Czechoslovak Mathematical Journal

Gary Chartrand; Heather Gavlas; Kelly Schultz; Steven J. Winters
On strong digraphs with a prescribed ultracenter

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 1, 83–94

Persistent URL: http://dml.cz/dmlcz/127340

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

ON STRONG DIGRAPHS WITH A PRESCRIBED ULTRACENTER

GARY CHARTRAND, HEATHER GAVLAS, KELLY SCHULZ, Kalamazoo,[1]
STEVE J. WINTERS, Oshkosh

(Received July 11, 1994)

Summary. The (directed) distance from a vertex u to a vertex v in a strong digraph D is the length of a shortest u-v (directed) path in D. The eccentricity of a vertex v of D is the distance from v to a vertex furthest from v in D. The radius $\mathrm{rad}D$ is the minimum eccentricity among the vertices of D and the diameter $\mathrm{diam}D$ is the maximum eccentricity. A central vertex is a vertex with eccentricity $\mathrm{rad}\,D$ and the subdigraph induced by the central vertices is the center $C(D)$. For a central vertex v in a strong digraph D with $\mathrm{rad}\,D < \mathrm{diam}D$, the central distance $c(v)$ of v is the greatest nonnegative integer n such that whenever $d(v,x) \leqslant n$, then x is in $C(D)$. The maximum central distance among the central vertices of D is the ultraradius $\mathrm{urad}D$ and the subdigraph induced by the central vertices with central distance $\mathrm{urad}D$ is the ultracenter $UC(D)$. For a given digraph D, the problem of determining a strong digraph H with $UC(H) = D$ and $C(H) \neq D$ is studied. This problem is also considered for digraphs that are asymmetric.

1. INTRODUCTION

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path. The *eccentricity* $e(v)$ of a vertex v of G is the distance between v and a vertex furthest from v. The minimum eccentricity among the vertices of G is the *radius* $\mathrm{rad}\,G$ of G, and the maximum eccentricity is the *diameter* $\mathrm{diam}\,G$. A vertex whose eccentricity is $\mathrm{rad}\,G$ is called a *central vertex.* The subgraph of G induced by its central vertices is the *center* $C(G)$ of G. The center of a connected graph has been the subject of much study. In [4], Winters introduced a subgraph of $C(G)$ which is, in a certain sense, more central than the center itself.

For a central vertex v in a connected graph G with $\mathrm{rad}\,G < \mathrm{diam}\,G$ the *central distance* $c(v)$ is the greatest nonnegative integer n such that whenever $d(v,x) \leqslant n$

[1] Research supported in part by Office of Naval Research Grant N00014-91-J-1060

for a vertex x of G, then x is a central vertex. The maximum central distance among the central vertices of G is the *ultraradius* $\operatorname{urad} G$ of G, and the subgraph of $C(G)$ induced by those central vertices v with $c(v) = \operatorname{urad} G$ is the *ultracenter* $UC(G)$ of G. Chartrand, Novotny and Winters studied the ultracenter further in [1]. Among the results presented is that for every graph G, there exists a connected graph H such that $UC(H) = G$ and $C(H) \neq G$. Furthermore, the minimum order of such a graph H is 4 more than the order of G. It is the object of this paper to study the analogous concepts for digraphs.

The (directed) *distance* $d(u, v)$ from a vertex u to a vertex v in a strong digraph D is the length of a shortest u–v (directed) path in D. The *eccentricity* $e(v)$ of a vertex v of D is the distance from v to a vertex furthest from v. The minimum eccentricity among the vertices of D is called the *radius* $\operatorname{rad} D$ of D and the maximum eccentricity is the *diameter* $\operatorname{diam} D$. A vertex v in a strong digraph D is called a *central vertex* if $e(v) = \operatorname{rad} D$. The subdigraph induced by the central vertices of D is called the *center* $C(D)$ of D. Two vertices u and v are *adjacent* in a digraph D if D contains at least one of the arcs (u, v) and (v, u). If (u, v) is an are of D, then u is *adjacent to* v, and v is *adjacent from* u. A digraph D is *asymmetric* if whenever u and v are adjacent in D, then exactly one of the arcs (u, v) and (v, u) is present in D. Chartrand, Johns, and Tian [2] showed for every asymmetric digraph D, there exists a strong asymmetric digraph H with $C(H) = D$. In [3], Shaikh showed for every (not necessarily asymmetric) digraph D, there exists a strong digraph H such that $C(H) = D$.

Let v be a central vertex of a strong digraph D with $\operatorname{rad} D < \operatorname{diam} D$. The *central distance* $c(v)$ of v is the largest nonnegative integer n such that whenever $d(v, x) \leqslant n$ the vertex x is in the center of D. Let $m = \max\{c(v)\}$, where the maximum is taken over all central vertices v of D. The subdigraph of $C(D)$ induced by those vertices v with $c(v) = m$ is called the *ultracenter* of D, which we denote by $UC(D)$. The number m is referred to as the *ultraradius* of D and is denoted by $\operatorname{urad} D$.

For example, each vertex of the digraph D of Figure 1 is labeled with its eccentricity. Thus, $\operatorname{rad} D = 6$ and $\operatorname{diam} D = 9$. Furthermore, each central vertex of D is labeled with its central distance and so $\operatorname{urad} D = 3$.

Let D be a strong digraph with $\operatorname{rad} D < \operatorname{diam} D$. If v is a vertex with central distance k then there is a path $P\colon v = v_0, v_1, v_2, v_3, \ldots, v_{k+1}$ of length $k+1$ from v to a vertex v_{k+1} not in the center of D. Thus, $c(v_i) = k - i$ for $0 \leqslant i \leqslant k$. The following theorem is a consequence of this observation.

Theorem 1. *Let D be a strong digraph with $\operatorname{rad} D < \operatorname{diam} D$ and $\operatorname{urad} D = m$. For each integer i $(0 \leqslant i \leqslant m)$, there exists a central vertex u_i with $c(u_i) = i$.*

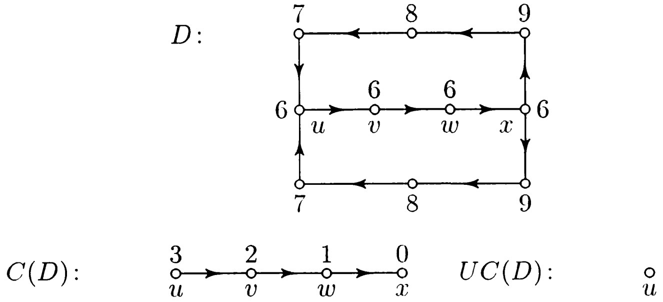

Figure 1. The center and ultracenter of a strong digraph

Let D be a strong digraph with $\operatorname{rad} D < \operatorname{diam} D$ and $\operatorname{urad} D = m$. If some vertex of $UC(D)$ is adjacent to a noncentral vertex, then, by definition, $m = 0$. So, $C(D) = UC(D)$. Thus, if $m \geqslant 1$, then there are no vertices in $UC(D)$ adjacent to noncentral vertices. In a special case, we can provide information about the structure of the ultracenter of a strong digraph.

Theorem 2. *Let D be a strong digraph with $\operatorname{rad} D < \operatorname{diam} D$. If there is a unique central vertex of D that is not in the ultracenter of D, then $UC(D)$ is connected.*

Proof. Let x be the central vertex of D that does not belong to the ultracenter. Suppose, to the contrary, that $UC(D)$ is disconnected. Let v be a vertex of D such that $d(x, v) = e(x)$. Suppose, first, that v does not belong to $C(D)$. Let w be a vertex of $UC(D)$ that is adjacent to x, that is, $c(w) = 1$. Consequently, $\operatorname{urad} D = 1$ and every vertex of $UC(D)$ is adjacent to x. Hence $d(w, v) = 1 + d(x, v)$, and so $e(w) > e(x)$, which is impossible. Therefore, v belongs to $UC(D)$.

Let u be a vertex in $UC(D)$ such that u and v belong to distinct components of $UC(D)$. Then each u–v path contains x. Thus, $e(u) \geqslant d(u, v) \geqslant 1 + d(x, v) = 1 + e(x)$, contradicting the fact that u and x have the same eccentricity. $\square$

2. The ultracentral appendage number of digraphs

The minimum number of vertices needed to be added to a digraph D to produce a strong digraph H such that $UC(H) = D$ and $C(H) \neq D$ is called the *ultracentral appendage number* of D and is denoted by $ua(D)$. Such a digraph H is called a *minimum ultracentral superdigraph* of D. Since H contains a central vertex that is not in D, and H contains a noncentral vertex, $ua(D) \geqslant 2$. The *central appendage number* $A(D)$ of a digraph D is the minimum number of vertices that must be added to D to produce a digraph H such that $C(H) = D$. The central appendage number was studied by Shaikh [3], who showed that $0 \leqslant A(D) \leqslant 3$ for every digraph D.

For example, consider $D \cong 2K_1$, where $V(D) = \{u, v\}$. The strong digraph H of Figure 2 has the property that $UC(H) = D$ but $C(H) \neq D$. In fact, $C(H)$ contains the vertices x and y as well. Thus, $ua(D) \leqslant 3$. If $ua(D) = 2$, then there is a unique central vertex of a minimum ultracentral superdigraph H that is not in the ultracenter of H. So by Theorem 2 $UC(H)$ is connected, producing a contradiction. Therefore, $ua(D) \geqslant 3$, which gives $ua(D) = 3$.

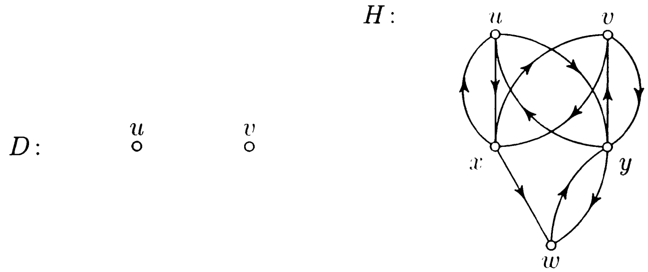

Figure 2. A digraph ultracentral appendage number 3

We now show that *every* digraph has ultracentral appendage number 2 or 3.

Theorem 3. *The ultracentral appendage number of every digraph D is well-defined and $2 \leqslant ua(D) \leqslant 3$.*

Proof. Let D be a digraph and let H be the strong digraph obtained from D by adding the vertices x_1, x_2, and y and the arcs indicated in Figure 3. Thus, x_1 and x_2 are adjacent to and from every vertex of D. Observe that all vertices of D and x_1 and x_2 are central vertices, while y is not. Also $UC(H) = D$. Thus $ua(D)$ is well-defined and $ua(D) \leqslant 3$. We have previously noted that $ua(D) \geqslant 2$ for every digraph D and thus $2 \leqslant ua(D) \leqslant 3$. □

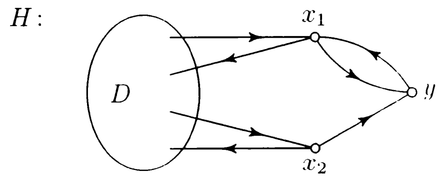

Figure 3

Now we show that the bounds presented in Theorem 3 are sharp.

Theorem 4. *If D is a nontrivial digraph containing a vertex v such that $d_D(u, v) \leqslant 2$ for all vertices u in D, then $ua(D) = 2$.*

Proof. By Theorem 3, $ua(D) \geqslant 2$. Let $D' = D - v$. The strong digraph H shown in Figure 4 is an ultracentral superdigraph for D. The vertex x is then adjacent to and from every vertex of D'. Thus, $ua(D) \leqslant 2$. □

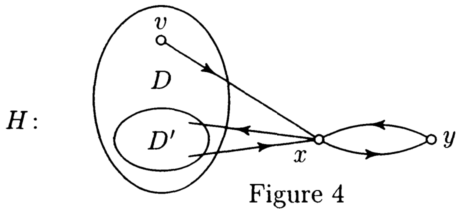

Figure 4

We have already seen that if $D \cong 2K_1$, then $ua(D) = 3$. We next show the existence of an infinite class of digraphs with ultracentral appendage number 3.

Theorem 5. *If D is a digraph containing no vertex that is reachable from all other vertices of D, then $ua(D) = 3$.*

Proof. Assume, to the contrary, that $ua(D) = 2$. Let H be a minimum ultracentral superdigraph for D, where x is the central vertex of H that is not in D and y is the noncentral vertex of H. Let w be a vertex of H such that $d(x, w) = e(x)$. Since x must be adjacent to y and $e(x) > 1$, we must have $w \in V(D)$. Let u be a vertex of D different from w. If some shortest $u - w$ path contains x, then $d(u, w) \geqslant 1 + e(x)$, which gives $e(u) > e(x)$, producing a contradiction. Therefore, there exists a $u - w$ path in D for every vertex u of D, giving the desired result. □

Corollary 6. *If D is a disconnected digraph, then $ua(D) = 3$.*

3. The asymmetric ultracentral appendage number of asymmetric digraphs

In this section we consider only asymmetric digraphs. For an asymmetric digraph D, we define the *(asymmetric) ultracentral appendage number* $ua^*(D)$ of D as the minimum number of vertices to be added to D to produce an asymmetric digraph H with $UC(H) = D$ and $C(H) \neq D$. The (asymmetric) central appendage number $A^*(D)$ was studied by Chartrand, Johns, and Tian [2], who showed that $0 \leqslant A^*(D) \leqslant 4$ for all digraphs D.

Theorem 7. *For every asymmetric digraph D, $ua^*(D)$ exists and $3 \leqslant ua^*(D) \leqslant 5$.*

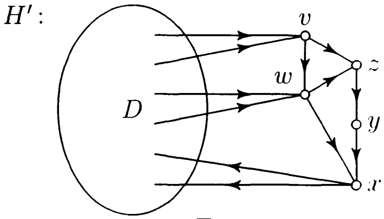

Figure 5

P r o o f. Let D be an asymmetric digraph. The digraph H' of Figure 5 is obtained by adding the vertices v, w, x, y, z and all those arcs so that every vertex of D is adjacent to both v and w, and adjacent from x. Therefore, H' is strong and asymmetric with $UC(H') = D$ and $C(H') \neq D$. Thus $ua^*(D)$ exists and $ua^*(D) \leqslant 5$.

Since every ultracentral superdigraph of D contains a central vertex that is not in D and a noncentral vertex, $ua^*(D) \geqslant 2$. Suppose, to the contrary, that $ua^*(D) = 2$. Then there is a minimum ultracentral superdigraph H of D containing two vertices that are not in D. Let x be the central vertex that is not in D and let y be the noncentral vertex of H. Necessarily x is adjacent to y, and every vertex of D is adjacent to x. Let $z \in V(H)$ such that $e(x) = d(x, z)$.

Since $e(x) > 1$, we have that $z \in V(D)$. Consequently, $d(x, z) = d(y, z) + 1$ Certainly,

$$\max_{w \in V(D)} d(y, w) = d(y, z).$$

Since $e(y) > e(x)$, it follows that $e(y) = d(y, x)$. Since $d(y, x) = 2$, it follows that $e(y) = 2$, which implies that y belongs to $UC(H)$, producing a contradiction. □

Next, we show that the lower bound given in Theorem 7 for $ua^*(D)$ cannot be improved in general. For example, consider $D \cong K_1$, and let $V(D) = \{u\}$. The asymmetric digraph H' in Figure 6 has the property that $UC(H') = D$ but $C(H') \neq D$. Thus $ua^*(D) = 3$. However, $A^*(D) = 0$ in this case.

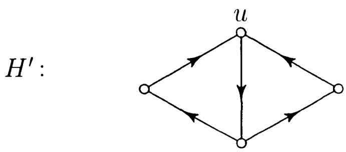

Figure 6. $ua^*(K_1) = 3$

If D is an asymmetric disconnected digraph, then we can improve the upper bound presented in Theorem 7.

Theorem 8. *For every disconnected digraph D, $3 \leqslant ua^*(D) \leqslant 4$.*

Proof. Assume that D is an asymmetric disconnected digraph, where D_1 is one component of D and D_2 is the union of the remaining components. By Theorem 7, $ua^*(D) \geqslant 3$. The digraph H in Figure 7 is obtained by adding to D the four vertices u, v, x, y and the arcs (u,v), (u,y), (v,x), (v,y), as well as all those arcs such that x is adjacent to every vertex of D_1, y is adjacent to every vertex of D_2, and u and v are adjacent from every vertex of D. Since $UC(H) = D$ and $C(H) \neq D$, if follows that $ua^*(D) \leqslant 4$. □

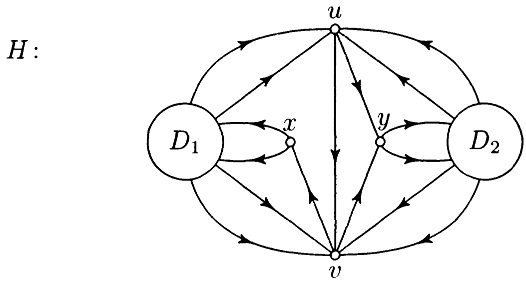

Figure 7

We have seen that there exists an asymmetric digraph D with $ua^*(D) = 3$. We now show that an asymmetric digraph exists with ultracentral appendage number 4.

Theorem 9. *There exists an asymmetric digraph D with $ua^*(D) = 4$.*

Proof. Let $D \cong 2K_1$. By Theorem 8, either $ua^*(D) = 3$ or $ua^*(D) = 4$. Suppose, to the contrary, that $ua^*(D) = 3$. Let H be a minimum ultracentral superdigraph (necessarily of order 5) such that $UC(H) = D$ and $C(H) \neq D$. Let $V(H) = \{u,v,w,x,y\}$ and suppose that $UC(H) = \langle\{u,v\}\rangle$. By Theorem 2, there must be two central vertices of H that are not in the ultracenter of H. Suppose that w and x are these vertices. We consider two cases.

Case 1. Suppose that $\operatorname{urad} H = 2$. Then $c(u) = c(v) = 2$. Also, exactly one of w and x must have central distance 1, say $c(w) = 1$ and so $c(x) = 0$. This situation is illustrated in Figure 8.

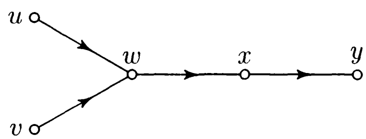

Figure 8. A subdigraph of H

No further arcs from u or v can be present in H since $c(u) = c(v) = 2$. Thus, $d(u, v) \geqslant 3$ and so $\operatorname{rad} H \geqslant 3$. Since $e(x) < e(y)$, at least one of the arcs (x, u) and (x, v) must be present in H, say (x, u). Since $e(x) \geqslant 3$, neither (x, v) nor (y, v) is present in H. This, however, implies that H is not strong, producing a contradiction.

Case 2. Suppose that $\operatorname{urad} H = 1$. Thus, both w and x are adjacent to y. Also, since H is strong, y must be adjacent to at least one of u and v, say u. Furthermore, each of u and v is adjacent to at least one vertex of $C(H)$. We consider two subcases according to the number of vertices of $C(H)$ to which u and v are adjacent.

Subcase 2.1. Suppose that each of u and v is adjacent to exactly one vertex of $C(H)$. First, suppose that u and v are adjacent to the same vertex, say w. Since H is strong, every vertex of H is adjacent from at least one vertex. Consequently, x is adjacent from w. Thus far we have the digraph shown in Figure 9.

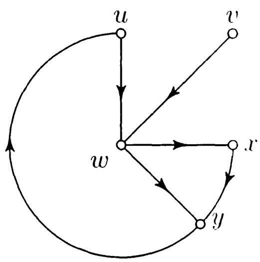

Figure 9. A subdigraph of H

The vertex v is adjacent from at least one vertex as well. Necessarily, at least one of x and y is adjacent to v. In either case, $e(w) = 2$, which implies that $e(u) = e(v) = 2$. However, then, h contains a u–v path of length 2, which is impossible.

Therefore u and v are adjacent to distinct vertices, say u is adjacent to w, and v is adjacent to x. Now either y or w is adjacent to v. If y is adjacent to v, then $e(y) = 2$, which is impossible. Thus, w is adjacent to v, so $e(w) = 2$. Thus, $e(u) = e(v) = 2$ and x is adjacent to u (see Figure 10).

Since $d(x, v) = 2$, the arc (x, w) belongs to H. At present, however, $d(u, x) = 3$, and no further arcs can be added. This contradicts that fact that $e(u) = 2$.

Subcase 2.2. Suppose that at least one of u and v is adjacent to two vertices of $C(H)$. In this case, y is not adjacent to v, for otherwise $e(y) = 2$. This implies that not both u and v are adjacent to both w and x. Since v is adjacent from some vertex, it follows that u is adjacent to w and x; while v is adjacent to one of w and x, and adjacent from the other. Suppose that v is adjacent to w (see Figure 11).

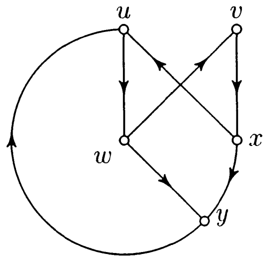

Figure 10. A subdigraph of H

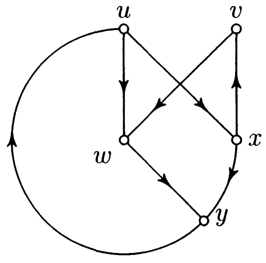

Figure 11. A subdigraph of H

Then $e(x) = 2$ and $e(u) = e(v) = 2$. This, however, implies that $d(v, u) = 2$, which is not the case. If v is adjacent to x, than $e(w) = 2$; so $e(u) = e(v) = 2$. However, than, $d(v, u) = 2$, and again this is not the case. $\square$

Thus, $ua^*(2K_1) = 4$ while $ua(2K_1) = 3$. We next describe a sufficient condition for a disconnected asymmetric digraph to have ultracentral appendage number 3.

Theorem 10. *Let* $D \cong D_1 \cup D_2$, *where* D_1 *and* D_2 *are strong asymmetric digraphs such that* $\operatorname{diam} D_1 \leqslant 3$, $\operatorname{diam} D_2 \leqslant 3$, *and* $D_1 \not\cong K_1$. *Then* $ua^*(D) = 3$.

Proof. By Theorem 8, $ua^*(D) \geqslant 3$. The digraph H of Figure 12 obtained by adding the vertices x, y, and z and all those arcs such that x is adjacent from and y is adjacent to every vertex of D_1, y is adjacent from and x is adjacent to every vertex of D_2, and z is adjacent to a single vertex of D_1. Then each vertex of H has eccentricity 3, except z, in which case $e(z) = 4$. Thus, H has the desired properties and $ua^*(D) \leqslant 3$. So $ua^*(D) = 3$. $\square$

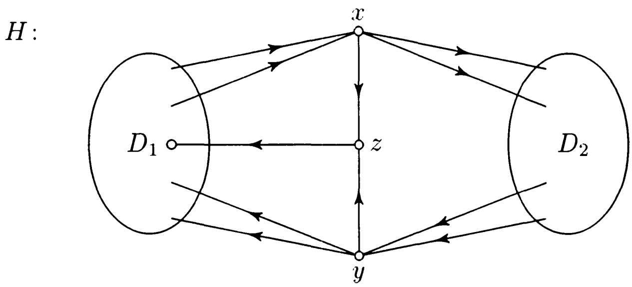

Figure 12

We now turn our attention to connected asymmetric digraphs.

Theorem 11. *If* D *is a strong asymmetric digraph with* $\operatorname{diam} D = 2$, *then* $ua^*(D) = 3$.

Proof. We construct the strong digraph H of Figure 13 by adding three vertices x, y and z and all those arcs such that y and z are adjacent to every vertex of D, and x is adjacent from every vertex of D. Since $UC(H) = D$ and $C(H) = \langle V(D) \cup \{x, z\}\rangle$, it follows that $ua^*(D) = 3$. □

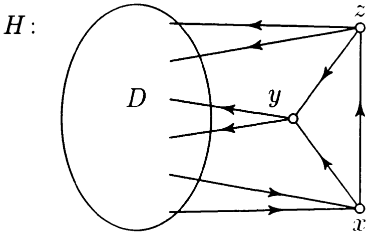

Figure 13

We now show that there is a connected asymmetric digraph having ultracentral appendage number 4.

Theorem 12. *There exists a connected asymmetric digraph D with $ua^*(D) = 4$.*

Proof. Let D be the digraph shown in Figure 14. We show that $ua^*(D) = 4$. We now construct the asymmetric digraph F of Figure 14 by adding the vertices t, x, y, and z to D together with the indicated arcs. Then the central vertices of F are u, v, w, x, and y, and the ultracentral vertices are u, v, and w. Thus $UC(F) = D$; so $ua^*(D) \leqslant 4$. Consequently, it remains only to show that $ua^*(D) \neq 3$.

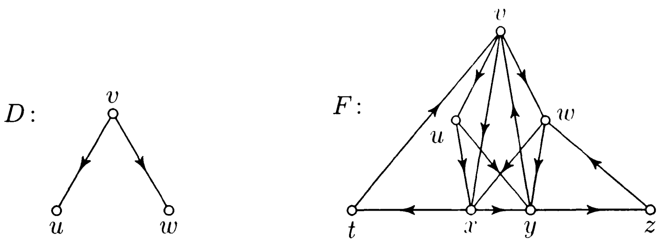

Figure 14

Assume, to the contrary, that $ua^*(D) = 3$. Let H be a minimum ultracentral superdigraph for D with $V(H) = \{u, v, w, x, y, z\}$. We consider three cases.

Case 1. Assume that there are exactly two vertices not in the center of H, say y and z. Thus, all vertices of D are adjacent to x and, without loss of generality, $(x, y) \in E(H)$ (see Figure 15). Since u is in $UC(H)$ and y and z are not central vertices, neither (u,y) nor (u,z) is present in H. Thus, $d_H(u,w) > 2$ and $\operatorname{rad} H > 2$. Since v is adjacent to u, w, and x and $e(v) > 2$, it follows that $(x,z) \notin E(H)$. Since H is strong, $(y,z) \in E(H)$ and at least one of (y,v) and (z,v) is an arc of H. If $(y,v) \in E(H)$, then $e(y) = 2$, producing a contradiction; while if $(z,v) \in E(H)$, then $e(z) = 3$ and z is a central vertex of H, also producing a contradiction.

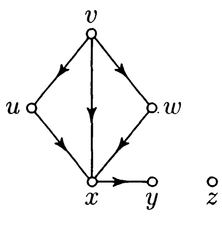

Figure 15

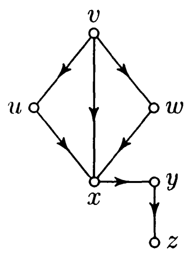

Figure 16

Case 2. Assume that there is exactly one vertex, say z, not in the center of H and $\operatorname{urad} H = 2$. Then there is a vertex, say y, such that $c(y) = 0$ and a vertex, say x, with $c(x) = 1$. Therefore, all vertices of D are adjacent to x and (x,y) and (y,z) are present in H (see Figure 16), while (u,y), (u,z), and (x,z) cannot be present in H. Thus, $\operatorname{rad} H > 2$ because $d(u,w) > 2$. Since H is strong, at least one of (y,v) and (z,v) is in H. If $(y,v) \in E(H)$, then $e(y) = 2$, producing a contradiction. If $(z,v) \in E(H)$, then $e(z) = 3$ and z is a central vertex of H, again producing a contradiction.

Case 3. Assume that $\operatorname{urad} H = 1$ and that there is exactly one vertex, say z, not in the center of H. Consequently, (x,z) and (y,z) are in H. Suppose that $\operatorname{rad} H = 2$. Then $d(u,w) = 2$. Thus, without loss of generality, the arcs (u,x) and (x,w) are present in H. Similarly, (w,y) and (y,u) are present in H, giving $d(w,u) = 2$. Also, since $d(u,v) = 2$, it follows that $(x,v) \in E(H)$. Similarly, $(y,v) \in E(H)$ since $d(w,v) = 2$. This produces the subdigraph in Figure 17. Now there are no arcs that can be added to allow $d(v,z)$ to be less than 3, producing a contradiction. Thus, $\operatorname{rad} H \geqslant 3$. Since each of u, v, and w must be adjacent to one of x and y, it follows that x or y, say x, must be adjacent from at least two of u, v, and w. We consider three subcases.

Subcase 3.1. Assume that all three vertices u, v, and w are adjacent to x. Since $\operatorname{rad} H \geqslant 3$, it follows that $e(v) \geqslant 3$. Thus, $d(v,y) = 3$ (see Figure 18). However, this is impossible.

Subcase 3.2. Assume that only u and v are adjacent to x. Since H is strong, w is adjacent to y. Thus, $e(v) = 2$, producing a contradiction.

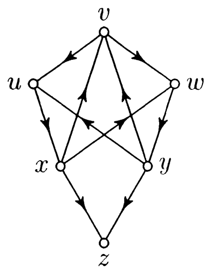

Figure 17

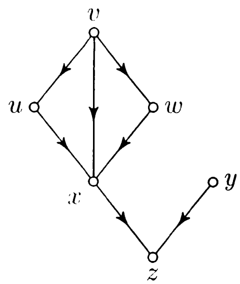

Figure 18

Subcase 3.3. Assume that only u and w are adjacent to x. Since H is strong $(v, y) \in E(H)$. Thus, $e(v) = 2$, again producing a contradiction. $\square$

We close with one lingering question: Does there exist an asymmetric digraph D with $ua^*(D) = 5$? If such a digraph D exists, it must surely be connected. Indeed, if D is strong, then $\operatorname{diam} D \geqslant 3$.

References

[1] *G. Chartrand, K. Novotny, and S.J. Winters*: The ultracenter and central fringe of a graph. Networks. To appear.

[2] *G. Chartrand, G.L. Johns, and S. Tian*: Directed distance in digraphs: centers and peripheries. Congr. Numer. *89* (1992), 89–95.

[3] *M.P. Shaikh*: On digraphs with prescribed centers and peripheries. J. Undergrad. Math. *25* (1993), 31–42.

[4] *S.J. Winters*: Distance Associated with Subgraphs and Subdigraphs. Ph.D. Dissertation, Western Michigan University, 1993.

Authors' addresses: G. Chartrand, H. Gavlas, K. Schulty, Western Michigan University, Department of Mathematics and Statistics, Kalamazoo, Michigan 49008-5152, U.S.A.; S.J. Winters, University of Wisconsin Oshkosh, U.S.A.