Czechoslovak Mathematical Journal

Ján Jakubík

Lateral and Dedekind completions of strongly projectable lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 3, 511–523

Persistent URL: http://dml.cz/dmlcz/127375

Terms of use:

© Institute of Mathematics AS CR, 1997

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 47 (122) (1997), Praha

LATERAL AND DEDEKIND COMPLETIONS OF STRONGLY PROJECTABLE LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received February 22, 1995)

For a lattice ordered group G we denote by G^L and G^D the lateral completion or the Dedekind completion of G, respectively. (For definitions, cf. Section 1 below.)

The main result of [2] is the following theorem:

(∗) (Bernau) Let G be an archimedean lattice ordered group. Then the relation

$$G^{DL} = G^{LD} \tag{1}$$

is valid.

This solved a problem proposed by Conrad [4].

In the present paper the validity of (1) for strongly projectable lattice ordered groups will be proved.

Let us remark that an archimedean lattice ordered group need not be strongly projectable; also, a strongly projectable lattice ordered group need not be archimedean. Thus our result neither implies (∗) nor is implied by (∗).

For each lattice ordered group G the lateral completion G^L is defined uniquely up to isomorphism (cf. Conrad [4], Bernau [1]). Hence, in fact, the relation (1) is to be considered in the sense of isomorphism (leaving all the elements of G fixed).

1. Preliminaries

In the whole paper G denotes a lattice ordered group.

An indexed system $(g_i)_{i\in I}$ $(I \neq \emptyset)$ of elements of G is called disjoint if $g_i \geqslant 0$ for each $i \in I$, and $g_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

G is said to be laterally complete if each indexed disjoint system in G has the supremum in G.

If G is an ℓ-subgroup of a lattice ordered group H such that for each $h \in H$ with $0 < h$ there exists $g \in G$ with $0 < g \leqslant h$, then G is called a dense ℓ-subgroup of H.

1.1. Definition. (Cf. Conrad [4].) A lattice ordered group H is said to be a *lateral completion* of G if the following conditions are satisfied:

(i) H is laterally complete.
(ii) G is a dense ℓ-subgroup of H.
(iii) If H_1 is an ℓ-subgroup of H such that $G \subseteq H_1$ and H_1 is laterally complete, then $H_1 = H$.

1.2. Theorem. (Bernau [1].) *Each lattice ordered group possesses a lateral completion. If H and H' are lateral completions of G, then there exists an isomorphism φ of H onto H' such that $\varphi(g) = g$ for each $g \in G$.*

Thus, up to isomorphism, the lateral completion of G is uniquely determined; we denote it by G^L.

Let $X \subseteq G$. The system of all upper bounds (or lower bounds, respectively) of X in G will be denoted by $U(X)$ (or $(L(X))$. A pair (A, B) of nonempty subsets A and B of G will be said to be a cut in G if $A = L(B)$ and $B = U(A)$. A cut (A, B) will be called a D-cut if the relations

$$\bigwedge_{a\in A, b\in B} (b - a) = 0,$$
$$\bigwedge_{a\in A, b\in B} (-a + b) = 0$$

are valid in G.

1.3. Definition. A lattice ordered group G is said to be *D-complete* if for each D-cut (A, B) in G there exists $g \in G$ such that the relation

$$\sup A = g = \inf B$$

is valid.

1.4. Definition. A lattice ordered group H is called a *Dedekind completion* of G if the following conditions are satisfied:

(i) H is D-complete.
(ii) G is an ℓ-subgroup of H.
(iii) For each $h \in H$ there are subsets X and Y of G such that the relations

$$\sup X = h = \inf Y$$

are valid in H.

From the results of Everett [5] (cf. also Fuchs [5], Chap. V, §10) we obtain

1.5. Theorem. *Each lattice ordered group possesses a Dedekind completion. If H and H' are Dedekind completions of G, then there exists an isomorphism φ of H onto H' such that $\varphi(g) = g$ for each $g \in G$.*

1.6. Theorem. (Conrad [4].) *Let G be a dense ℓ-subgroup of a laterally complete lattice ordered group H. Next, let H_0 be the intersection of all ℓ-subgroups H_i of H such that $G \subseteq H_i$ and H_i is laterally complete. Then H_0 is a lateral completion of G.*

2. Auxiliary results

If G is a dense ℓ-subgroup of a lattice ordered group H, then we express this fact by writing $G \subseteq_d H$.

It is obvious that if H' is a Dedekind completion of G, then $G \subseteq_d H'$.

2.1. Lemma. *Let $G \subseteq_d G'$. Suppose that H' is a lateral completion of G'. Then there is a lateral completion H of G such that $H \subseteq_d H'$.*

Proof. We have $G' \subseteq_d H'$, hence $G \subseteq_d H'$. Now it suffices to apply 1.6. □

2.2. Lemma. *Let H be a lattice ordered group such that $G \subseteq_d H$. Next, let H_0 be the set of all $h \in H$ such that there exist $X, Y \subseteq G$ having the property that the relation*

$$\sup X = h = \inf Y$$

is valid in H. Then H_0 is an ℓ-subgroup of H.

Proof. Let $h \in H$ and let X, Y be as above. Further, let $h' \in H$, $X' \subseteq G$, $Y' \subseteq G$ be such that $\sup X' = h' = \inf Y'$ is valid in H. Then we have

$$\sup\{x + x'\}_{x \in X, x' \in X'} = h + h' = \inf\{y + y'\}_{y \in Y, y' \in Y'}$$

in H. Analogous relations remain valid if $+$ is replaced by $\vee$ or by $\wedge$. Also,

$$\sup\{-y\}_{y \in Y} = -h = \inf\{-x\}_{x \in X}.$$

Hence H_0 is an ℓ-subgroup of H. □

2.3. Lemma. *Let H be a lattice ordered group, $\{x_i\}_{i\in I} \subseteq H$, $\{y_j\}_{j\in J} \subseteq H$, $h \in H$,*

$$\sup\{x_i\}_{i\in I} = h = \inf\{y_j\}_{j\in J}.$$

Then

$$\bigwedge_{i\in I, j\in J} (y_j - x_i) = 0 = \bigwedge_{i\in I, j\in J} (-x_i + y_j).$$

Proof. We have

$$0 = \bigwedge_{j\in J} y_j - \vee_{i\in I} x_i = \bigwedge_{j\in J} y_j + \bigwedge_{i\in I}(-x_i) = \\ = \bigwedge_{j\in J, i\in I} (y_j - x_i).$$

The other relation can be verified analogously. □

An ℓ-subgroup H_1 of a lattice ordered group H_2 will be called regular if, whenever $X \subseteq H_1$, $Y \subseteq H_1$, $x \in H_1$, $y \in H_1$ and the relations

$$\sup X = x, \quad \inf Y = y$$

are valid in H_1, then these relations are valid also in H_2.

2.4. Lemma. (Bernau [1].) *Let H_1 be a dense ℓ-subgroup of H_2. Then H_1 is a regular ℓ-subgroup of H_2.*

2.5. Lemma. *Let G, H and H_0 be as in 2.2. Assume that H is D-complete. Then H_0 is D-complete as well.*

Proof. Let (A, B) be a D-cut in H_0. We denote by B_1 the set of all upper bounds of A in H, and by A_1 the set of all lower bounds of B_1 in H. Then (A_1, B_1) is a cut in H and $A \subseteq A_1$, $B \subseteq B_1$. The relations

$$\bigwedge_{a\in A, b\in B} (b - a) = 0 = \bigwedge_{a\in A, b\in B} (-a + b)$$

are valid in H_0. In view of 2.4, these relations are valid also in H (since, obviously, H_0 is a dense ℓ-subgroup of H). Then the inclusions $A \subseteq A_1$, $B \subseteq B_1$ imply

$$\bigwedge_{a_1\in A_1, b_1\in B_1} (b_1 - a_1) = \bigwedge_{a_1\in A_1, b_1\in B_1} (-a_1 + b_1) = 0.$$

Thus (A_1, B_1) is a D-cut in H. Since H is D-complete, there exists $h \in H$ with

$$\sup A_1 = h = \inf B_1.$$

From the definition of B_1 and from $h = \inf B_1$ we see that the relation

$$h = \sup A$$

is valid in H. Since $A \subseteq H_0$, for each $a \in A$ there exists a subset $X(a)$ of G such that the relation

$$a = \sup X(a)$$

holds in H_0. Thus according to the definition of H_0 this relation holds also in H. Similarly, there exists $Y \subseteq G$ such that $h = \inf Y$ is valid in H. Denote $X = \bigcup_{a \in A} X(a)$. Then we have

$$h = \sup A = \sup\{\sup X(a)\}_{a \in A} = \sup X$$

in H. This yields that $h \in H_0$. Now, since (A, B) is a cut in H_0, we obtain that $h \in A \cap B$ and

$$h = \inf B$$

in H_0. Therefore in view of 1.3, H_0 is D-complete. □

2.6. Lemma. *Let G, H and H_0 be as in 2.2. Then H_0 is a Dedekind completion of G.*

P r o o f. We apply the conditions (i), (ii) and (iii) from 1.4. In view of 2.2, H_0 is an ℓ-subgroup of H and clearly $G \subseteq H_0$; thus G is an ℓ-subgroup of H_0. According to 2.5, H_0 is D-complete. Hence (i) and (ii) from 1.4 are satisfied. The definition of H_0 yields that (iii) from 1.4 also holds. □

2.7. Corollary. *Let G be a dense ℓ-subgroup of a lattice ordered group H and let H' be a Dedekind completion of H. Then there exists a Dedekind completion H_0 of G such that*

(i) $H_0 \subseteq_d H'$;

(ii) *if H_0^1 is a dense ℓ-subgroup of H' such that $G \subseteq H_0^1$ and if H_0^1 is D-complete, then $H_0 \subseteq H_0^1$.*

3. Strong projectability

For $X \subseteq G$ we denote by X^δ the polar of X in G; i.e.,

$$X^\delta = \{g \in G | g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

A lattice ordered group H is said to be strongly projectable if each polar of H is a direct factor of H.

If we have a direct product decomposition

$$G = \prod_{i \in I} G_i$$

and if $g \in G$, $i \in I$, then the component of g in G_i will be denoted by $g(G_i)$ or by $g(i)$. We identify the element $g(G_i)$ with the element g' of G such that $g'(G_i) = g(G_i)$ and $g'(G_{i(1)}) = 0$ for each $i(1) \in I$ with $i(1) \neq i$.

It is well-known that if $0 < g \in G$, then $g(i)$ is the greatest element of the set $G_i \cap [0, g]$.

3.1. Lemma. *Let G be laterally complete and strongly projectable. Let $(A_i)_{i \in I}$ be an indexed system of direct factors of G such that $A_{i(1)} \cap A_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. Put*

$$B = \Big(\bigcup_{i \in I} A_i\Big)^\delta.$$

Then $G = B \times \prod_{i \in I} A_i$.

Proof. G is strongly projectable and hence B is a direct factor of G. Consider the mapping

$$\varphi : G \to B \times \prod_{i \in I} A_i$$

such that

$$\varphi(x)(A_i) = x(A_i) \quad \text{for each } i \in I,$$

$$\varphi(x)(B) = x(B).$$

Then φ is a homomorphism of G into $B \times \prod_{i \in I} A_i$.

Let $x \in G$, $\varphi(x) = 0$. Then $\varphi(|x|) = 0$. Thus $|x| \wedge a_i = 0$ whenever $i \in I$ and $0 \leqslant a_i \in A_i$. Hence $|x| \in B$ yielding that $|x|(B) = |x|$. But $|x|(B) = 0$ and therefore $|x| = 0 = x$. Hence φ is an isomorphism of G into $B \times \prod_{i \in I} A_i$.

For proving that φ is surjective it suffices to verify that if $0 \leqslant x^i \in A_i$ for $i \in I$ and $0 \leqslant b \in B$, then there exists $g \in G$ such that $\varphi(g)(i) = x^i$ for $i \in I$ and $\varphi(g)(B) = b$.

Choose $0 \leqslant x^i \in A_i$ $(i \in I)$ and $0 \leqslant b \in B$. Since G is laterally complete there exists $g \in G$ such that

$$g = b \vee \Bigl(\bigvee_{i \in I} x^i\Bigr).$$

It is easy to verify that

$$x^i = \max([0, g] \cap A_i)$$

for $i \in I$ and that

$$b = \max([0, g] \cap B).$$

Hence $\varphi(g)(i) = x^i$ for $i \in I$ and $\varphi(g)(B) = b$. Therefore φ is an isomorphism of G onto $B \times \prod_{i \in I} A_i$, which completes the proof. □

3.2. Lemma. *Let G be laterally complete and strongly projectable. Next, let H be a Dedekind completion of G. Then H is laterally complete.*

Proof. Let $(h_i)_{i \in I}$ be a disjoint indexed system of elements of H. Let $i \in I$. There exists $X_i \subseteq G^+$ such that

$$h_i = \sup X_i$$

is valid in H. Then

$$x_{i(1)} \wedge x_{i(2)} = 0$$

whenever $x_{i(1)} \in X_{i(1)}$, $x_{i(2)} \in X_{i(2)}$ and $i(1), i(2)$ are distinct elements of I. Put

$$A_i = X_i^{\delta\delta} \quad \text{for} \quad i \in I,$$
$$B = \Bigl(\bigcup_{i \in I} A_i\Bigr)^{\delta}.$$

We have

$$A_{i(1)} \cap A_{i(2)} = \{0\}$$

if $i(1)$ and $i(2)$ are distinct elements of I. Thus according to 3.1 we obtain

$$G = B \times \prod_{i \in I} A_i.$$

In [10] it was proved that if an abelian lattice ordered group G^1 is a direct product of lattice ordered groups G_i^1 $(i \in I)$ and if G^2 is a Dedekind completion of G^1, then there are Dedekind completions G_i^2 of G_i^1 $(i \in I)$ such that G^2 is a direct product of G_i^2 $(i \in I)$. It is easy to verify that this result remains valid for the non-abelian case as well.

Hence there exists a direct product decomposition

$$H = B^0 \times \prod_{i \in I} A_i^0$$

such that B^0 is a Dedekind completion of B and A_i^0 is a Dedekind completion of A_i $(i \in I)$.

Since $h_i \in A_i^0$ for $i \in I$, we infer that there exists $h \in H$ such that

$$h(B^0) = 0, \quad h(A_i^0) = h_i \quad \text{for } i \in I.$$

Then the relation $h = \vee_{i \in I} h_i$ is valid in H and therefore H is laterally complete. □

3.3. Lemma. *Let G be strongly projectable and let H be a lateral completion of G, $0 \leqslant h \in H$. Then there exists a disjoint indexed system $(x_i)_{i \in I}$ in G such that the relation $h = \vee_{i \in I} x_i$ is valid in H.*

This was proved in [8].

3.4.1. Lemma. *Let G be strongly projectable and let H be a lateral completion of G. Then H is strongly projectable.*

Proof. Let $\emptyset \neq X \subseteq H$. The polar of X in H will be denoted by $X^{\perp}$. There exists $X_1 \subseteq H^+$ such that

$$X^{\perp} = X_1^{\perp}, \quad X^{\perp\perp} = X_1^{\perp\perp}.$$

In view of 3.3, for each $x_1 \in X_1$ there exists a subset $Y(x_1)$ of G^+ such that the relation

$$x_1 = \sup Y(x_1)$$

is valid in H. Put

$$Y = \bigcup_{x_1 \in X_1} Y(x_1).$$

Then we have

$$Y^{\perp} = X_1^{\perp}$$

and hence $Y^{\perp\perp} = X_1^{\perp\perp}$. Since G is strongly projectable, we obtain

$$G = Y^{\delta\delta} \times Y^{\delta}.$$

It is easy to verify that $Y^{\perp}$ and $Y^{\perp\perp}$ are Dedekind completions of Y^{δ} or of $Y^{\delta\delta}$, respectively. Then according to [9] (cf. the quotation in the proof of 3.2) we get

$$H = Y^{\perp\perp} \times Y^{\perp}.$$

Hence H is strongly projectable. □

3.4.2. Lemma. *Let G be strongly projectable and let H be a Dedekind completion of G. Then H is strongly projectable.*

The proof is as in 3.4.1 with the following distinction: the existence of $Y(x_1)$ with the desired properties is a consequence of the definition of the Dedekind completion (i.e., we need not apply 3.3).

3.5. Lemma. *Suppose that G is strongly projectable and D-complete. Let H be a lateral completion of G. Assume that $0 < h \in H$, $b \in G$, $h \leqslant b$. Then $h \in G$.*

Proof. We have $0 \leqslant -h + b$. Since G is strongly projectable, according to 3.3 there are disjoint indexed systems $(g_i^1)_{i\in I}$ and $(g_j^2)_{j\in J}$ in G such that the relations

$$h = \bigvee_{i\in I} g_i^1, \tag{1}$$

$$-h + b = \bigvee_{j\in J} g_j^2 \tag{2}$$

are valid in H. From (2) we infer that the following relations hold in H:

$$-h = \bigvee_{j\in J} (g_j^2 - b),$$

$$h = \bigwedge_{j\in J} g_j^3, \tag{3}$$

where $g_j^3 = b - g_j^2$. Hence $g_j^3 \in G$ for each $j \in J$. Next, (1) and (3) yield by simple calculation that the relations

$$\bigwedge_{i\in I, j\in J} (g_j^3 - g_i^1) = 0 = \bigwedge_{i\in I, j\in J} (-g_i^1 + g_j^3) \tag{4}$$

are valid in H. Hence these relations hold in G as well.

Denote

$$B_1 = U(\{g_i^1\}_{i\in I}), \quad A_1 = L(B_1),$$

where the symbols U and L are taken with respect to G. Then (A_1, B_1) is a cut in G. Clearly

$$(5.1) \qquad \{g_i^1\}_{i\in I} \subseteq A_1,$$

$$(5.2) \qquad \{g_j^3\}_{j\in J} \subseteq B_1.$$

From (4) we obtain that the relations

$$\bigwedge_{a_1\in A_1, b_1\in B_1} (b_1 - a_1) = 0 = \bigwedge_{a_1\in A_1, b_1\in B_1} (-a_1 + b_1)$$

hold in G. Hence (A_1, B_1) is a D-cut in G. Now we apply the assumption that G is D-complete. Thus there is $g^0 \in G$ such that

$$(6) \qquad \sup A_1 = g^0 = \inf B_1$$

is valid in G. Since G is dense in H (this is a consequence of 3.3), the relations (6) hold also in H. Then from (5.1) we get $h \leqslant g^0$ and from (5.2) we obtain $h \geqslant g^0$. Therefore $h = g^0$, which completes the proof. □

3.6. Lemma. *Let G be strongly projectable and D-complete. Suppose that H is a lateral completion of G. Then H is D-complete.*

Proof. Let H_1 be a Dedekind completion of H. We have to show that $H_1 = H$. It suffices to verify that $H_1^+ \subseteq H$.

Let $0 \leqslant h_1 \in H_1$. There exist subsets A_1 and B_1 of H such that the relations

$$(1) \qquad \sup A_1 = h_1 = \inf B_1$$

are valid in H_1. Choose $b_1 \in B_1$. There exists a disjoint indexed system $(b_i)_{i\in I}$ of elements of G such that the relation

$$(2) \qquad b_! = \vee_{i\in I} b_i$$

holds in H.

It follows from the Axiom of Choice that there exists a disjoint indexed system $(b_i)_{i\in I'}$ of elements of G such that $I' \subseteq I$ and, whenever $0 < g \in G$, then $g \wedge b_i > 0$ for some $i \in I'$.

Let the symbol $\perp$ have the same meaning as above (i.e., it is applied for denoting polars in H). For each $i \in I'$ we put

$$C_i = \{b_i\}^{\perp\perp}.$$

Since H is laterally complete it is strongly projectable and hence in view of 3.1 we have

$$H = \prod_{i \in I'} C_i. \tag{3.1}$$

Hence according to [9] (cf. the quotation in the proof of 3.2),

$$H_1 = \prod_{i \in I'} C_i^1, \tag{3.2}$$

where C_i^1 is a Dedekind completion of C_i $(i \in I')$.

From (2) we obtain that if $i \in I$, then

$$b_i(C_i) = b_i, \quad b_{i(1)}(C_i) = 0 \quad \text{for } i(1) \in I' \setminus \{i\}.$$

These relations remain valid if C_i is replaced by C_i^1 $(i \in I)$.

Since $H \subseteq_d H_1$, the relation (2) holds in H_1 as well. Then

$$h_1 = h_1 \wedge b_1 = \vee_{i \in I}(h_1 \wedge b_i) \tag{4}$$

is valid in H_1.

Let $i \in I$ be fixed. From (4) we obtain that

$$h_1(C_i^1) = h_1 \wedge b_i \geqslant 0.$$

Thus $h_1 \wedge b_i \in C_i^1$.

There exist $A_i, B_i \subseteq C_i$ such that the relations

$$\sup A_i = h_1 \wedge b_i = \inf B_i$$

are valid in C_i^1. Denote

$$A_i^* = A_i \cap G, \quad B_i^* = B_i \cap G.$$

Since $b_i \in A$, in view of 3.5 we have $h \in H$ for each $h \in H$ with $0 \leqslant h \leqslant b_i$. This yields that the relations

$$\sup A_i^* = h_1 \wedge b_i = \inf B_i^*$$

hold in C_i^1. Thus

(5) $$\bigwedge_{x\in A_i^*, y\in B_i^*} (y-x) = 0 = \bigwedge_{x\in A_i^*, y\in B_i^*} (-x+y).$$

However, $A_i^*, B_i^* \subseteq G$ and thus, since G is D-complete, we infer from (5) in the obvious way that there exists $z \in G$ such that

(6) $$\sup A_i^* = z = \inf B_i^*$$

is valid in G.

Since $G \subseteq_d H$ (cf. 3.3) we get $G \subseteq_d H_1$ and thus (6) holds also in H_1. We obtain $z = h_1 \wedge b_i$. Therefore $h_1 \wedge b_i \in G$ for each $i \in I$.

The indexed system $(h_1 \wedge b_i)_{i\in I}$ of elements of G is disjoint, hence there exist $h_0 \in H$ such that

(7) $$h_0 = \vee_{i\in I}(h \wedge b_i)$$

is valid in H. Since $H \subseteq_d H_1$, the relation (7) holds also in H_1. Then (4) yields that $h_1 = h_0 \in H$, which completes the proof. □

4. Isomorphisms of G^{DL} and G^{LD}

Let G be a lattice ordered group. We denote by
H—a lateral completion of G;
H_1—a Dedekind completion of H;
K—a Dedekind completion of G;
K_1—a lateral completion of K.

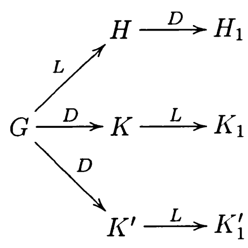

Figure 1

4.1. Theorem. *Let G be a strongly projectable lattice ordered group and let H, H_1, K, K_1 be as above. Then there exists an isomorphism φ of K_1 onto H_1 such that $\varphi(g) = g$ for each $g \in G$.*

Proof. (Cf. Fig. 1.) Since $G \subseteq_d H$ and $H \subseteq_d H_1$, in view of 2.7 there exists a Dedekind completion K' of G such that $K' \subseteq_d H_1$. In view of 3.4.2, K' is strongly projectable.

According to 3.2 and 3.4.1, H_1 is laterally complete. Then 2.1 yields that there is a lateral completion K_1' of K' such that $K_1' \subseteq_d H_1$.

Since $G \subseteq_d K' \subseteq_d H_1$ we get $G \subseteq_d K' \subseteq_d K_1'$. This and the lateral completeness of K_1' yield (cf. 2.1) that $H \subseteq_d K_1'$.

By applying the definition of K_1' and 3.6 we obtain that $H_1 \subseteq K_1'$. Therefore we have

$$H_1 = K_1'. \tag{1}$$

There exists an isomorphism φ_1 of K onto K' such that $\varphi(g) = g$ for each $g \in G$. Next, there exists an isomorphism φ of K_1 onto K_1' such that $\varphi(x) = x$ for each $x \in K$. In particular, $\varphi(g) = g$ for each $g \in G$. To complete the proof it suffices to apply the relation (1). □

References

[1] *S. J. Bernau*: The lateral completion of an arbitrary lattice group. J. Austral. Math. Soc. *19* (1975), 263–289.

[2] *S. J. Bernau*: Lateral and Dedekind completion of archimedean lattice groups. J. London Math. Soc. *12* (1976), 320–322.

[3] *D. Byrd and T. J. Lloyd*: A note on lateral completion in lattice ordered groups. J. London Math. Soc. *1* (1969), 358–362.

[4] *P. F. Conrad*: Lateral completion of lattice ordered groups. Proc. London Math. Soc. *19* (1969), 444–480.

[5] *P. F. Conrad*: The essential closure of an archimedean lattice group. Duke Math. J. *38* (1971), 151–160.

[6] *C. J. Everett*: Sequence completion of lattice moduls. Duke Math. J. *11* (1944), 109–119.

[7] *L. Fuchs*: Partially ordered algebraic systems. Pergamon Press, Oxford-London-New York-Paris, 1963.

[8] *J. Jakubík*: Representations and extensions of ℓ-groups. Czechoslovak Math. J. *13* (1963), 267–283. (In Russian.)

[9] *J. Jakubík*: Orthogonal hull of a strongly projectable lattice ordered group. Czechoslovak Math. J. *28* (1978), 484–504.

[10] *J. Jakubík*: Maximal Dedekind completion of an abelian lattice ordered group. Czechoslovak Math. J. *28* (1978), 611–631.

[11] *H. Nakano*: Modern spectral theory. Tokyo, 1950.

[12] *A. G. Pinsker*: Extended semiordered groups and spaces. Uchen. zapiski Leningrad. Gos. Ped. Inst. *86* (1949), 236–365. (In Russian.)

Author's address: Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia.