^{n-j}(x^{-1}D)^{n-j}x^{-\mu-1/2}(\mathfrak{H}_\mu\psi)(x)\|_\infty \|x^m T_{\mu,j}(\mathfrak{H}_\mu f)(x)\|_2.
\end{aligned}
$$

By [6, Lemma 1] and [5, Theorem 2.2 and Proposition 2.15], there exist $r, s \in \mathbb{N}$ (not depending on φ) such that

$$\|((1-S_\mu)^k f)\natural\varphi\|_{\mu,q} \leqslant C\|\mathfrak{H}_\mu\psi\|_{\mu,r}\|\mathfrak{H}_\mu f\|_{\mu,q} = C\|\psi\|_{\mu,r}\|f\|_{\mu,q} \leqslant C\|\varphi\|_{\mu,s}\|f\|_{\mu,q}.$$

This shows that $(1-S_\mu)^k f \in \mathcal{O}^\natural_{s,q}$, and that the embeddings

$$(1-S_\mu)^k \mathscr{H}^q_\mu \hookrightarrow \mathcal{O}^\natural_{s,q} \hookrightarrow \operatorname*{ind\,lim}_{p\to\infty} \mathcal{O}^\natural_{p,q} = \mathcal{O}^\natural_q$$

are continuous. Since $k \in \mathbb{N}$ is arbitrary, we conclude that

$$\operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k \mathscr{H}^q_\mu \hookrightarrow \mathcal{O}^\natural_q$$

is also continuous. This completes the proof. □

As a consequence of Proposition 4.10, we obtain some results about continuity, topological properties, and structure in $\mathcal{O}^\natural_q$ ($q \in \mathbb{N}$).

Corollary 4.11. *For each $q \in \mathbb{N}$, the following holds.*

(i) *The embedding $\mathcal{O}^\natural_q \hookrightarrow \mathscr{H}'_\mu$ is continuous, when $\mathscr{H}'_\mu$ is endowed with either its weak* or its strong topology.*

(ii) $\mathcal{O}_q^\sharp$ *is the strong dual of* $\operatorname{proj}\lim\limits_{k\to\infty}(1 - S_\mu)^{-k}\mathscr{H}_\mu^{-q}$.

P r o o f. To prove (i) it suffices to show that the mapping $(1 - S_\mu)^k\mathscr{H}_\mu^q \hookrightarrow \mathscr{H}_\mu'$ is continuous for every $k \in \mathbb{N}$, when either the weak* or the strong topology are considered on $\mathscr{H}_\mu'$. Indeed, for $k \in \mathbb{N}$, $T \in (1 - S_\mu)^k\mathscr{H}_\mu^q$, and $\varphi \in \mathscr{H}_\mu$ we may write

$$|\langle\varphi, T\rangle| = |\langle(1 - S_\mu)^k\varphi, (1 - S_\mu)^{-k}T\rangle| \leqslant \|(1 - S_\mu)^k\varphi\|_{\mu,-q}\|(1 - S_\mu)^{-k}T\|_{\mu,q}.$$

The space $\mathscr{H}_\mu$ being continuously contained in $\mathscr{H}_\mu^{-q}$ [5, Propositions 2.15 and 2.4], this establishes (i).

Part (ii) follows from Lemma 4.9 (iv) and Proposition 4.10. □

Corollary 4.12. *For every $q \in \mathbb{N}$, the space $\mathcal{O}_q^\sharp$ is complete, reflexive and bornological.*

P r o o f. Let $q \in \mathbb{N}$. The Fréchet space $\operatorname{proj}\lim\limits_{k\to\infty}(1 - S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ is barrelled [12, II-7.1, Corollary], bornological [12, II-8.1] and semireflexive [12, IV-5.8]. Hence, $\operatorname{proj}\lim\limits_{k\to\infty}(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ is reflexive [12, IV-5.6], and its strong dual $\mathcal{O}_q^\sharp$ is reflexive [12, IV-5.6, Corollary 1], complete [12, IV-6.1], and bornological [12, IV-5.6 and IV-6.6]. □

Proposition 4.13. *Let $q \in \mathbb{N}$ and $T \in \mathscr{H}_\mu'$. Then $T \in \mathcal{O}_q^\sharp$ if, and only if, there exists $k \in \mathbb{N}$, $c_j \in \mathbb{C}$ $(0 \leqslant j \leqslant k)$, and $\varphi \in \mathscr{H}_\mu^q$, such that $T = \sum\limits_{j=0}^{k} c_j S_\mu^j\varphi$.*

P r o o f. Fix $q \in \mathbb{N}$. Assume that $\varphi \in \mathscr{H}_\mu^q$, and let $k, j \in \mathbb{N}$, with $0 \leqslant j \leqslant k$. We claim that $(1 - S_\mu)^{-k}S_\mu^j\varphi \in \mathscr{H}_\mu^q$, whence $S_\mu^j\varphi \in (1 - S_\mu)^k\mathscr{H}_\mu^q$. In fact, we have

$$(1 - S_\mu)^{-k}S_\mu^j\varphi = \mathfrak{H}_\mu'((1 + x^2)^{-k}\mathfrak{H}_\mu'(S_\mu^j\varphi)) = \mathfrak{H}_\mu'((-1)^j x^{2j}(1 + x^2)^{-k}(\mathfrak{H}_\mu\varphi)).$$

For $0 \leqslant j \leqslant k$ the function $x^{2j}(1 + x^2)^{-k}$ lies in $\mathcal{O}_{q,q}$ [6, Proposition 4], which establishes our claim.

Now, let $T \in \mathscr{H}_\mu'$. If $T = \sum\limits_{j=0}^{k} c_j S_\mu^j\varphi$ for some $k \in \mathbb{N}$, $c_j \in \mathbb{C}$ $(0 \leqslant j \leqslant k)$, and $\varphi \in \mathscr{H}_\mu^q$ then, as just proved, $T \in (1 - S_\mu)^k\mathscr{H}_\mu^q$. Conversely, suppose $T \in \mathcal{O}_q^\sharp$, so that $T \in (1 - S_\mu)^k\mathscr{H}_\mu^q$ for some $k \in \mathbb{N}$. If $\varphi \in \mathscr{H}_\mu^q$ is such that $(1 - S_\mu)^{-k}T = \varphi$ then $(\mathfrak{H}_\mu'T)(x) = (1 + x^2)^{-k}(\mathfrak{H}_\mu'\varphi)(x)$, whence $T = (1 - S_\mu)^k\varphi = \sum\limits_{j=0}^{k} \binom{k}{j}(-1)^j S_\mu^j\varphi$, with $S_\mu^j\varphi \in (1 - S_\mu)^k\mathscr{H}_\mu^q$. □

Next we characterize the bounded subsets of $\mathcal{O}_q^\sharp$ ($q \in \mathbb{N}$).

Proposition 4.14. *Let $q \in \mathbb{N}$. A set $B \subset \mathcal{O}_q^\sharp$ is bounded if, and only if, there exists $k \in \mathbb{N}$ such that $B \subset (1 - S_\mu)^k \mathscr{H}_\mu^q$ and B is bounded in $(1 - S_\mu)^k \mathscr{H}_\mu^q$.*

Proof. The sufficiency is clear. To prove the necessity, we argue by contradiction. Assume that $B \setminus (1 - S_\mu)^k \mathscr{H}_\mu^q \neq \emptyset$ for all $k \in \mathbb{N}$. Then there exist sequences $\{k_j\}_{j=1}^\infty$ in $\mathbb{N}$ and $\{f_j\}_{j=1}^\infty$ in B such that $f_j \in (1 - S_\mu)^{k_{j+1}} \mathscr{H}_\mu^q \setminus (1 - S_\mu)^{k_j} \mathscr{H}_\mu^q$. Choose a bounded closed convex neighborhood U_1 of zero in $(1 - S_\mu)^{k_1} \mathscr{H}_\mu^q$ such that $f_1 \notin U_1$. Once $U_1, U_2, \ldots, U_j$ have been chosen, apply [8, Lemma 1] to find a bounded closed convex neighborhood U_{j+1} of zero in $(1 - S_\mu)^{k_{j+1}} \mathscr{H}_\mu^q$ such that $U_j \subset U_{j+1}$ and $f_1 \notin i U_{j+1}$ for $1 \leqslant i \leqslant j + 1$. Then $\bigcup_{j \in \mathbb{N}} U_j$ is a convex neighborhood of the origin in $\mathcal{O}_q^\sharp$ which does not absorb $\{f_j\}_{j=1}^\infty \subset B$; this is the expected contradiction. Hence, $B \subset (1 - S_\mu)^k \mathscr{H}_\mu^q$ for some $k \in \mathbb{N}$.

Now suppose that B is not bounded in any $(1 - S_\mu)^j \mathscr{H}_\mu^q$ ($j \geqslant k$). Write $V_k = \{T \in (1 - S_\mu)^k \mathscr{H}_\mu^q : \|(1 - S_\mu)^{-k} T\|_{\mu,q} \leqslant 1\}$, and choose $g_j \in B \setminus j V_k$ ($j = k, k + 1$). By [8, Lemma 1] there is a bounded closed convex neighborhood V_{k+1} of zero in $(1 - S_\mu)^{k+1} \mathscr{H}_\mu^q$ satisfying $V_k \subset V_{k+1}$ and $g_j \notin j V_{k+1}$ ($j = k, k + 1$). An inductive procedure allows to define sequences $\{g_j\}_{j=k}^\infty \subset B$ and $\{V_j\}_{j=k}^\infty$ such that V_j is a bounded closed convex neighborhood of zero in $(1 - S_\mu)^j \mathscr{H}_\mu^q$ and $g_i \notin i V_j$ ($k \leqslant i \leqslant j$). Therefore $\bigcup_{j \in \mathbb{N}} V_j$ is a convex neighborhood of zero in $\mathcal{O}_q^\sharp$ which does not absorb $\{g_j\}_{j=k}^\infty \subset B$. This contradiction completes the proof. □

In the sense indicated by Proposition 4.15 below, $\mathscr{H}_\mu$ is a dense subspace of $(\mathcal{O}_{q+1}^\sharp)'$ ($q \in \mathbb{N}$).

Proposition 4.15. *Let $q \in \mathbb{N}$, and for each $\varphi \in \mathscr{H}_\mu$ define*

$$\langle L_\varphi, T \rangle = \int_0^\infty \mathfrak{H}_\mu(T \sharp \varphi)(x) \, \mathrm{d}x \quad (T \in \mathcal{O}_{q+1}^\sharp).$$

*Then $\mathscr{L}_\mu = \{L_\varphi : \varphi \in \mathscr{H}_\mu\}$ is a (weakly *, strongly) dense subspace of $(\mathcal{O}_{q+1}^\sharp)'$.*

Proof. Fix $\varphi \in \mathscr{H}_\mu$, $p \in \mathbb{N}$, and $T \in \mathcal{O}_{p,q+1}^\sharp$. Then [5, Theorem 2.2]

$$\begin{aligned} |\langle L_\varphi, T \rangle| &\leqslant \int_0^\infty |\mathfrak{H}_\mu(T \sharp \varphi)(x)| \, \mathrm{d}x \leqslant \int_0^\infty \frac{1}{1 + x} |(1 + x) \mathfrak{H}_\mu(T \sharp \varphi)(x)| \, \mathrm{d}x \\ &\leqslant \left\{ \int_0^\infty \frac{\mathrm{d}x}{1 + x^2} \right\}^{1/2} \left\{ \int_0^\infty (1 + x)^2 |\mathfrak{H}_\mu(T \sharp \varphi)(x)|^2 \, \mathrm{d}x \right\}^{1/2} \\ &\leqslant C \|\mathfrak{H}_\mu(T \sharp \varphi)\|_{\mu,q+1} = C \|T \sharp \varphi\|_{\mu,q+1} \leqslant C \|T\|_{p,q+1}^\sharp \|\varphi\|_{\mu,p}. \end{aligned}$$

Hence, $L_\varphi\colon \mathscr{O}^\sharp_{p,q+1} \longrightarrow \mathbb{C}$ is continuous. The arbitrariness of $p \in \mathbb{N}$ yields $L_\varphi\colon \mathscr{O}^\sharp_{q+1} \longrightarrow \mathbb{C}$ continuous, so that $\mathscr{L}_\mu = \{L_\varphi\colon \varphi \in \mathscr{H}_\mu\}$ is a subspace of $(\mathscr{O}^\sharp_{q+1})'$.

Since $\mathscr{O}^\sharp_{q+1}$ is reflexive, the strong topology of $(\mathscr{O}^\sharp_{q+1})'$ is the Mackey topology, and convex subsets of $(\mathscr{O}^\sharp_{q+1})'$ have the same closures in both the weak* and the strong topologies [12, IV-3.3]. Thus, we only need to prove that $\mathscr{L}_\mu$ is weakly* dense in $(\mathscr{O}^\sharp_{q+1})'$. This may be accomplished by showing that $\mathscr{L}_\mu$ separates points of $\mathscr{O}^\sharp_{q+1}$ with respect to the duality $\langle(\mathscr{O}^\sharp_{q+1})', \mathscr{O}^\sharp_{q+1}\rangle$ [12, IV-1.3]. In fact, let $T \in \mathscr{O}^\sharp_{q+1}$ be such that

$$0 = \langle L_\varphi, T\rangle = \int_0^\infty x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)(\mathfrak{H}_\mu\varphi)(x)\,\mathrm{d}x$$

for all $\varphi \in \mathscr{H}_\mu$ (Proposition 4.3). Then

$$\int_0^\infty x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)\varphi(x)\,\mathrm{d}x = 0$$

for all $\varphi \in \mathscr{H}_\mu$, whence $x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x) = 0$ $(x \in I)$. Here we have used the fact that $x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)$ is a continuous function on I [6, Proposition 5]. Therefore $T = 0$, which completes the proof. □

Given two topological vector spaces $\mathscr{A}, \mathscr{B}$, denote by $\mathscr{L}_s(\mathscr{A}, \mathscr{B})$ (respectively, $\mathscr{L}_b(\mathscr{A}, \mathscr{B})$) the space $\mathscr{L}(\mathscr{A}, \mathscr{B})$ endowed with the topology of pointwise convergence (respectively, of bounded convergence). Fix $q \in \mathbb{N}$. By Proposition 4.7, $\mathscr{O}^\sharp_q$ may be regarded as a subspace of $\mathscr{L}(\mathscr{H}_\mu, \mathscr{H}^q_\mu)$. Hence, besides the inductive topology τ, it is natural to consider on $\mathscr{O}^\sharp_q$ the topologies τ_s and τ_b, induced by $\mathscr{L}_s(\mathscr{H}_\mu, \mathscr{H}^q_\mu)$ and $\mathscr{L}_b(\mathscr{H}_\mu, \mathscr{H}^q_\mu)$, respectively. We shall prove that all these topologies coincide on $\mathscr{O}^\sharp_q$ (Proposition 4.19). For this purpose we need an auxiliary result about the Hankel convolution, stated as Lemma 4.17, which we shall derive from the corresponding property of the multiplication (Lemma 4.16).

Given $k, q \in \mathbb{N}$, let us consider the spaces

$$\begin{aligned}(1+x^2)^{-k}\mathscr{H}^{-q}_\mu &= \{(1+x^2)^{-k}u \in \mathscr{H}'_\mu\colon u \in \mathscr{H}^{-q}_\mu\},\\ (1+x^2)^{k}\mathscr{H}^{q}_\mu &= \{T \in \mathscr{H}'_\mu\colon (1+x^2)^{-k}T \in \mathscr{H}^{q}_\mu\},\end{aligned}$$

endowed with the topology induced by $\mathscr{H}^{-q}_\mu$ (respectively, $\mathscr{H}^q_\mu$) through the multiplier $(1+x^2)^{-k}$. In is apparent that for each $k, q \in \mathbb{N}$, the Hilbert spaces $(1+x^2)^{-k}\mathscr{H}^{-q}_\mu$ and $(1+x^2)^{k}\mathscr{H}^{q}_\mu$ constitute a dual system with respect to the bilinear form

$$\begin{aligned}(1+x^2)^{-k}\mathscr{H}^{-q}_\mu \times (1+x^2)^{k}\mathscr{H}^{q}_\mu &\longrightarrow \mathbb{C},\\ (u,v) &\longmapsto \langle u,v\rangle = \left\langle (1+x^2)^{k}u, (1+x^2)^{-k}v\right\rangle.\end{aligned}$$

By [5, Theorem 2.2], the Hankel transformation defines an isomorphism between $(1+x^2)^{-k}\mathscr{H}_\mu^{-q}$ and $(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$, respectively $(1+x^2)^k\mathscr{H}_\mu^q$ and $(1-S_\mu)^k\mathscr{H}_\mu^q$. In particular, $\operatorname{ind}\lim_{k\to\infty}(1+x^2)^k\mathscr{H}_\mu^q$ is the strong dual of $\operatorname{proj}\lim_{k\to\infty}(1+x^2)^{-k}\mathscr{H}_\mu^{-q}$ (Lemma 4.9, (iv)).

Lemma 4.16. *Fix $q \in \mathbb{N}$. Assume that $B \subset \operatorname{proj}\lim_{k\to\infty}(1+x^2)^{-k}\mathscr{H}_\mu^{-q}$ is bounded, and let Δ denote the unit ball of $\mathscr{H}_\mu^{-q}$. There exists $\varphi \in \mathscr{H}_\mu$ such that $B \subset x^{-\mu-1/2}\varphi(x)\Delta$.*

P r o o f. To every $k \in \mathbb{N}$ there corresponds $C_k > 0$ such that $BC_k^{-1}(1+x^2)^{-k}\Delta$. Let B^0, respectively Δ^0, denote the polar set of B, respectively Δ, and let $v \in C_k(1+x^2)^k\Delta^0 = (C_k^{-1}(1+x^2)^{-k}\Delta^0)$. If $u \in C_k^{-1}(1+x^2)^{-k}\Delta$ then $C_k(1+x^2)^k u \in \Delta$, $C_k^{-1}(1+x^2)^{-k}v \in \Delta^0$, and

$$|\langle u,v\rangle| = |\langle (1+x^2)^k u, (1+x^2)^{-k}v\rangle| \leqslant 1.$$

As $B \subset C_k^{-1}(1+x^2)^{-k}\Delta$, it follows that $|\langle u,v\rangle| \leqslant 1$ for all $u \in B$ and $v \in C_k(1+x^2)^k\Delta^0$. Therefore, $C_k(1+x^2)^k\Delta^0 \subset B^0$.

Now, let $\psi \in C^\infty(\mathbb{R})$ satisfy $0 \leqslant \psi(t) \leqslant 1$ $(t \in \mathbb{R})$, $\psi(t) = 0$ $(t \leqslant 0)$, and $\psi(t) = 1$ $(t \geqslant 1)$. For $d > 0$ and $h \in \Delta^0$ there holds

$$\|\psi(x-d)(1+x^2)^{-1}h(x)\|_{\mu,q} \leqslant C\|h\|_{\mu,q}\sum_{j=0}^{q}\|(x^{-1}D)^j\psi(x-d)(1+x^2)^{-1}\|_\infty \leqslant \frac{C}{1+d^2}$$

with C independent of d and h, whence $\lim_{d\to\infty}\sup_{h\in\Delta^0}\|\psi(x-d)(1+x^2)^{-1}h(x)\|_{\mu,q} = 0$. Thus, given $k \in \mathbb{N}$ there exists a sequence $\{d_k\}_{k\in\mathbb{N}}$ such that $d_k > 0$, $d_{k+1} > 1+d_k$, and

$$\sup_{h\in\Delta^0}\|\psi(x-d_k)(1-x^2)^{-1}h(x)\|_{\mu,q} \leqslant 2^{-k-1}C_{k+1} \quad (k \in \mathbb{N}).$$

Then, for $k \in \mathbb{N}$ and $h \in \Delta^0$,

$$\begin{aligned}\psi(x-d_k)(1+x^2)^k h(x) &= (1+x^2)^{k+1}\psi(x-d_k)(1+x^2)^{-1}\\ &\quad\times h(x) \in 2^{-k-1}C_{k+1}(1+x^2)^{k+1}\Delta^0.\end{aligned}$$

Define

$$\varphi^{-1}(x) = x^{-\mu-1/2}\left[\frac{1}{2}MC_1 + \sum_{k=1}^{\infty}\psi(x-d_k)(1+x^2)^k\right] \quad (x \in I),$$

where $M > 0$ is chosen so that $M\Delta^0 \subset (1+x^2)\Delta^0$ [6, Proposition 4].

Note that $\varphi \in \mathscr{H}_\mu$. Moreover, if $h \in \Delta^0$ we have

$$\begin{aligned}
x^{\mu+1/2}\varphi^{-1}(x)h(x) &= \frac{1}{2}C_1Mh(x) + \sum_{k=1}^{\infty}\psi(x-d_k)(1+x^2)^k h(x) \\
&\in \frac{1}{2}C_1M\Delta^0 + \sum_{k=1}^{\infty}2^{-k-1}C_{k+1}(1+x^2)^{k+1}\Delta^0 \\
&\subset \sum_{k=1}^{\infty}2^{-k}C_k(1+x^2)^k\Delta^0 \subset \left(\sum_{k=1}^{\infty}2^{-k}\right)B^0 = B^0.
\end{aligned}$$

That is, $\Delta^0 \subset x^{-\mu-1/2}\varphi(x)B^0$.

On the other hand, $x^{-\mu-1/2}\varphi(x)(1+x^2)^k\mathscr{H}_\mu^q \subset \mathscr{H}_\mu^q$ $(k \in \mathbb{N})$, because $x^{-\mu-1/2}\mathscr{H}_\mu \subset \mathscr{O}_{q,q}$ [6, Proposition 6]. Consequently, $B^0 \subset \operatorname{ind}\lim\limits_{k\to\infty}(1+x^2)^k\mathscr{H}_\mu^q \subset x^{\mu+1/2}\varphi^{-1}(x)\mathscr{H}_\mu^q$.

Now, let $u \in B \subset \operatorname{proj}\lim\limits_{k\to\infty}(1+x^2)^{-k}\mathscr{H}_\mu^{-q} \subset \mathscr{H}_\mu' \subset \mathscr{B}_\mu'$. Then $x^{\mu+1/2}\varphi^{-1}(x)u(x) \in \mathscr{B}_\mu'$, and for $w \in \mathscr{B}_\mu \cap \Delta^0 \subset \mathscr{B}_\mu \cap x^{-\mu-1/2}\varphi(x)B^0$ we find that

$$\left|\left\langle x^{\mu+1/2}\varphi^{-1}(x)u(x), w(x)\right\rangle\right| = \left|\left\langle u(x), x^{\mu+1/2}\varphi^{-1}(x)w(x)\right\rangle\right| \leqslant 1.$$

The space $\mathscr{B}_\mu$ being dense in $\mathscr{H}_\mu^q$ [5, Proposition 2.12], this shows that

$$x^{\mu+1/2}\varphi^{-1}(x)u(x) \in \Delta.$$

Since $u \in B$ is arbitrary, we conclude that $B \subset x^{-\mu-1/2}\varphi(x)\Delta$. The proof is thus complete. □

In view of the remark preceding Lemma 4.16, and applying Corollary 4.4, we immediately obtain

Lemma 4.17. *Fix $q \in \mathbb{N}$. Assume that $B \subset \operatorname{proj}\lim\limits_{k\to\infty}(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ is bounded, and let Δ denote the unit ball of $\mathscr{H}_\mu^{-q}$. There exists $\varphi \in \mathscr{H}_\mu$ such that $B \subset \varphi\sharp\Delta$.*

Lemma 4.18. *For every $q \in \mathbb{N}$, the inductive topology of $\mathscr{O}_q^\sharp$ is generated by the system of seminorms $\Sigma = \{\|\cdot\sharp\varphi\|_{\mu,q}\}_{\varphi\in\mathscr{H}_\mu}$.*

Proof. Let $q \in \mathbb{N}$. As above, we denote by τ the inductive topology of $\mathscr{O}_q^\sharp$. Note that the Σ-topology is just the topology of pointwise convergence τ_s that $\mathscr{O}_q^\sharp$ inherits from $\mathscr{L}(\mathscr{H}_\mu, \mathscr{H}_\mu^q)$.

For each $p \in \mathbb{N}$, the estimate $\|T\sharp\varphi\|_{\mu,q} \leqslant \|T\|_{p,q}^\sharp\|\varphi\|_{\mu,p}$, valid whenever $T \in \mathscr{O}_{p,q}^\sharp$ and $\varphi \in \mathscr{H}_\mu$, shows that the identity mapping $\mathscr{O}_{p,q}^\sharp \hookrightarrow (\mathscr{O}_q^\sharp, \tau_s)$ is continuous. Thus, τ is finer than τ_s.

Conversely, choose a τ-neighborhood U of zero in $\mathcal{O}_q^\sharp = \operatorname*{ind\,lim}_{k\to\infty}(1-S_\mu)^k\mathscr{H}_\mu^q$. Then there is a bounded set $B \subset \operatorname*{proj\,lim}_{k\to\infty}(1-S_\mu)^{-k}\mathscr{H}_\mu^{-q}$ such that $B^0 \subset U$ (Corollary 4.11). By Lemma 4.17, there exists $\varphi \in \mathscr{H}_\mu$ satisfying $B \subset \varphi\sharp\Delta$, where Δ denotes the unit ball in $\mathscr{H}_\mu^{-q}$. For $T \in (1-S_\mu)^k\mathscr{H}_\mu^q \subset \mathscr{H}_\mu'$ and $\psi \in \mathscr{H}_\mu$, we have $\langle T, \varphi\sharp\psi\rangle = \langle T\sharp\varphi, \psi\rangle$ [10, Proposition 3.5]. Since $\mathscr{H}_\mu$ is dense in $\mathscr{H}_\mu^{-q}$ [5, Proposition 2.12], for each $T \in \mathcal{O}_q^\sharp$ we may write

$$\begin{aligned}\sup\{|\langle F,T\rangle|: F\in B\} &\leqslant \sup\{|\langle T,\varphi\sharp u\rangle|: u\in\Delta\}\\ &= \sup\{|\langle T,\varphi\sharp\psi\rangle|: \psi\in\mathscr{H}_\mu, \|\psi\|_{\mu,-q}\leqslant 1\}\\ &= \sup\{|\langle T\sharp\varphi,\psi\rangle|: \psi\in\mathscr{H}_\mu, \|\psi\|_{\mu,-q}\leqslant 1\} = \|T\sharp\varphi\|_{\mu,q}.\end{aligned}$$

Thus $\{T \in \mathcal{O}_q^\sharp : \|T\sharp\varphi\|_{\mu,q} \leqslant 1\} \subset B^0 \subset U$, and therefore τ_s is finer than τ. □

Proposition 4.19. *Let $q \in \mathbb{N}$. The bounded convergence topology and the pointwise convergence topology coincide on $\mathcal{O}_q^\sharp$ as a subspace of $\mathscr{L}(\mathscr{H}_\mu, \mathscr{H}_\mu^q)$, and they equal the inductive topology of $\mathcal{O}_q^\sharp$. In particular, the inductive topology of $\mathcal{O}_q^\sharp$ is generated by any one of the families of seminorms $\{\|\cdot\sharp\varphi\|_{\mu,q}\}_{\varphi\in\mathscr{H}_\mu}$, $\{\sup_{\varphi\in\mathfrak{B}}\|\cdot\sharp\varphi\|_{\mu,q}\}_{B\in\mathfrak{B}_\mu}$, where $\mathfrak{B}_\mu$ denotes the family of all bounded subsets of $\mathscr{H}_\mu$.*

Proof. The notation for the spaces and topologies that will be used here was introduced after Proposition 4.15.

The identity mapping $\mathcal{O}_{p,q}^\sharp \hookrightarrow \mathscr{L}_b(\mathscr{H}_\mu^p, \mathscr{H}_\mu^q)$ $(p \in \mathbb{N})$ is continuous. Thus, $\mathcal{O}_{p,q}^\sharp \hookrightarrow \mathscr{L}_b(\mathscr{H}_\mu, \mathscr{H}_\mu^q)$ $(p \in \mathbb{N})$ is continuous [5, Proposition 2.15], and consequently so is $(\mathcal{O}_q^\sharp, \tau) \hookrightarrow \mathscr{L}_b(\mathscr{H}_\mu, \mathscr{H}_\mu^q)$. This means that $\tau_b \subset \tau$. As $\tau = \tau_s$ (Lemma 4.18), and since $\tau_s \subset \tau_b$, we conclude that $\tau = \tau_s = \tau_b$. □

Remark 4.20. $\mathcal{O}_{q+1}^\sharp$ is neither Montel nor nuclear. For the proof take a sequence $B = \{k_n\}_{n\in\mathbb{N}}$ in $\mathscr{H}_\mu$ such that $\|k_n\|_{\mu,q+1} = 1$ and $(\mathfrak{H}_\mu k_n)(x) = 0$ $(x \geqslant 1/n)$. This B is bounded in $\mathcal{O}_{q+1}^\sharp$ (Proposition 4.14). Moreover, if $\varphi \in \mathscr{H}_\mu$ is such that $(\mathfrak{H}_\mu\varphi)(x) = x^{\mu+1/2}$ $(0 < x < 1)$, then [5, Theorem 2.2] and the exchange formula for the Hankel convolution yield

$$\begin{aligned}(4.4)\quad \|k_n\sharp\varphi\|_{\mu,q+1}^2 = \|\mathfrak{H}_\mu(k_n\sharp\varphi)\|_{\mu,q+1}^2 &= \sum_{i+j=0}^{q+1}\int_0^\infty |x^iT_{\mu,j}\mathfrak{H}_\mu(k_n\sharp\varphi)(x)|^2\,dx\\ &= \sum_{i+j=0}^{q+1}\int_0^{1/n} |x^{i+j+\mu+1/2}(x^{-1}D)^jx^{-\mu-1/2}(\mathfrak{H}_\mu k_n)(x)|^2\,dx\\ &= \sum_{i+j=0}^{q+1}\int_0^\infty |x^iT_{\mu,j}(\mathfrak{H}_\mu k_n)(x)|^2\,dx = \|\mathfrak{H}_\mu k_n\|_{\mu,q+1}^2\\ &= \|k_n\|_{\mu,q+1}^2 = 1.\end{aligned}$$

From (4.4) and Proposition 4.19 we infer that B does not have any cluster point in $\mathcal{O}^\sharp_{q+1}$. In particular, B is closed in $\mathcal{O}^\sharp_{q+1}$. Also by (4.4) and Proposition 4.19, B cannot contain zero-convergent subsequences in $\mathcal{O}^\sharp_{q+1}$.

In the notation of Proposition 4.15, for every $\varphi \in \mathcal{H}_\mu$ we have

$$
\begin{aligned}
|\langle L_\varphi, k_n\rangle| &= \left|\int_0^\infty \mathfrak{H}_\mu(k_n \sharp \varphi)(x)\,\mathrm{d}x\right| = \left|\int_0^\infty x^{-\mu-1/2}(\mathfrak{H}_\mu k_n)(x)(\mathfrak{H}_\mu\varphi)(x)\,\mathrm{d}x\right| \\
&\leqslant n^{-1/2}\sup_{x\in I}|x^{-\mu-1/2}(\mathfrak{H}_\mu\varphi)(x)|\left\{\int_0^\infty |(\mathfrak{H}_\mu k_n)(x)|^2\,\mathrm{d}x\right\}^{1/2} \qquad (4.5)\\
&\leqslant n^{-1/2}\sup_{x\in I}|x^{-\mu-1/2}(\mathfrak{H}_\mu\varphi)(x)|\,\|k_n\|_{\mu,q+1} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Now (4.5) and Proposition 4.15 imply that B converges weakly to 0 in $\mathcal{O}^\sharp_{q+1}$. As convergence in $\mathcal{O}^\sharp_{q+1}$ is stronger than weak convergence, we conclude that B does not contain any convergent subsequence in $\mathcal{O}^\sharp_{q+1}$. Therefore B is not compact, and $\mathcal{O}^\sharp_{q+1}$ is not Montel.

Since $\mathcal{O}^\sharp_{q+1}$ is complete (Corollary 4.12) and B is closed in $\mathcal{O}^\sharp_{q+1}$ we find that B is not precompact, so that $\mathcal{O}^\sharp_{q+1}$ is not nuclear [13, Proposition III-50.2].

5. Multipliers on $\mathcal{H}_\mu^p$ ($p \in \mathbb{Z}$)

Let $p, q \in \mathbb{N}$. In this Section we present new properties of the spaces $\mathcal{O}_{p,q}$ of multipliers from $\mathcal{H}_\mu^p$ into $\mathcal{H}_\mu^q$ and of $\mathcal{H}_\mu^{-q}$ into $\mathcal{H}_\mu^{-p}$, investigated in [6] and treated in Section 2 above. As an interesting consequence, the space $\mathcal{O}$ of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ is expressed as a projective-inductive limit of Hilbert spaces.

According to Proposition 4.1 and the remark following Proposition 4.6, we have

$$
\mathcal{O}_{p,q+1} \hookrightarrow \mathcal{O}_{p,q} \hookrightarrow \mathcal{O}_{p+1,q} \quad (p, q \in \mathbb{N}),
$$

with continuous embeddings. Hence, for $q \in \mathbb{N}$ we may consider

$$
\mathcal{O}_q = \operatorname*{ind\,lim}_{p\to\infty} \mathcal{O}_{p,q}.
$$

The generalized Hankel transformation makes $\mathcal{O}^\sharp_q$ and $x^{\mu+1/2}\mathcal{O}_q$ isomorphic, where the latter space is topologized so that the mapping $\varphi \longmapsto x^{\mu+1/2}\varphi(x)$ defines an isomorphism from $\mathcal{O}_q$ onto $x^{\mu+1/2}\mathcal{O}_q$. We may also consider

$$
\mathcal{O} = \operatorname*{proj\,lim}_{q\to\infty} \mathcal{O}_q.
$$

Here $\mathcal{O}$ is the space of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$, topologized in such a way that the generalized Hankel transformation is an isomorphism between $\mathcal{O}'_{\mu,\sharp}$ and $x^{\mu+1/2}\mathcal{O}$ [6, Proposition 14].

Via the generalized Hankel transformation, for $q \in \mathbb{N}$ the properties of $\mathcal{O}_q$ enumerated below can be immediately derived from the corresponding ones of the spaces $\mathcal{O}_q^\sharp$ studied in Section 4. Proofs will be omitted.

Proposition 5.1. *For each $q \in \mathbb{N}, \mathcal{O}_q$ is the space of all continuous multipliers from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu^q$ and from $\mathcal{H}_\mu^{-q}$ into $\mathcal{H}'_\mu$.*

Proposition 5.2. *Given $q \in \mathbb{N}$, let $\mathcal{MD}(\mathcal{H}_\mu, \mathcal{H}_\mu^q)$ denote the space all those $F \in \mathcal{H}'_\mu$ such that $x^{-\mu-1/2}\varphi(x)F(x) \in \mathcal{H}_\mu^q$ for all $\varphi \in \mathcal{H}_\mu$, and the mapping $\varphi \longmapsto x^{-\mu-1/2}\varphi(x)F(x)$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu^q$. Then, $\mathcal{MD}(\mathcal{H}_\mu, \mathcal{H}_\mu^q) = x^{\mu+1/2}\mathcal{O}_q$.*

The spaces $(1 + x^2)^k\mathcal{H}_\mu^q$ and $(1 + x^2)^{-k}\mathcal{H}_\mu^{-q}(k, q \in \mathbb{N})$ were defined prior to Lemma 4.16.

Proposition 5.3. *Let $q \in \mathbb{N}$.*

(i) *The identity $x^{\mu+1/2}\mathcal{O}_q = \operatorname{ind}\lim_{k\to\infty}(1+x^2)^k\mathcal{H}_\mu^q$ holds. Moreover, $x^{\mu+1/2}\mathcal{O}_q$ is the strong dual of $\operatorname{proj}\lim_{k\to\infty}(1 + x^2)^{-k}\mathcal{H}_\mu^{-q}$.*

(ii) *The embedding $x^{\mu+1/2}\mathcal{O}_q \hookrightarrow \mathcal{H}'_\mu$ is continuous, when $\mathcal{H}'_\mu$ is endowed with either its weak* or its strong topology.*

(iii) *The space $\mathcal{O}_q$ is complete, reflexive and bornological, but neither Montel nor nuclear.*

(iv) *Let $T \in \mathcal{H}'_\mu$. Then $T \in x^{\mu+1/2}\mathcal{O}_q$ if, and only if, there exists $k \in \mathbb{N}$, $c_j \in \mathbb{C}$ $(0 \leqslant j \leqslant k)$, and $\varphi \in \mathcal{H}_\mu^q$, such that $T = \sum_{j=0}^{k} c_j x^{2j}\varphi(x)$.*

(v) *A set $B \subset \mathcal{O}_q$ is bounded if, and only if, there exists $k \in \mathbb{N}$ such that $x^{\mu+1/2}B \subset (1+x^2)^k\mathcal{H}_\mu^q$ and $x^{\mu+1/2}B$ is bounded in $(1 + x^2)^k\mathcal{H}_\mu^q$.*

(vi) *For every $\varphi \in \mathcal{H}_\mu$, define*

$$\langle M_\varphi, T\rangle = \int_0^\infty T(x)\varphi(x)\,\mathrm{d}x \quad (T \in \mathcal{O}_{q+1}).$$

Then $\mathcal{M}_\mu = \{M_\varphi : \varphi \in \mathcal{H}_\mu\}$ is a (weakly, strongly) dense subspace of $(\mathcal{O}_{q+1})'$.*

(vii) *The bounded convergence topology and the pointwise convergence topology coincide on $\mathcal{O}_q$ as a subspace of $\mathcal{L}(\mathcal{H}_\mu, \mathcal{H}_\mu^q)$, and they equal the inductive topology of $\mathcal{O}_q$. In particular, the inductive topology of $\mathcal{O}_q$ is generated by any one of the families of seminorms $\{\|\cdot\varphi(x)\|_{\mu,q}\}_{\varphi\in\mathcal{H}_\mu}$, $\{\sup_{\varphi\in B}\|\cdot\varphi(x)\|_{\mu,q}\}_{B\in\mathfrak{B}_\mu}$, where $\mathfrak{B}_\mu$ denotes the family of all bounded subsets of $\mathcal{H}_\mu$.*

6. Joint continuity of multiplication and of Hankel convolution of distributions

Our purpose here is to analyze the joint continuity of the Hankel convolution as an $\mathscr{H}'_\mu$-valued mapping. We will establish first the corresponding property for the ordinary product in $\mathscr{H}'_\mu$.

An auxiliary result is required.

Lemma 6.1. *Given $q \in \mathbb{N}$ there exist $p \in \mathbb{N}$, $p \geqslant q$, and a sequence $\{\varphi_m\}_{m\in\mathbb{N}} \subset \mathscr{H}_\mu$ such that*

$$\sup_{m\in\mathbb{N}} \|x^{-\mu-1/2}\varphi_m(x)\|_{p,q} = 1 \quad \textit{and} \quad \lim_{m\to\infty} \|\varphi_m(x)\mathrm{e}^{-x^2}\|_{\mu,q+1} = \infty.$$

P r o o f. Fix $q \in \mathbb{N}$, and define

$$h_m(x) = m^{\mu+1/2-q}\mathrm{e}^{-(mx)^2} \quad (m \in \mathbb{N}, x \in I).$$

Then, for $j \in \mathbb{N}$, $0 \leqslant j \leqslant q$, we have

$$\begin{aligned}\sup_{x\in I}|x^{j+\mu+1/2}(x^{-1}D)^j h_m(x)| &\leqslant 2^q m^{j-q}\sup_{x\in I}|(mx)^{j+\mu+1/2}\mathrm{e}^{-(mx)^2}| \\ &\leqslant 2^q \sup_{x\in I}|x^{j+\mu+1/2}\mathrm{e}^{-x^2}| \quad (m \in \mathbb{N}).\end{aligned}$$

Note that $\sup_{x\in I}|x^{j+\mu+1/2}\mathrm{e}^{-x^2}| < \infty$ $(0 \leqslant j \leqslant q)$, because $f_j(x) = x^{j+\mu+1/2}\mathrm{e}^{-x^2}$ $(x \in I)$ is continuous and the limits $\lim_{x\to 0+} f_j(x)$, $\lim_{x\to+\infty} f_j(x)$ exist and are finite $(0 \leqslant j \leqslant q)$. By [6, Proposition 4], there exists $p \in \mathbb{N}$, $p \geqslant q$, such that $h_m \in \mathscr{O}_{p,q}$ $(m \in \mathbb{N})$, with $S = \sup_{m\in\mathbb{N}} \|h_m\|_{p,q} < \infty$.

Now, set

$$\varphi_m(x) = S^{-1}m^{\mu+1/2-q}x^{\mu+1/2}\mathrm{e}^{-(mx)^2} \quad (m \in \mathbb{N},\ x \in I).$$

Clearly $\varphi_m \in \mathscr{H}_\mu$ ($m \in \mathbb{N}$), and $\sup_{m\in\mathbb{N}} \|x^{-\mu-1/2}\varphi_m(x)\|_{p,q} = S^{-1} \sup_{m\in\mathbb{N}} \|h_m\|_{p,q} = 1$. Moreover,

$$\begin{aligned}
\|\varphi_m(x)\mathrm{e}^{-x^2}\|^2_{\mu,q+1} &\geqslant \int_0^\infty |T_{\mu,q+1}(\varphi_m(x)\mathrm{e}^{-x^2})|^2\,\mathrm{d}x \\
&= S^{-2}m^{2\mu+1-2q}\int_0^\infty |x^{q+\mu+3/2}(x^{-1}D)^{q+1}\mathrm{e}^{-(m^2+1)x^2}|^2\,\mathrm{d}x \\
&= S^{-2}4^{q+1}m^{2\mu+1-2q}(m^2+1)^{2(q+1)}\int_0^\infty |x^{q+\mu+3/2}\mathrm{e}^{-(m^2+1)x^2}|^2\,\mathrm{d}x \\
&= S^{-2}4^{q+1}(1+m^{-2})^{2(q+1)}m\int_0^\infty |x^{q+\mu+3/2}\mathrm{e}^{-(1+m^{-2})x^2}|^2\,\mathrm{d}x \\
&\geqslant S^{-2}4^{q+1}(1+m^{-2})^{2(q+1)}m\int_0^\infty x^{2q+2\mu+3}\mathrm{e}^{-4x^2}\,\mathrm{d}x \\
&= S^{-2}4^{-\mu-3/2}(1+m^{-2})^{2(q+1)}m\Gamma(\mu+q+2) \quad (m\in\mathbb{N}).
\end{aligned}$$

Thus, $\lim_{m\to\infty} \|\varphi_m(x)\mathrm{e}^{-x^2}\|_{\mu,q+1} = \infty$. □

From now on β and σ will refer to the strong and the weak* topologies of $\mathscr{H}'_\mu$, respectively.

In [6, Proposition 16] we established that the mapping $(\theta, T) \longmapsto \theta T$ is $\mathscr{O}$-hypocontinuous from $\mathscr{O} \times (\mathscr{H}'_\mu, \beta)$ into $(\mathscr{H}'_\mu, \beta)$. However, the following holds.

Proposition 6.2. *The product $(\theta, T) \longmapsto \theta T$ is not jointly continuous from $\mathscr{O} \times (\mathscr{H}'_\mu, \beta)$ into $(\mathscr{H}'_\mu, \sigma)$. Hence, it is not jointly continuous from $\mathscr{O} \times (\mathscr{H}'_\mu, \beta)$ into $(\mathscr{H}'_\mu, \beta)$, either.*

Proof. The polar set P of the singleton $\{x^{\mu+1/2}\mathrm{e}^{-x^2}\} \subset \mathscr{H}_\mu$ is a σ-neighborhood of zero in $\mathscr{H}'_\mu$. If the multiplication were jointly continuous from $\mathscr{O} \times (\mathscr{H}'_\mu, \beta)$ into $(\mathscr{H}'_\mu, \sigma)$, then we could find zero-neighborhoods U, V in $\mathscr{O}$, $(\mathscr{H}'_\mu, \beta)$, respectively, with $UV \subset P$. For some $q \in \mathbb{N}$ there exists a zero-neighborhood G in $\mathscr{O}_q$ such that $G \cap \mathscr{O} \subset U$. Let $p \in \mathbb{N}$ correspond to q as in Lemma 6.1. Choose a ball $B_{p,q}(\varepsilon)$ centered at zero, with radius ε, in $\mathscr{O}_{p,q}$, satisfying $B_{p,q}(\varepsilon) \subset G$, and a sequence $\{\varphi_m\}_{m\in\mathbb{N}} \subset \mathscr{H}_\mu$ such that

$$\sup_{m\in\mathbb{N}} \|x^{-\mu-1/2}\varphi_m(x)\|_{p,q} < \varepsilon \quad \text{and} \quad \lim_{m\to\infty} \|\varphi_m(x)\mathrm{e}^{-x^2}\|_{\mu,q+1} = \infty.$$

For each $m \in \mathbb{N}$ and $g \in V$ we have $x^{-\mu-1/2}\varphi_m(x)g(x) \in UV \subset P$, so that

$$\left|\left\langle x^{-\mu-1/2}\varphi_m(x)g(x), x^{\mu+1/2}\mathrm{e}^{-x^2}\right\rangle\right| = \left|\left\langle g(x), \varphi_m(x)\mathrm{e}^{-x^2}\right\rangle\right| \leqslant 1.$$

This means that all functions $\varphi_m(x)\mathrm{e}^{-x^2}$ ($m \in \mathbb{N}$) lie in the polar set Q of V. Since Q is bounded in $\mathscr{H}_\mu$, so is the sequence $\{\varphi_m(x)\mathrm{e}^{-x^2}\}_{m\in\mathbb{N}}$, and we conclude that $\sup \|\varphi_m(x)\mathrm{e}^{-x^2}\|_{\mu,q+1} < \infty$ [5, Proposition 2.15]. This contradiction completes the proof. □

In [6, Proposition 19] we proved that the Hankel convolution is $\mathcal{O}'_{\mu,\sharp}$-hypocontinuous from $\mathcal{O}'_{\mu,\sharp} \times (\mathcal{H}'_\mu, \beta)$ into $(\mathcal{H}'_\mu, \beta)$. It is not jointly continuous from $\mathcal{O}'_{\mu,\sharp} \times (\mathcal{H}'_\mu, \beta)$ into $(\mathcal{H}'_\mu, \beta)$, however.

Proposition 6.3. *The Hankel convolution is not jointly continuous from $\mathcal{O}'_{\mu,\sharp} \times (\mathcal{H}'_\mu, \beta)$ into $(\mathcal{H}'_\mu, \sigma)$, neither from $\mathcal{O}'_{\mu,\sharp} \times (\mathcal{H}'_\mu, \beta)$ into $(\mathcal{H}'_\mu, \beta)$.*

Proof. This follows immediately from Proposition 6.2 and [10, Proposition 5.2]. □

References

[1] *J. J. Betancor and I. Marrero*: Multipliers of Hankel transformable generalized functions. Comment. Math. Univ. Carolin. *33* (1992), 389–401.
[2] *J. J. Betancor and I. Marrero*: The Hankel convolution and the Zemanian spaces $\mathcal{B}_\mu$ and $\mathcal{B}'_\mu$. Math. Nachr. *160* (1993), 277–298.
[3] *J. J. Betancor and I. Marrero*: Structure and convergence in certain spaces of distributions and the generalized Hankel convolution. Math. Japon. *38(6)* (1993), 1141–1155.
[4] *J. J. Betancor and I. Marrero*: Some properties of Hankel convolution operators. Canad. Math. Bull. *36* (1993), 398–406.
[5] *J. J. Betancor and I. Marrero*: A Hilbert-space approach to Hankel-transformable distributions. Appl. Anal. *52* (1994), 103–124.
[6] *J. J. Betancor and I. Marrero*: Multipliers and convolution operators on Hankel transformable Hilbert spaces. Rend. Sem. Mat. Univ. Pol. Torino *51(2)* (1993), 119–137.
[7] *J Kucera*: Convolution of temperate distributions. Publ. Math. *33* (1986), 323–327.
[8] *J. Kucera and K. McKennon*: The topology on certain spaces of multipliers of temperate distributions. Rocky Mountain J. Math. *7* (1977), 377–383.
[9] *J. Kucera and K. McKennon*: Continuity of multiplication of distributions. Internat. J. Math. Sci. *4* (1981), 819–822.
[10] *I. Marrero and J. J. Betancor*: Hankel convolution of generalized functions. Rend. Mat. Appl. *15* (1995), 351–380.
[11] *K. McKennon*: The multiplication functions of Kucera. Czechoslovak Math. J. *29* (1979), 169–177.
[12] *H. H. Schaefer*: Topological Vector Spaces. Grad. Texts in Math., No. 3 (3rd printing), Springer, Berlin, 1971.
[13] *F. Tréves*: Topological Vector Spaces, Distributions, and Kernels. Academic Press, Now York, 1967.
[14] *A. H. Zemanian*: The Hankel transformation of certain distributions of rapid growth. SIAM J. Appl. Math. *14* (1966), 678–690.
[15] *A. H. Zemanian*: Generalized Integral Transformations. Interscience, New York, 1968.

Authors' address: Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna (Tenerife), Islas Canarias, Spain.