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RETRACT VARIETIES OF MONOUNARY ALGEBRAS

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In [1] the notion of order variety was defined as follows: an order variety is a class $\mathscr{K}$ of ordered sets which contains all retracts of members of $\mathscr{K}$ and all direct products of nonempty families of members of $\mathscr{K}$.

Analogously to [1], a class $\mathscr{K}$ of monounary algebras will be said to be a retract variety if it is closed with respect to isomorphisms and if it contains all retracts of members of $\mathscr{K}$ and all direct products of nonempty families of members of $\mathscr{K}$.

Retracts of monounary algebras were studied in [2] and [3].

We denote by $\mathfrak{R}$ the collection of all retract varieties of monounary algebras. This collection is considered to be partially ordered by the class-theoretical inclusion.

The aim of the present paper is to investigate the properties of the partially ordered collection $\mathfrak{R}$. The main results are Theorems 2.5′, 2.11′ and 3.10.

1. Retract variety generated by $\mathscr{K}$

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a retract of (A, f) if there is a mapping h of A onto M such that h is an endomorphism of (A, f) and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M.

The symbol $\mathscr{U}$ will denote the class of all monounary algebras. It is obvious that $\emptyset$ and $\mathscr{U}$ are the least and the greatest element of $\mathfrak{R}$, respectively.

A class $\mathscr{C}$ of monounary algebras is said to be retract (product) closed if it is closed with respect to isomorphisms and if it contains all retracts (direct products) of members of $\mathscr{C}$. Let $\mathscr{K}$ be a class of monounary algebras. We denote by $R(\mathscr{K})$ ($P(\mathscr{K})$) the class of monounary algebras whose elements are only all retracts (direct products) of members of $\mathscr{K}$ and their isomorphic images. It is easy to see that $R(\mathscr{K})$ ($P(\mathscr{K})$) is retract (product) closed.

Further, $\cong$ means an isomorphism between algebraic structures.

1.1. Lemma. *Let $\mathscr{K} \subseteq \mathscr{U}$. Then*

(i) $R^2(\mathscr{K}) = R(\mathscr{K})$,

(ii) $P^2(\mathscr{K}) = P(\mathscr{K})$,

(iii) $PR(\mathscr{K}) \subseteq RP(\mathscr{K})$.

Proof. The properties (i) and (ii) are obvious. Assume that $(A, f) \in PR(\mathscr{K})$. Then there are $I \neq \emptyset$ and $(A_i, f) \in R(\mathscr{K})$ for each $i \in I$ such that $(A, f) \cong \prod_{i \in I} (A_i, f)$. Thus, if $i \in I$, then there are $(B_i, f) \in \mathscr{K}$ and a retraction g_i of (B_i, f) onto (A_i, f). Define a mapping $g\colon \prod_{i \in I} B_i \to \prod_{i \in I} A_i$ by putting, whenever $b \in \prod_{i \in I} B_i$,

$$(g(b))(i) = g_i(b(i)) \text{ for each } i \in I.$$

Obviously, g is a homomorphism. Further, if $a \in \prod_{i \in I} A_i$, then

$$(g(a))(i) = g_i(a(i)) = a(i)$$

by the properties of g_i, i.e.,

$$g(a) = a.$$

Therefore g is a retraction of $\prod_{i \in I} (B_i, f)$ onto $\prod_{i \in I} (A_i, f)$ and hence $(A, f) \cong \prod_{i \in I} (A_i, f) \in RP(\mathscr{K})$. $\square$

A class $\mathscr{C}$ of monounary algebras is said to be a retract variety if it is retract closed and product closed. Let $\mathscr{K}$ be a class of monounary algebras. We denote by $V(\mathscr{K})$ the class of all monounary algebras such that any of them is a member of every retract variety $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{K}$. It is easy to see that $V(\mathscr{K})$ is a retract variety.

1.2. Definition. Under the above notation, if $\mathscr{K} \subseteq \mathscr{U}$, then $V(\mathscr{K})$ will be called a retract variety generated by $\mathscr{K}$.

1.3. Proposition. *If $\mathscr{K} \subseteq \mathscr{U}$, then $V(\mathscr{K}) = RP(\mathscr{K})$.*

Proof. According to 1.1(i) we have

$$R(RP(\mathscr{K})) = R^2(P(\mathscr{K})) = RP(\mathscr{K}). \tag{1}$$

Further, 1.1(iii) and (ii) yield

$$P(RP(\mathscr{K})) = PR(P(\mathscr{K})) \subseteq RP(P(\mathscr{K})) = R(P^2(\mathscr{K})) = RP(\mathscr{K}). \tag{2}$$

Thus $RP(\mathscr{K})$ is a retract variety by (1) and (2). Suppose that $\mathscr{V} \subseteq \mathscr{U}$ is a retract variety such that $\mathscr{K} \subseteq \mathscr{V}$. Then

$$RP(\mathscr{K}) \subseteq RP(\mathscr{V}) = \mathscr{V}.$$

$\square$

1.4. Notation. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ the set of all integers. For $n \in \mathbb{N}$ let $\mathbb{Z}_n$ be the set of all integers modulo n and consider the following monounary algebras:

$\underline{\mathbb{Z}} = (\mathbb{Z}, f)$, where $f(i) = i + 1$ for each $i \in \mathbb{Z}$;
$\underline{\mathbb{N}} = (\mathbb{N}, f)$, where $f(i) = i + 1$ for each $i \in \mathbb{N}$;
$\underline{n} = (\mathbb{Z}_n, f)$, where $f(i) \equiv i + 1 \pmod{n}$ for each $i \in \mathbb{Z}_n$.

1.5. Notation. Let $\mathscr{K} = \{\mathscr{A}_i = (A_i, f)\colon i \in I\} \subseteq \mathscr{U}$. The symbol

$$\sum \mathscr{K} = \sum_{i \in I} \mathscr{A}_i$$

will denote the disjoint sum of the algebras $\mathscr{A}_i$.

Let us remark that by constructing retract varieties each monounary algebra $\mathscr{A}_i$ can be replaced by a monounary algebra $\mathscr{B}_i$ with $\mathscr{B}_i \cong \mathscr{A}_i$.

Next, by applying this convention, we denote (for any $\mathscr{A} \in \mathscr{U}$ and any cardinal $\varkappa$) by the symbol $\varkappa \cdot \mathscr{A}$ the monounary algebra $\sum\limits_{i \in I} \mathscr{A}_i$, where card $I = \varkappa$ and $\mathscr{A} \cong \mathscr{A}_i$ for each $i \in I$.

1.6. Lemma. (i) $V(\underline{1}) = \underline{1}$.
(ii) *If* $n \in \mathbb{N} - \{1\}$, *then* $V(\underline{n}) = \{\varkappa \cdot \underline{n}\colon \varkappa \in \mathrm{Card}, \varkappa \neq 0\}$.

Proof. The assertion (i) is obvious. Let $n \in \mathbb{N} - \{1\}$. In view of 1.3, $V(\underline{n}) = RP(\underline{n})$. Consider $\underline{n}^\lambda$, where $\lambda \in \mathrm{Card} - \{0\}$. If $x \in \underline{n}^\lambda$, then $x(i)$ will be the natural i-th projection of x; we obtain

$$(f^n(x))(i) = f^n(x(i)) = x(i),$$

i.e.,

$$f^n(x) = x.$$

Therefore each element of $\underline{n}^\lambda$ belongs to some n-element cycle. Thus if λ is finite then $\underline{n}^\lambda$ consists of $\frac{1}{n} \cdot n^\lambda$ cycles. If λ is an infinite cardinal, then $\underline{n}^\lambda$ consists of 2^λ cycles. Hence for each $\delta \in \mathrm{Card}$ there is $\lambda \in \mathrm{Card}$ such that $\underline{n}^\lambda$ consists of at least δ n-element cycles. By retraction we can get an arbitrary non-zero number of n-element cycles, thus (ii) is valid. $\square$

1.7. Corollary. *For each $n \in \mathbb{N}$ there exists a monounary algebra $\mathscr{B}_n$ such that, whenever $n, m \in \mathbb{N}$, $n \neq m$, then $V(\mathscr{B}_n) \nsubseteq V(\mathscr{B}_m)$.*

Proof. Take a system $\{V(\underline{n})\colon n \in \mathbb{N}\}$. From 1.6 it follows that $V(\underline{n}) \nsubseteq V(\underline{m})$ for any $n, m \in \mathbb{N}$, $n \neq m$. □

1.8. Corollary. *For each $n \in \mathbb{N}$ there exists a monounary algebra $\mathscr{A}_n$ such that, whenever $n, m \in \mathbb{N}$, $n < m$, then $V(\mathscr{A}_n) \subsetneqq V(\mathscr{A}_m)$.*

Proof. Let $\mathscr{A}_n = \underline{2} + \underline{4} + \underline{8} + \ldots + \underline{2^n}$. Since $\mathscr{A}_n \in R(\mathscr{A}_{n+1})$ for each $n \in \mathbb{N}$, we obtain

$$V(\mathscr{A}_n) \subseteq V(\mathscr{A}_{n+1}). \tag{1}$$

Further, $\mathscr{A}_{n+1} \notin V(\mathscr{A}_n)$, thus

$$V(\mathscr{A}_n) \neq V(\mathscr{A}_{n+1}). \tag{2}$$

□

In Section 2 stronger results than 1.7 and 1.8 will be proved.

1.9. Lemma. *Let $\mathscr{V} = V\big(\alpha \cdot \underline{\mathbb{Z}} + \beta \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa_n \cdot \underline{n}\big)$, where $\{\alpha, \beta\} \cup \{\varkappa_n\colon n \in \mathbb{N}\} \subset$ Card. Then there are $\{\alpha', \beta'\} \cup \{\varkappa'_n\colon n \in \mathbb{N} - \{1\}\} \subseteq \{0, 1\}$ and $\varkappa'_1 \in \{0, 1, 2\}$ such that $\mathscr{V} = V\big(\alpha' \cdot \underline{\mathbb{Z}} + \beta' \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa'_n \cdot \underline{n}\big)$.*

Proof. Put

$$\varkappa'_1 = \begin{cases} \varkappa_1 & \text{if } \varkappa_1 \in \{0, 1\}, \\ 2 & \text{otherwise}. \end{cases}$$

If γ is some of the symbols $\alpha, \beta, \varkappa_n (n \in \mathbb{N} - \{1\})$, then we denote

$$\gamma' = \begin{cases} 0 & \text{if } \gamma = 0, \\ 1 & \text{otherwise}. \end{cases}$$

Further let

$$\mathscr{A} = (A, f) = \alpha \cdot \underline{\mathbb{Z}} + \beta \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa_n \cdot \underline{n},$$
$$\mathscr{A}' = (A', f) = \alpha' \cdot \underline{\mathbb{Z}} + \beta' \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa'_n \cdot \underline{n}.$$

These definitions imply that $\mathscr{A}' \in R(\mathscr{A})$, thus

$$V(\mathscr{A}') \subseteq V(\mathscr{A}). \tag{1}$$

Further, there exists a set I with card $I \geqslant \gamma$ for each $\gamma \in \{\alpha, \beta\} \cup \{\varkappa_n : n \in \mathbb{N}\}$. Put $\iota =$ card I and

$$\mathscr{D} = (D, f) = (\mathscr{A}')^\iota.$$

If (B, f) is a connected component of (D, f), then (B, f) is a connected component of a product $\prod_{i \in I} (B_i, f)$, where (B_i, f) is a connected component of $\mathscr{A}'$; for (B_i, f) there are the following possibilities:

$$\begin{aligned}
&(B_i, f) = \underline{\mathbb{Z}} \quad (\text{if } \alpha' = 1),\\
&(B_i, f) = \underline{\mathbb{N}} \quad (\text{if } \beta' = 1),\\
&(B_i, f) = \underline{n} \quad \text{for some } n \in \mathbb{N} \quad (\text{if } \varkappa'_n = 1).
\end{aligned}$$

Thus (B, f) satisfies one of the following conditions:

(2.1) $(B, f) \cong \underline{\mathbb{Z}}$,

(2.2) $(B, f) \cong \underline{\mathbb{N}}$,

(2.3) $(B, f) \cong \underline{d}$, where $d =$ l.c.m.$(n_1, \dots, n_k), k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{N}$ and $\varkappa'_{n_1} = \dots = \varkappa'_{n_k} = 1$.

Let $\alpha \neq 0$, i.e., $\alpha' \neq 0$ and consider $(B_i, f) = \underline{\mathbb{Z}}$ for each $i \in I$. Then $\prod_{i \in I} (B_i, f)$ consists of connected components isomorphic to $\underline{\mathbb{Z}}$; since $\iota \geqslant \alpha$, we obtain

(3.1) there are at least α connected components (B, f) with the property 2.1.

Analogously, if $\beta \neq 0$, i.e., $\beta' \neq 0$, then

(3.2) there are at least β connected components (B, f) with the property 2.2,

and if $n \in \mathbb{N} - \{1\}$ with $\varkappa_n \neq 0$, i.e., $\varkappa'_n \neq 0$, then

(3.3) there are at least $\varkappa_n$ connected components (B, f) isomorphic to $\underline{n}$.

Further, if $\varkappa_1 = 1$, i.e., $\varkappa'_1 = 1$, then (D, f) contains only one connected component isomorphic to $\underline{1}$. If $\varkappa_1 > 1$, i.e., $\varkappa'_1 = 2$, then there are at least 2^ι connected components isomorphic to $\underline{1}$, thus $\iota \geqslant \varkappa_1$ implies

(3.4) there are at least $\varkappa_1$ connected components (B, f) isomorphic to $\underline{1}$.

From (3.1)–(3.4) we obtain

(4) (A, f) is isomorphic to a subalgebra (E, f) of (D, f).

Then $(E, f) \in R(D, f)$ in view of [2], 1.3 and (2.1)–(2.3). Hence $(A, f) \in R(D, f)$, thus

(5) $$\mathscr{A} \in R(\mathscr{D}) \subseteq RP(\mathscr{A}') = V(\mathscr{A}').$$

Therefore (1) and (5) imply

$$V(\mathscr{A}) = V(\mathscr{A}').$$

□

1.10. Lemma. *Let $\emptyset \neq I \subseteq \mathbb{N}$ and suppose that i does not divide j for each $i, j \in I$, $i \neq j$. If $\mathscr{D} \in V\big(\sum_{i\in I} \underline{i}\big)$ and $k \in I$, then there is a connected component $\mathscr{B}$ of $\mathscr{D}$ such that $\mathscr{B} \cong \underline{k}$.*

Proof. Let the assumption be valid, $\mathscr{D} \in V\big(\sum_{i\in I} \underline{i}\big)$ and $k \in I$. We have $\mathscr{D} \in RP\big(\sum_{i\in I} \underline{i}\big)$, thus $\mathscr{D} \in R(\mathscr{A})$ and $\mathscr{A} = \big(\sum_{i\in I} \underline{i}\big)^{\lambda}$ for some cardinal $\lambda \neq 0$. Let (C, f) be a connected component of $\mathscr{A}$. Then either

(1.1) $$(C, f) \cong \underline{\mathbb{Z}}$$

or

(1.2) $$(C, f) \cong \underline{d}, \text{ where } d = \text{l.c.m.}(l_1, \dots, l_m), \{l_1, \dots, l_m\} \subseteq I.$$

Further, there exists a connected component (B, f) of $\mathscr{A}$ such that $(B, f) \cong \underline{k}$. We have $\mathscr{D} \in R(\mathscr{A})$, i.e., there is $(D', f) = \mathscr{D}' \cong \mathscr{D}$ such that $\mathscr{D}'$ is a subalgebra of $\mathscr{A}$ and D' is a retract of $\mathscr{A}$. Suppose that no connected component of $\mathscr{D}$ is isomorphic to $\underline{k}$. Then $B \subseteq A - D'$. Since D' is a retract of $\mathscr{A}$, $(B, f) \cong \underline{k}$, we obtain according to [2], 1.3 that there exists a connected component of (D', f) isomorphic to some $\underline{d_1}$ such that d_1 divides k. In view of (1.2)

$$d_1 = \text{l.c.m.}(l_1, \dots, l_m),$$

thus l_1 divides $k, \dots, l_m$ divides k. Then the assumption of the lemma yields $l_1 = \dots = l_m = k$ and $d_1 = k$, which is a contradiction. □

If P is a poset, $x, y \in P$, then the symbol $x \parallel y$ will mean that x and y are incomparable.

The aim of this section is to describe monounary algebras $\mathscr{A}_\alpha$ and $\mathscr{B}_\alpha$ for each $\alpha \in \mathrm{Ord}$ such that if $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$, then

(i) $V(\mathscr{A}_\alpha) \parallel V(\mathscr{A}_\beta)$,

(ii) $V(\mathscr{B}_\alpha) \subsetneqq V(\mathscr{B}_\beta)$.

In this part we will use the notion of the degree of an element $x \in A$, where $(A, f) \in \mathscr{U}$; for this notion cf. e.g. [4], [2]. The degree of $x \in A$ is an ordinal or the symbol ∞; it is denoted by $s_f(x)$. The following two assertions are consequences of the definition of $s_f(x)$.

2.1. Lemma. *Let $\{(D_i, f) : i \in I\} \subseteq \mathscr{U}$, $d \in \prod_{i \in I} D_i$. Then*

(i) *$s_f(d) \leqslant s_f(d(i))$ for each $i \in I$,*

(ii) *if $\gamma \in \mathrm{Ord}$, $s_f(d(i)) \in \{\gamma, \infty\}$ for each $i \in I$ and $s_f(d(j)) = \gamma$ for some $j \in I$, then $s_f(d) = \gamma$.*

2.2 Lemma. *For each $\alpha \in \mathrm{Ord}$ there exists a connected monounary algebra $\mathscr{A}_\alpha = (A_\alpha, f)$ and distinct elements $c_\alpha, a_\alpha \in A_\alpha$ with the following properties:*

(a) *$f(a_\alpha) = c_\alpha = f(c_\alpha)$,*

(b) *$s_f(a_\alpha) = \alpha$,*

(c) *if $x \in A_\alpha - \{c_\alpha\}$, then $f^n(x) = a_\alpha$ for some $n \in \mathbb{N} \cup \{0\}$.*

2.3. Lemma. *Let $\alpha \in \mathrm{Ord}$ and $(D, f) \in P(\mathscr{A}_\alpha)$. Then*
(i) *if $x \in D$, $f^2(x) = f(x) \neq x$, then $s_f(x) = \alpha$.*

P r o o f. If $(D, f) \in P(\mathscr{A}_\alpha)$, then there is $I \neq \emptyset$ such that $(D, f) = \mathscr{A}_\alpha^{\operatorname{card} I}$. Let $i \in I$. Then

$$(f^2(x))(i) = (f(x))(i), \text{ i.e., } f^2(x(i)) = f(x(i)),$$

$f(x(i)) \in \mathscr{A}_\alpha$ is an element of a one–element cycle, hence $f(x(i)) = c_\alpha$. From 2.2(a) and (c) we obtain

$$x(i) \in \{c_\alpha, a_\alpha\}. \tag{1}$$

Suppose

$$x(i) = c_\alpha \text{ for each } i \in \mathbb{N}. \tag{2}$$

Then

$$(f(x))(i) = f(x(i)) = f(c_\alpha) = c_\alpha = x(i),$$
$$f(x) = x,$$

a contradiction. Therefore

(3) $$\text{there is } j \in I \text{ with } x(j) = a_\alpha.$$

According to 2.1(ii) we have $s_f(x) = \alpha$. □

2.4. Lemma. *If* $\alpha \in \mathrm{Ord}$ *and* $(D, f) \in V(A_\alpha)$, *then* (i) *of* 2.3 *is valid.*

Proof. Let $\alpha \in \mathrm{Ord}$, $(D, f) \in R(E, f)$, $(E, f) \in P(\mathscr{A}_\alpha)$. By 2.3,

(1) $$\text{if } e \in E,\ f^2(e) = f(e) \neq e, \ \text{ then } \ s_f(e) = \alpha.$$

Assume that $x \in D$, $f^2(x) = f(x) \neq x$. We can suppose that (D, f) is a subalgebra of (E, f); instead of (D, f) we will write now (D, g). Since (D, g) is a subalgebra of (E, f), we have $s_g(t) \leqslant s_f(t)$ for each $t \in D$. By (1), $s_f(x) = \alpha \geqslant s_g(x)$. We want to show that $s_g(x) = \alpha$. Let us prove the assertion

(2) $$\text{if } t \in D, \text{ then } s_f(t) = s_g(t)$$

by induction with respect to $s_f(t)$.

(a) If $s_f(t) = 0$, then $s_g(t) \leqslant s_f(t) = 0$.

(b) Let $t \in D, s_f(t) = \beta, s_g(t) = \gamma$. According to [2], 1.3, for each $y \in f^{-1}(t)$ there exists $z \in f^{-1}(t) \cap D$ such that $s_f(y) \leqslant s_f(z)$, hence the induction hypothesis implies $s_f(y) \leqslant s_f(z) = s_g(z) < \gamma$. Therefore

(3) $$\text{if } y \in f^{-1}(t), \text{ then } s_f(y) < \gamma$$

and (3) yields

$$\beta = s_f(t) \leqslant \gamma = s_g(t) \leqslant s_f(t) = \beta.$$

By (2), $s_g(x) = \alpha$ and (i) holds. □

2.5. Proposition. $V(\mathscr{A}_\alpha) \parallel V(\mathscr{A}_\beta)$ *for each* $\alpha, \beta \in \mathrm{Ord}$, $\alpha \neq \beta$.

Proof. Let $\alpha, \beta \in \mathrm{Ord}$, $\alpha \neq \beta$. Then $\mathscr{A}_\beta \in V(\mathscr{A}_\beta)$ and

(1) $$f^2(a_\beta) = f(a_\beta) = a_\beta,$$

hence 2.4 (for β instead of α) and (1) yield $s_f(a_\beta) = \beta$. Then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$, because in the opposite case $s_f(a_\beta) = \alpha$ in view of 2.4. Hence

$$V(\mathscr{A}_\beta) \nsubseteq V(\mathscr{A}_\alpha). \tag{2}$$

Analogously,

$$V(\mathscr{A}_\alpha) \nsubseteq V(\mathscr{A}_\beta), \tag{3}$$

thus $V(\mathscr{A}_\alpha) \parallel V(\mathscr{A}_\beta)$. □

2.5′. Theorem. *For each $\alpha \in \mathrm{Ord}$ there exists a monounary algebra $\mathscr{A}_\alpha$ such that, whenever $\alpha, \beta \in \mathrm{Ord}$, $\alpha \neq \beta$, then $V(\mathscr{A}_\alpha) \parallel V(\mathscr{A}_\beta)$.*

2.6. Notation. *If $\alpha \in \mathrm{Ord}$, then put*

$$\mathscr{B}_\alpha = \sum_{\beta \in \mathrm{Ord},\ \beta \leqslant \alpha} \mathscr{A}_\beta.$$

2.7. Lemma. *If $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$, then $V(\mathscr{B}_\alpha) \subseteq V(\mathscr{B}_\beta)$.*

Proof. Let $\alpha < \beta$. Then $0 \leqslant \alpha$ and $A_0 \subseteq B_\alpha$, $A_0 = \{c_0, a_0\}$. Consider the mapping $h\colon B_\beta \to B_\alpha$ defined as follows:

$$h(x) = \begin{cases} x & \text{if } x \in B_\alpha, \\ c_0 & \text{otherwise}. \end{cases}$$

Obviously, h is the retraction endomorphism of $\mathscr{B}_\beta$ onto $\mathscr{B}_\alpha$, thus $\mathscr{B}_\alpha \in R(\mathscr{B}_\beta)$, which implies $V(\mathscr{B}_\alpha) \subseteq V(\mathscr{B}_\beta)$. □

2.8. Lemma. *Let $\alpha \in \mathrm{Ord}$ and $(D, f) \in P(\mathscr{B}_\alpha)$. Then*
(i) *if $x \in D, f(x) \neq x$, then $s_f(x) \leqslant \alpha$.*

Proof. Suppose that $\alpha \in \mathrm{Ord}$ and that $(D, f) = \mathscr{B}_\alpha^{\operatorname{card} I}$ for some nonempty set I. Let $x \in D, f(x) \neq x$. Then there is $i \in I$ such that $x(i) \notin \{c_\beta\colon \beta \in \mathrm{Ord}, \beta \leqslant \alpha\}$. According to the definition of $s_f(x)$ we get

$$s_f(x(i)) \leqslant s_f(f^k(x(i))) \text{ for each } k \in \mathbb{N} \cup \{0\}, \tag{1}$$

thus $s_f(x(i)) \leqslant \alpha$ by 2.2. Then 2.1(i) implies

$$s_f(x) \leqslant \alpha.$$

□

2.9. Lemma. *If $\alpha \in \mathrm{Ord}$ and $(D, f) \in V(\mathscr{B}_\alpha)$, then* (i) *of* 2.8 *is valid.*

Proof. Let the assumption hold. By 1.3, $(D, f) \in RP(\mathscr{B}_\alpha)$. The assertion is a consequence of 2.8 and of the definition of a retract. □

2.10. Lemma. *If $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$, then $V(\mathscr{B}_\alpha) \neq V(\mathscr{B}_\beta)$.*

Proof. Let $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$. According to 2.9 we have

$$\{s_f(x) \colon x \in (D, f) \in V(\mathscr{B}_\alpha)\} \subseteq \{\gamma \in \mathrm{Ord} : \gamma \leqslant \alpha\} \cup \{\infty\}. \tag{1}$$

Since $\mathscr{B}_\beta \in V(\mathscr{B}_\beta)$ and since there is $y \in B_\beta$ with $s_f(y) = \beta$, we obtain with respect to (1) that $V(\mathscr{B}_\alpha) \neq V(\mathscr{B}_\beta)$. □

2.11. Proposition. *If $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$, then $V(\mathscr{B}_\alpha) \subsetneq V(\mathscr{B}_\beta)$.*

Proof. Immediately from 2.7 and 2.10. □

2.11′. Theorem. *For each $\alpha \in \mathrm{Ord}$ there exists a monounary algebra $\mathscr{B}_\alpha$ such that, whenever $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$, then $V(\mathscr{B}_\alpha) \subsetneq V(\mathscr{B}_\beta)$.*

3. Atomic retract varieties

Retract variety $\mathscr{V}$ will be called atomic if $\mathscr{V} \neq \emptyset$ and, whenever $\mathscr{V}'$ is a retract variety with $\emptyset \neq \mathscr{V}' \subseteq \mathscr{V}$, then $\mathscr{V}' = \mathscr{V}$.

It is obvious that atomic retract varieties must be of the form $V(\{\mathscr{A}\})$, where $\mathscr{A} = (A, f) \in \mathscr{U}$; we will write $V(A, f) = V(\mathscr{A})$ instead of $V(\{\mathscr{A}\})$.

In the following lemmas 3.1–3.4 suppose that $\mathscr{A} = (A, f) \in \mathscr{U}$.

3.1. Lemma. *Assume that there is $n \in \mathbb{N}$ and a connected component (K, f) of (A, f) with an n-element cycle and such that* card *$K > n$. Then $V(\mathscr{A})$ is not atomic.*

Proof. Let B be the set-theoretical union of all cycles of $\mathscr{A}$. According to the definition of a retract, $(B, f) \in R(A, f)$, therefore

$$\emptyset \neq V(B, f) \subseteq V(A, f).$$

Let $(D, f) \in V(B, f)$. In view of 1.3, $(D, f) \in R(E, f)$, where $(E, f) \in (B, f)^{\operatorname{card} I}$ for some $I \neq \emptyset$. We have

$$\text{if } e \in E, \text{ then } \operatorname{card} f^{-1}(e) = 1. \tag{1}$$

Since $(D, f) \in R(E, f)$, (1) implies

$$\text{(2)} \qquad \text{if } d \in D, \text{ then card } f^{-1}(d) = 1.$$

From the assumption we obtain that there is $a \in K \subseteq A$ such that card $f^{-1}(a) \geqslant 2$, hence we get (in view of (2))

$$\text{(3)} \qquad (A, f) \notin V(B, f).$$

Therefore

$$\emptyset \subset V(B, f) \subset V(A, f)$$

and $V(A, f)$ is not atomic. □

3.2. Lemma. *Assume that there is a connected component (K, f) of (A, f) such that*

(a) $\underline{\mathbb{Z}} \cong (C, f)$ *for some subalgebra* (C, f) *of* (K, f),
(b) $C \neq K$.

Then $V(A, f)$ is not atomic.

Proof. By way of contradiction, assume that $V(A, f)$ is atomic. Then in view of 3.1, each connected component of (A, f) containing a cycle consists of the cycle. Let $B = C \cup B_1$, where B_1 is the set of all cyclic elements of A. Obviously, $(B, f) \in R(A, f)$, thus

$$\emptyset \neq V(B, f) \subseteq V(A, f).$$

By the same method as in the proof of 3.1 we obtain

$$(A, f) \notin V(B, f),$$
$$\emptyset \subset V(B, f) \subset V(A, f),$$

which is a contradiction. □

3.3. Lemma. *If there is $x \in A$ with* card $f^{-1}(x) \geqslant 2$, *then $V(A, f)$ is not atomic.*

Proof. Suppose that $V(A, f)$ is atomic. In view of 3.1 and 3.2, we obtain that the following assertion is valid:

(1) if (K, f) is a connected component of (A, f) and (K, f) contains a subalgebra isomorphic to $\underline{\mathbb{Z}}$ or to $\underline{n}$ for some $n \in \mathbb{N}$, then card $f^{-1}(x) = 1$ for each $x \in K$.

Consider the set L consisting of all $a \in A$ such that (i) $f^{-1}(a) = \emptyset$, and (ii) there is b belonging to the same connected component as a with card $f^{-1}(b) \geqslant 2$.

Assume that there is $x_0 \in A$ with card $f^{-1}(x_0) \geqslant 2$. Then, by (1), the connected component (K, f) containing x_0 contains no subalgebra isomorphic to $\underline{\mathbb{Z}}$ or to $\underline{n}$ for some $n \in \mathbb{N}$, thus there is $a_0 \in K$ with $f^{-1}(a_0) = \emptyset$. Further, the fact that a_0 and x_0 belong to the same connected component and the relation card $f^{-1}(x_0) \geqslant 2$ imply that $a_0 \in L$. Therefore $L \neq \emptyset$. If $a \in L$, then

$$\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) \geqslant 2\} \neq \emptyset;$$

put

$$k(a) = \min\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) \geqslant 2\}.$$

Further let

$$\begin{aligned} m &= \min\,\{k(a)\colon a \in L\},\\ J &= \{a \in L\colon k(a) = m\},\\ W &= \{f^m(a)\colon a \in J\}. \end{aligned}$$

For each $v \in W$ such that $f^{-m}(v) \subseteq J$ we choose a fixed element of the set $f^{-m}(v)$ and denote this fixed element by $\bar{v}$. Then we define

$$\begin{aligned} I =&\{a \in J\colon f^{-m}(f^m(a)) \nsubseteq J\} \cup\\ &\cup \{a \in J\colon f^{-m}(f^m(a)) \subseteq J, a \neq \overline{f^m(a)}\},\\ B' =&\{a, f(a), \dots, f^{m-1}(a)\colon a \in I\},\\ B =&A - B'. \end{aligned}$$

Further we will proceed by presenting some lemmas and after proving them, we will return to the proof of 3.3. □

3.3.1. Lemma. *(B, f) is a subalgebra of (A, f).*

Proof. It follows from the definition of B and B'. (It can be shown analogously as in 5.1, [2].) □

3.3.2. Lemma. *$(B, f) \in R(A, f)$.*

Proof. [2], Thm. 1.3. implies that it suffices to prove the following assertion:

(a) If $y \in f^{-1}(B) - B$, then there is $z \in B$ such that $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.

Let $y \in f^{-1}(B) - B$. Then there is $a \in I$ with $y = f^{m-1}(a)$. We get

(1) $$s_f(y) = m - 1.$$

Consider two cases (one of them occurs):

(2.1) $$f^{-m}(f^m(a)) \nsubseteq J;$$

(2.2) $$f^{-m}(f^m(a)) \subseteq J, a \neq \overline{f^m(a)}.$$

Denote $v = f^m(a)$. If (2.2) is valid, then obviously

$$f^{-m}(v) \cap B \neq \emptyset.$$

Let (2.1) hold. If card $f^{-m}(v) = 1$, then $f^{-m}(v) = \{a\}$ and the relation card $f^{-1}(v) \geqslant 2$ yields that there is $a' \in L$ with

$$k(a') < k(a) = m,$$

which is a contradiction. Hence card $f^{-m}(v) > 1$, $a \neq \bar{v} \in B$. We have shown

(3) $$f^{-m}(v) \cap B \neq \emptyset.$$

Take $u \in f^{-m}(v) - \{a\}, f^{m-1}(u) = z$. Then $z \in B$ and

(4) $$s_f(z) \geqslant m - 1.$$

By (1) we obtain that (a) is valid. □

3.3.3. Corollary. $V(B, f) \subseteq V(A, f)$.

3.3.4. Lemma. *If $(D, f) \in V(B, f)$ and $x \in D$, then $k(x) > m$.*

P r o o f. In view of the definition of (B, f),

(1) $$k(x) > m \text{ for each } x \in B.$$

Let $(D, f) \in R(E, f)$, $(E, f) = (B, f)^{\operatorname{card} I}$ for some $I \neq \emptyset$. Take $e \in E$. Then

(2) $$k(e(i)) > m \text{ for each } i \in I,$$

which implies

(3) $$k(e) > m.$$

Since $(D, f) \in R(E, f)$, the definition of a retract and (3) yield that $k(x) > m$ for each $x \in D$. □

3.3.5. Corollary. $(A, f) \notin V(B, f)$.

Let us return to the proof of 3.3. There it was assumed that $V(A, f)$ is atomic. Now the assertion that $V(\mathscr{A})$ is not atomic is a consequence of 3.3.3 and 3.3.5, because

$$\emptyset \subsetneqq V(B, f) \subsetneqq V(A, f).$$

We have got a contradiction, which completes the proof of 3.3. □

3.4. Corollary. *If $\mathscr{V}$ is atomic, then there are cardinals α, β and $\varkappa_n$ for each $n \in \mathbb{N}$ such that*

$$\mathscr{V} = V\left(\alpha \cdot \underline{\mathbb{Z}} + \beta \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa_n \cdot \underline{n}\right).$$

3.5. Corollary. *If $\mathscr{V}$ is atomic, then there are $\{\alpha, \beta\} \cup \{\varkappa_n : n \in \mathbb{N} - \{1\}\} \subseteq \{0, 1\}$ and $\varkappa_1 \in \{0, 1, 2\}$ such that $\{\alpha, \beta\} \cup \{\varkappa_n : n \in \mathbb{N}\} \neq \{0\}$ and*

(i) $\mathscr{V} = V\big(\alpha \cdot \underline{\mathbb{Z}} + \beta \cdot \underline{\mathbb{N}} + \sum_{n \in \mathbb{N}} \varkappa_n \cdot \underline{n}\big)$.

Proof. The assertion follows from 3.4 and 1.9. □

3.6. Lemma. *If $\mathscr{V}$ is atomic, then $\mathscr{V} = V(\mathscr{A})$ for some $\mathscr{A} \in \mathscr{U}$ satisfying one of the following conditions:*

(a) $\mathscr{A} = \sum_{i \in I} \underline{i}$, *where $\emptyset \neq I \subseteq \mathbb{N}$ and i does not divide j for each $i, j \in I, i \neq j$;*

(b) $\mathscr{A} = \underline{\mathbb{Z}}$;

(c) $\mathscr{A} = \underline{\mathbb{N}}$.

Proof. Let $\mathscr{V}$ be atomic. Then $\mathscr{V}$ satisfies (i) of 3.5. First let $\varkappa_1 \neq 0$. Then $\mathscr{B} = \underline{1} \in R(\mathscr{A})$ by [2], 1.3, thus $\emptyset \neq V(\mathscr{B}) \subseteq V(\mathscr{A})$, which implies $V(\mathscr{B}) = V(\mathscr{A})$.

Suppose that $\varkappa_1 = 0$ and that there is $i_0 \in I$ with $\varkappa_{i_0} \neq 0$. Then $\mathscr{B} = \sum_{n \in \mathbb{N}} \varkappa_n$. Now $\underline{n} \in R(\mathscr{A})$ according to [2], 1.3, hence $V(\mathscr{B}) \subseteq V(\mathscr{A})$, and therefore $V(\mathscr{B}) = V(\mathscr{A})$. If $\mathscr{B}$ is not in the form required in (a), then there are nonempty sets J, I and $\mathscr{C} \in \mathscr{U}$ as follows:

$$J = \{j \in \mathbb{N} : \varkappa_j = 1 \ \& \ (\exists n \in \mathbb{N} - \{j\})(\varkappa_n = 1 \ \& \ n \text{ divides } j)\},$$

$$I = \mathbb{N} - \{j \in \mathbb{N} : \varkappa_j = 0\} - J,$$

$$\mathscr{C} = \sum_{i \in I} \varkappa_i \cdot \underline{i} = \sum_{i \in I} \underline{i}.$$

Then $\mathscr{C} \in R(\mathscr{B})$ according to [2], 1.3, thus, $\emptyset \neq V(\mathscr{C}) \subseteq V(\mathscr{B})$. Hence $V(\mathscr{C}) = V(\mathscr{A})$ and i does not divide j for each $i, j \in I$, $i \neq j$.

Now let $\varkappa_n = 0$ for each $n \in \mathbb{N}$. If $\alpha = 1$, then $\underline{\mathbb{Z}} \in R(\mathscr{A})$, thus $V(\underline{\mathbb{Z}}) \subseteq V(\mathscr{A})$ and $V(\underline{\mathbb{Z}}) = V(\mathscr{A})$, i.e., (b) is valid. If $\alpha = 0$, then we have (c). □

3.7. Lemma. *If $\mathscr{A}$ fulfils* (b) *or* (c) *of* 3.6, *then $V(\mathscr{A})$ is atomic.*

Proof. Suppose that $\emptyset \neq \mathscr{W} \subseteq V(\mathscr{A})$ and that $\mathscr{B} \in \mathscr{W}$. Consider the case $\mathscr{A} = \underline{\mathbb{Z}}$; the other case is analogous. Then

$$\mathscr{B} \in V(\mathscr{A}) = \{\lambda \cdot \underline{\mathbb{Z}} \colon \lambda \in \mathrm{Card} - \{0\}\}.$$

This implies that $\mathscr{A} \in R(\mathscr{B}), V(\mathscr{A}) \subseteq V(\mathscr{B}) \subseteq \mathscr{W}$. Hence $V(\mathscr{A})$ is atomic. □

3.8. Lemma. *If $\mathscr{A}$ fulfils (a) of* 3.6, *then $V(\mathscr{A})$ is atomic.*

Proof. Let the assumption be valid and suppose that $\emptyset \neq \mathscr{W} \subseteq V(\mathscr{A}), \mathscr{B} \in \mathscr{W}$. If $1 \in I$, then $\mathscr{A} = \underline{1}$ and $\mathscr{W} = V(\mathscr{A})$. Assume that $1 \notin I$. By 1.10 for each $i \in I$ there is a connected component $\mathscr{C}_i$ of $\mathscr{B}$ such that $\mathscr{C}_i \cong \underline{i}$. Therefore $\mathscr{A} \in R(\mathscr{B})$, which implies $V(\mathscr{A}) \subseteq V(\mathscr{B}) \subseteq \mathscr{W}$. □

3.9. Theorem. *$\mathscr{V}$ is atomic if and only if there is $\mathscr{A} \in \mathscr{U}$ such that $\mathscr{V} = V(\mathscr{A})$ and $\mathscr{A}$ fulfils one of the conditions* (a)–(c) *of* 3.6.

Proof. The assertion is a consequence of 3.6–3.8. □

3.10. Theorem. *There are exactly $2^{\aleph_0}$ atomic retract varieties of monounary algebras.*

Proof. In view of 3.9 the number of atomic retract varieties is less than or equal to $2^{\aleph_0}$. Hence we have to verify that the number of those atomic retract varieties $V(\mathscr{A})$ of $\mathfrak{R}$ for which $\mathscr{A}$ has the form described in the condition (a) of 3.6 is at least $2^{\aleph_0}$.

Let $\mathscr{S}$ be the set of all monounary algebras $\mathscr{A}$ satisfying (a) of 3.6. Then it is clear that card $\mathscr{S} = 2^{\aleph_0}$. Thus we have to show that if $\mathscr{A}$ and $\mathscr{A}'$ are distinct elements of $\mathscr{S}$, then $V(\mathscr{A}) \neq V(\mathscr{A}')$.

To this aim, let us suppose that $\mathscr{A} = \sum\limits_{i \in I} \underline{i}$, $\mathscr{A}' = \sum\limits_{i' \in I'} \underline{i'}$ and that $\emptyset \neq I \subseteq \mathbb{N}$, $\emptyset \neq I' \subseteq \mathbb{N}$, where

$$\text{(1)} \qquad i \text{ does not divide } j \text{ for each } i, j \in I, i \neq j,$$

$$\text{(1')} \qquad i' \text{ does not divide } j' \text{ for each } i', j' \in I', i' \neq j',$$

$$\text{(2)} \qquad \sum_{i \in I} \underline{i} \neq \sum_{i' \in I'} \underline{i'}.$$

In way of contradiction, assume that

$$\text{(3)} \qquad V(\mathscr{A}) = V(\mathscr{A}').$$

By (2), there is $k \in I$ with $k \notin I'$. Let $\mathscr{D} \in V(\mathscr{A})$. In view of 1.10 there exists a connected component $\mathscr{B}$ of $\mathscr{D}$ such that $\mathscr{B} \cong \underline{k}$. Then, since $\mathscr{D} \in V(\mathscr{A}')$, we obtain $\mathscr{D} \in RP(\mathscr{A}')$, $\mathscr{D} \in R((\mathscr{A}')^\lambda)$ for some $0 \neq \lambda \in \mathrm{Card}$, thus

$$k = \mathrm{l.c.m.}(i'_1, \dots, i'_m), \quad \{i'_1, \dots, i'_m\} \subseteq I'.$$

We have

$$i'_1 \text{ divides } k. \tag{4}$$

Further, $i'_1 \in I'$ and $\mathscr{D} \in V(\mathscr{A}')$, hence by applying 1.10 again we get that there is a connected component $\mathscr{B}'$ of $\mathscr{D}$ such that $\mathscr{B}' \cong \underline{i'_1}$. Now the relation $\mathscr{D} \in V(\mathscr{A})$ implies

$$i'_1 = \mathrm{l.c.m.}(i_1, \dots, i_l), \quad \{i_1, \dots, i_l\} \subseteq I.$$

Then

$$i_1 \text{ divides } i'_1, \ \dots, \ i_l \text{ divides } i'_1. \tag{5}$$

By (4), (5) and (1)

$$\begin{gathered} i_1 = k, \ \dots, \ i_l = k, \\ {i'}_1 = k, \end{gathered}$$

hence $k \in I'$, which is a contradiction. Therefore (3) fails to hold. This completes the proof. $\square$

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