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# GEOMETRICAL ASPECTS OF THE COVARIANT DYNAMICS OF HIGHER ORDER

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*Abstract.* We present some geometrical aspects of a higher-order jet bundle which is considered a suitable framework for the study of higher-order dynamics in continuous media. We generalize some results obtained by A. Vondra, [7]. These results lead to a description of the geometrical dynamics of higher order generated by regular equations.

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## Introduction

The present study is an attempt to emphasize some geometrical aspects of a possible mathematical model for the higher-order dynamics in continuous media as well as for the higher-order field theories.

The mathematicians agree (see [1], [2], [4], etc) that the most suitable framework for this application is a higher-order jet bundle associated to a fibered manifold. A physical field is a section of this "configuration manifold". The partial differential equations describing some higher-order dynamics are the kernels of some operators which appear as sections in a vector bundle of forms over that jet bundle, [1].

A. Vondra initiated such a study for a fibered manifold having the base of dimension 1, [5], [6], [7].

We consider a fibered manifold $(E, \pi_0, B)$, where $B$ is an orientable manifold of dimension $n \geqslant 1$ ("parameter space" containing $n-1$ "spatial variables" and a "time variable"), $E$ is a manifold of dimension $n+m$ and $\pi_0$ is a submersion of $E$ on $B$.

In [4] one argues the importance of a covariant approach that is the time variable and the other parameters on the whole.

To start the study it is necessary to define some associated structures and geometrical objects as $f(3,-1)$-structures, contact forms, connection of order $r$, dynamical connections.

Our approach means, in a more general context, to consider the $f(3,-1)$-structure on a jet bundle introduced by Vondra in the case $n=1$, [6].

The results of § 4 ($r=1$) generalize those obtained by Vondra in [7]. These results lead to a description of the geometrical dynamics of higher order generated by regular equations.

We shall use the standard multi-index notation. A multi-index is denoted by $I=(i_1,\dots,i_n)\in\mathbb{N}^n$. The length of $I$ is $|I|=i_1+\dots+i_n$ and its power is $w(I)=|I|!/I!$ where $I!=i_1!\dots i_n!$. $0=(0,\dots,0)$ is the null multi-index and $1_i=(0,\dots,1,\dots 0)$ with 1 at the $i$-th place. For $I=(i_1,\dots,i_n)$, $J=(j_1,\dots,j_n)$ we define the sum $I+J=(i_1+j_1,\dots,i_n+j_n)$. In particular, $Ij=jI=I+1_j=(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_n)$. For a family of objects $A=\{a_{i,J},|I|=m,|J|=1\}$ with $m,l$ fixed, we may define a new family $\sigma(A)=\{\sigma_L(A),|L|=m+1\}$ by

$$\sigma_L(A)=\frac{1}{w(L)}\sum_{I+J=L}w(I)w(J)a_{I,J}$$

(the sum is made for all multi-indexes $I,J$ with $I+J=L$). The family of objects $A=\{a_{I,j},|I|=m\}$ is identified with the family $A=\{a_{I,1_j},|I|=m\}$ for which $\sigma(A)=\{\sigma_L(A),|L|=m+1\}$, where

$$\sigma_L(A)=\frac{1}{w(L)}\sum_{I+J=L}w(I)a_{I,J},\ \ |I|=m,\ \ |J|=1.$$

All manifolds and mappings are supposed to be smooth and the summation convention is used as far as possible.

## 1. Geometric structures on $J^pE$

Let $(E,\pi_0,B)$ be a fibered manifold with $\dim B=n$, $\dim E=n+m$, $(U,x^i)$ a local chart on $B$ and $(U_0=\pi_0^{-1}(U),x^i,u^\alpha)$ the local fibered chart on $E$ adapted to $(U,x^i)$. If $(\overline{U}_0,\bar{x}^i,\bar{u}^\alpha)$ is another chart local fibered charts on $E$ adapted to $(\overline{U},\bar{x}^i)$ and $U\cap\overline{U}\neq\emptyset$ then the coordinate transformations are

$$\begin{aligned}(1.1)\qquad &\bar{x}^i=\bar{x}^i(x),\ \det\|\overline{B}^i_j\|\neq 0,\ \overline{B}^i_j=\frac{\partial\bar{x}^i}{\partial x^j};\\ &\bar{u}^\alpha=\bar{u}^\alpha(x,u),\ \det\|\overline{A}^\alpha_\beta\|\neq 0,\ \overline{A}^\alpha_\beta=\frac{\partial\bar{u}^\alpha}{\partial u^\beta}.\end{aligned}$$

Let $\Gamma(\pi_0)$ be the set of the sections of $\pi_0$ and for a local section on $U \subset B$, $s \in \Gamma_U(\pi_0)$, let us denote

$$u_I^\alpha(x) =: u_{i_1\ldots i_n}^\alpha(x) := \frac{\partial^{|I|} s^\alpha(x)}{(\partial x^1)^{i_1}\ldots(\partial x^n)^{i_n}}, \tag{1.2}$$

where $I = (i_1, \ldots, i_n)$ is a multi-index with $|I| \leqslant r$. The equivalence relation in $\Gamma_U(\pi_0)$ is introduced as follows: $s_1 \sim s_2$ iff $\underset{1}{u_I^\alpha}(x) = \underset{2}{u_I^\alpha}(x)$, $0 \leqslant |I| \leqslant r$, $x \in U$ and determines the $r$-jets of sections of $\pi_0$ in $x$, denoted by $j_x^r s$. Finally, the set of all such $r$-jets of sections of $\pi_0$ is a differentiable manifold denoted by $J^r E$;

$$\begin{aligned} &(J^r E, \pi_r, B), \text{ where} \\ &\pi_r\colon\ J^r E \to B,\ \pi_r(j_x^r s) = x, \end{aligned} \tag{1.3}$$

is a fibered manifold; for each pair $(p, r)$ such that $0 \leqslant p \leqslant r-1$, $(J^r E, \pi_{pr}, J^p E)$, where

$$\pi_{pr}\colon\ J^r E \to J^p E,\ \pi_{pr}(j_x^r s) = j_x^p s, \tag{1.4}$$

is a fiber bundle. In particular, $J^r E$ is an affine bundle over $J^{r-1}E$ and $J^0 E = E$. The local fibered chart on $J^r E$ induced by $(U, x^i)$ is $(U_r = \pi_r^{-1}(U), x^i, u_I^\alpha)$, $0 < |I| < r$.

For $f \in \mathscr{F}(J^r E)$ the partial derivative of $f$ in direction $x^i$ is defined by

$$(j^{r+1}s)^*(d_i f) = \partial_i(f \circ j^r s),\ \forall\ s \in \Gamma(\pi_0). \tag{1.5}$$

In the local chart $(U_r, x^i, u_I^\alpha)$ we have

$$d_i^r f = \partial_i f + \sum_{0 \leqslant |I| \leqslant r} u_{iI}^\alpha \partial_\alpha^I f, \tag{1.6}$$

where $0 \leqslant |I| \leqslant r$, $\partial_\alpha^I =: \frac{\partial}{\partial u_I^\alpha}$ and $i$ is identified with $1_i$, [3], [1].

For two local fibered charts on $J^r E$, $(\overline{U}_r, \bar{x}^i, \bar{u}_I^\alpha)$, $(U_r, x^i, u_J^\beta)$ with $\overline{U} \cap U \neq \emptyset$, the coordinate transformations are

$$\begin{aligned} \bar{x}^i &= \bar{x}^i(x), \\ \bar{u}^\alpha &= \bar{u}^\alpha(x, u), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{u}_L^\alpha &= \sigma_L(d_i(\bar{u}_I^\alpha)), \end{aligned} \tag{1.7}$$

where $|L| = 1 + |I|$ and $0 \leqslant |I| \leqslant r-1$. The natural local basis on $J^rE$ is $\{\partial_i, \partial_\alpha^I\}$ and the local co-basis is $\{dx^i, du_I^\alpha\}$, where $0 \leqslant |I| \leqslant r$.

The canonical projection (1.4), $\pi_{pr}\colon\ (x^i, u_I^\alpha) \in J^rE \mapsto (x^i, u_J^\alpha) \in J^pE$, with $0 \leqslant |I| \leqslant r$, $0 \leqslant |J| \leqslant p$, leads to the vector subbundles $V_{pr} = \mathrm{Ker}(\pi_{pr})_*$, $0 \leqslant p \leqslant r-1$, of the tangent bundle $T(J^rE)$. The local fiberes of $V_{pr}$ determine regular differential systems

$$V_{pr}\colon\ z \in J^rE \mapsto V_{pr}(z) \subset T_z(J^rE) \tag{1.8}$$

on $J^rE$ having the property

$$V_{r-1r}(z) \subset V_{r-2r}(z) \subset \ldots \subset V_{or}(z). \tag{1.9}$$

These differential systems are generated by the vector fields $\{\partial_\alpha^I\}, 0 \leqslant |I| \leqslant r$.

We call the contact form $\overset{p}{\theta}$, $1 \leqslant p \leqslant r-1$, the $V(J^pE)$-valued 1-form on $J^rE$ such that

$$\begin{aligned} &\overset{p}{\theta}\,((j^rs)_*\nu) = 0,\ \forall s \in \Gamma_U(\pi_0),\ \forall \nu \in TB, \\ &\overset{p}{\theta}\,(\xi) = (\pi_{pr})_*\xi,\ \forall \xi \in V(J^rE), [3]. \end{aligned} \tag{1.10}$$

By using the canonical local basis and co-basis we obtain

$$\overset{p}{\theta} = \sum_{|I|=p-1} \overset{p}{\theta}{}_I^\alpha \otimes \partial_\alpha^I, \tag{1.11}$$

where

$$\overset{p}{\theta}{}_I^\alpha = du_I^\alpha - u_{I,i}^\alpha dx^i, \quad |I| = p-1. \tag{1.12}$$

We can define a $V(J^rE)$-valued contact form $\theta_2$ on $J^rE$ by

$$\theta_2 = \sum_{p=1}^{r-1} \overset{p}{\theta} = \sum_{p=1}^{r-1} \sum_{|I|=p} \theta_I^\alpha \otimes \partial_\alpha^I. \tag{1.13}$$

Finally, we consider a contact map on $J^rE$ which is a $\pi_r^*(T^*B) \otimes T(J^{r-1}E)$-valued 1-form $\theta_1$ locally given by

$$\theta_1 = dx^i \otimes d_i^r, \quad \text{where } d_i^r = \partial_i + \sum_{0 \leqslant |I| \leqslant r} u_{iI}^\alpha \partial_\alpha^I. \tag{1.14}$$

We can also introduce some 1-forms $\overset{p}{J}$, $1 \leqslant p \leqslant r-1$, on $J^r E$, which are $T(J^p E) \otimes T(J^{p+1}E)$-valued and defined by

$$J^p = \sum_{|I|=p-1} \overset{p}{\theta}{}^{\alpha}_{I} \otimes \partial^{Ii}_{\alpha} \otimes d^{p+1}_{i}, \tag{1.15}$$

where

$$d^{p+1}_{i} = \partial_i + \sum_{0 \leqslant |I| \leqslant p+1} u^{\alpha}_{iI} \partial^{I}_{\alpha}. \tag{1.16}$$

For each $i \in \{1, \ldots, n\}$, let us define a $T(J^{p+1}E)$-valued 1-form on $J^r E$ by

$$\overset{p}{J}{}^{i} = \sum_{|I|=p-1} \overset{p}{\theta}{}^{\alpha}_{I} \otimes \partial^{Ii}_{\alpha}. \tag{1.17}$$

It follows from (1.17) that

$$\overset{p}{J}{}^{i} \circ \overset{p}{J}{}^{j} = 0; \ [\overset{p}{J}{}^{i}, \overset{p}{J}{}^{j}]_{FN} = 0, \tag{1.18}$$

where $[\ ,\ ]_{FN}$ is the Frölicher-Nijenhuis bracket defined for the vector valued forms. Consequently, the 1-form $\overset{p}{J}{}^{i}$ is an almost tangent structure called the almost tangent structure in direction $x^i$.

## 2. Connection of order $r$. Dynamical connection of order $r$

A connection of order $r$ on $(E, \pi_0, B)$ is a section $\Lambda\colon\ J^{r-1}E \to J^r E$ of the bundle $(J^r E, \pi_{r-1r}, J^{r-1}E)$. Any such connection is locally given by

$$\Lambda\colon\ (x^i, u^{\alpha}_{I}) \in J^{r-1}E \mapsto (x^i, u^{\alpha}_{I}, \Lambda^{\alpha}_{J}) \in J^r E,\ 0 \leqslant |I| \leqslant r-1,\ |J| = r,$$

where

$$\Lambda^{\alpha}_{J} = \Lambda^{\alpha}_{J}(x^i, u^{\alpha}_{I}).$$

The horizontal form $h^r$ of $\Lambda$ and the vertical form $v^r$ are given by

$$\begin{aligned} h^r &= \theta_1 \circ \Lambda = dx^i \otimes \left(\partial_i + \sum_{0 \leqslant |I| \leqslant r-2} u^{\alpha}_{iI} \partial^{I}_{\alpha} + \sum_{|J|=r-1} \Lambda^{\alpha}_{iJ} \partial^{J}_{\alpha}\right), \\ v^r &= \theta_2 \circ \Lambda = \sum_{0 \leqslant |I| \leqslant r-1} \theta^{\alpha}_{I} \otimes \partial^{I}_{\alpha} + \sum_{|J|=r-1} (du^{\alpha}_{J} - \Lambda^{\alpha}_{iJ} dx^i) \partial^{J}_{\alpha}. \end{aligned} \tag{2.1}$$

The $\pi_{r-1r}$-horizontal distribution $\operatorname{Im} h^r$ is called the *semispray distribution* $\Delta_r^{r-1}(\Lambda)$ and it is locally generated on $J^{r-1}E$ by the vector fields

$$\Gamma_i = \partial_i + \sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iI}\partial^I_\alpha + \sum_{|J|=r-1} \Lambda^\alpha_{iJ}\partial^J_\alpha. \tag{2.2}$$

The forms associated to $\Delta_r^{r-1}(\Lambda)$ are given by

$$\theta^\alpha_I = du^\alpha_I - u^\alpha_{iI}dx^i,\ \ \Psi^\alpha_J = du^\alpha_J - \Lambda^\alpha_{iJ}dx^i, \tag{2.3}$$

$0 \leqslant |I| \leqslant r-2$, $|J| = r-2$. The connection $\Lambda$ of order $r$ determines the direct sum decomposition

$$TJ^{r-1}E = \Delta_r^{r-1}(\Lambda) \oplus V(J^{r-1}E). \tag{2.4}$$

A section $s \in \Gamma_U(\pi_0)$ is called an integral section of $\Lambda$ if

$$j^r s = \Lambda \circ j^{r-1}s$$

on $U$. The condition of integrability is locally given by the relations

$$s^\alpha_J(x, s^\beta_I(x)) = \Lambda^\alpha_J(x, s^\beta_I(x)),\ \ |J| = r,\ \ 0 \leqslant |I| \leqslant r-1. \tag{2.5}$$

From (2.2) and (2.5) it results that $s$ is an integral section if and only if $j^{r-1}s$ is an integral map of $\Delta_r^{r-1}(\Lambda)$.

Let $\widetilde{\pi}_{1,r-1}\colon J^1(J^{r-1}E) \to J^{r-1}E$ be the 1-jet bundle of sections of the bundle $\pi_{r-2r-1}\colon J^{r-1}E \to J^{r-2}E$. If $(U_{r-1}, x^i, u^\alpha_I), 0 \leqslant |I| \leqslant r-2$, is a local chart on $J^{r-2}E$ and $s(x^i, u^\alpha_I) = (x^i, u^\alpha_I, s^\alpha_J(x, u^\alpha_I))$, $0 \leqslant |I| \leqslant r-2$, $|J| = r-1$, is a section of $\pi_{r-2r-1}$, then

$$j^1_{(x,u)}s = (x^i, u^\alpha_I, s^\alpha_J, s^\alpha_{Ji}, s^{\alpha I}_{J\beta}),\ \text{where} \\ s^\alpha_{Ji} = \frac{\partial s^\alpha_J}{\partial x^i},\ \ s^{\alpha I}_{J\beta} = \frac{\partial s^\alpha_J}{\partial u^\beta_I}. \tag{2.6}$$

A canonical chart on $J^1(J^{r-1}E)$ is given by $(\widetilde{U}_{1r-1} = \widetilde{\pi}_{1r-1}(U_{r-1}),\ x^i, u^\alpha_I, u^\alpha_J, u^\alpha_L, u^{\alpha I}_{J\beta})$, $0 \leqslant |I| \leqslant r-2, |J| = r-1, |L| = r$. The contact map on $J^1(J^{r-1}E)$ is

$$\widetilde{\theta}_1 = dx^i \otimes \left(\partial_i + \sum_{0\leqslant|I|\leqslant r-1} u^\alpha_{iI}\partial^I_\alpha\right) + \sum_{0\leqslant|I|\leqslant r-2} du^\alpha_I \otimes \left(\partial^I_\alpha + \sum_{|J|=r-1} u^{\beta I}_{J\alpha}\partial^J_\beta\right) \tag{2.7}$$

and the contact form is

$$\widetilde{\theta}_2 = \sum_{|J|=r-1} \left( du^\alpha_J - du^\alpha_{iJ} dx^i - \sum_{0\leqslant |I| \leqslant r-2} u^{\alpha I}_{J\beta} du^\beta_I \right) \otimes \partial^J_\alpha. \tag{2.8}$$

A dynamical connection on $J^{r-1}E$ is a section $F_d\colon J^{r-1}E \to J^1(J^{r-1}E)$ of $\widetilde{\pi}_{1,r-1}$. Locally, such a connection is given by

$$F_d\colon (x^i, u^\alpha_I) \in J^{r-1}E \mapsto (x^i, u^\alpha_I, u^\alpha_J, F^\alpha_L, F^{\alpha I}_{J\beta}) \in J^1(J^{r-1}E),$$

where

$$F^\alpha_L = F^\alpha_L(x^i, u^\beta_I), F^{\alpha I}_{J\beta} = F^{\alpha I}_{J\beta}(x^i, u^\gamma_I),\ 0 \leqslant |I| \leqslant r-2, |J| = r-1, |L| = r.$$

The horizontal form $h_{F_d}$ of $F_d$ and the vertical form $v_{F_d}$ are given by

$$\begin{aligned} h_{F_d} &= \widetilde{\theta}_1 \circ F_d = dx^i \otimes \left( \partial_i + \sum_{0\leqslant |I| \leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} F^\alpha_{iJ} \partial^J_\alpha \right) \\ &\quad + \sum_{0\leqslant |I| \leqslant r-2} du^\alpha_I \otimes (\partial^I_\alpha + \sum_{|J|=r-1} F^{\beta I}_{J\alpha} \partial^J_\beta), \\ v_{F_d} &= \widetilde{\theta}_2 \circ F_d = \sum_{|J|=r-1} \left( du^\alpha_J - F^\alpha_{iJ} dx^i - \sum_{0\leqslant |I| \leqslant r-2} F^{\alpha I}_{J\beta} du^\beta_I \right) \otimes \partial^I_\alpha. \end{aligned} \tag{2.9}$$

The horizontal distribution $\operatorname{Im} F_d$ on $J^{r-1}E$ is locally generated by the vector fields

$$\begin{aligned} \widetilde{\Gamma}_i &= \partial_i + \sum_{0\leqslant |I| \leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} F^\alpha_{iJ} \partial^J_\alpha, \\ \widetilde{H}^I_\alpha &= \partial^I_\alpha + \sum_{|J|=r-1} F^{\beta I}_{J\alpha} \partial^J_\beta, \quad 0 \leqslant |I| \leqslant r-2, \end{aligned} \tag{2.10}$$

or equivalently by the forms

$$\widetilde{\Psi}^\alpha_J = du^\alpha_J - \left( F^\alpha_{iJ} - \sum_{0\leqslant I \leqslant r-2} u^\beta_{iI} F^{\alpha I}_{J\beta} \right) dx^i - \sum_{0\leqslant |I| \leqslant r-2} F^{\alpha I}_{J\beta} du^\beta_I, \quad |J| = r-1. \tag{2.11}$$

## 3. $f(3,-1)$-structure on $J^{r-1}E$

**Theorem 3.1.** *A tensor field $H$ of type $(1,1)$ on $J^{r-1}E$ which satisfies the relations*

$$\theta_1 \circ H = 0, \quad \theta_2 \circ H = \theta_2, \quad H|_{V(J^{r-1}E)} = -1_{V(J^{r-1}E)} \tag{3.1}$$

*is a $f(3,-1)$-structure on $J^{r-1}E$.*

Proof. The endomorphism $H\colon T(J^{r-1}E) \to T(J^{r-1}E)$ in the local chart $(U_{r-1}, x^i, u^\alpha_I)$, $0 \leqslant |I| \leqslant r-1$, has the expression

$$\begin{aligned} H = {} & \Bigl(H^i_j dx^j + \sum_{0\leqslant |I|\leqslant r-2} H^{i,I}_\alpha \theta^\alpha_I + \sum_{|J|=r-1} H^{i,J} du^\alpha_J\Bigr) \otimes \partial_i \\ & + \sum_{0\leqslant |I|\leqslant r-2} \Bigl(H^\beta_{I,j} dx^j + \sum_{0\leqslant |L|\leqslant r-2} H^{\beta L}_{I\alpha} \theta^\alpha_L + \sum_{|J|=r-1} H^{\beta J}_{I\alpha} du^\alpha_J\Bigr) \partial^I_\beta \\ & + \sum_{|J|=r-1} \Bigl(H^\beta_{J,j} dx^j + \sum_{0\leqslant |I|\leqslant r-2} H^{\beta I}_{J\alpha} \theta^\alpha_I + \sum_{|K|=r-1} H^{\beta K}_{J\alpha} du^\alpha_K\Bigr) \otimes \partial^J_\beta. \end{aligned} \tag{3.2}$$

The condition $\theta_1 \circ H = 0$ yields $H^i_j = H^{i,I}_j = H^{i,J}_\alpha = 0$; $\theta_2 \circ H = \theta_2$ implies $H^\beta_{I,j} = H^{\beta J}_{I\alpha} = 0$, $H^{\beta L}_{I\alpha} = \delta^\beta_\alpha \delta^L_I$, $0 \leqslant |I| \leqslant r-2$, $0 \leqslant |L| \leqslant r-2$, $|J| = r-1$, where

$$\delta^L_I = \delta^{1_1}_{i_1} \dots \delta^{1_n}_{i_n}, \quad \text{for } I = (i_1, \dots, i_n),\ L = (l_1, \dots, l_n).$$

From the third condition (3.1) we obtain $H^{\beta K}_{J\alpha} = -\delta^\beta_\alpha \delta^K_J$, $|K| = |J| = r-1$, and consequently,

$$H = \sum_{0\leqslant |I|\leqslant r-2} \theta^\alpha_I \otimes \partial^I_\alpha + \sum_{|J|=r-1} \Bigl(H^\beta_{J,i} dx^i + \sum_{0\leqslant |I|\leqslant r-2} H^{\beta I}_{J\alpha} \theta^\alpha_I - du^\beta_J\Bigr) \otimes \partial^J_\beta. \tag{3.3}$$

In particular, we have

$$\begin{aligned} H(\partial_i) &= -\sum_{0\leqslant |I|\leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} H^\beta_{J,i} \partial^J_\beta; \\ H(\partial^I_\alpha) &= \partial^I_\alpha + \sum_{|J|=r-1} H^{\beta I}_{J\alpha} \partial^J_\beta, \quad 0 \leqslant |I| \leqslant r-2; \\ H(\partial^J_\alpha) &= -\partial^J_\alpha; \quad |J| = r-1. \end{aligned} \tag{3.4}$$

From (3.4) we obtain

$$H^2(\partial_i) = -\sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iI}\partial^I_\alpha - \sum_{|J|=r-1}\Bigl(\sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iI}H^{\beta I}_{J\alpha} + H^\beta_{J,i}\Bigr)\partial^J_\beta;$$
$$H^2(\partial^I_\alpha) = \partial^I_\alpha,\ 0\leqslant|I|\leqslant r-2;$$
$$H^2(\partial^J_\alpha) = \partial^J_\alpha,\ |J| = r-1.$$

Thus $H^3(\partial_i) = \partial_i, H^3(\partial^I_\alpha) = \partial^I_\alpha, H^3(\partial^J_\alpha) = \partial^J_\alpha$ and $H$ defines a $f(3,-1)$-structure on $J^{r-1}E$. □

**Corollary 3.2.** *The eigenspace of $H$ corresponding to the eigenvalue 1 is $\mathrm{Im}(H^2-H) = V_{\widetilde{\pi}_{1,r-1}}(J^{r-1}E)$. The eigenspace of $H$ corresponding to the eigenvalue 0 is $\mathrm{Im}(H^2-I)$. The eigenspace of $H$ corresponding to the eigenvalue $(-1)$ is $\mathrm{Im}(H^2+H)$.*

*The subbundle*

$$H'(J^{r-1}E) = \mathrm{Im}(H^2+H)\oplus\mathrm{Im}(H^2-I) \tag{3.5}$$

*is called the weak horizontal subbundle associated to $H$. His generators and the vector fields*

$$\overline{\Gamma}_i = \partial_i + \sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iI}\partial^I_\alpha + \sum_{|J|=r-1}\Bigl(H^\beta_{J,i} + \frac{1}{2}\sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iJ}H^{\beta I}_{J\alpha}\Bigr)\partial^J_\beta, \tag{3.6}$$
$$\overline{H}^I_\alpha = \partial^I_\alpha + \frac{1}{2}\sum_{|J|=r-1} H^{\beta I}_{J\alpha}\partial^J_\beta,\quad 0\leqslant|I|\leqslant r-2.$$

*Also we have*

$$T(J^{r-1}E) = H'(J^{r-1}E)\oplus V(J^{r-1}E). \tag{3.7}$$

**Theorem 3.3.** *Each $f(3,-1)$-structure $H$ on $J^{r-1}E$ defined in Theorem 3.1 induces a canonical dynamical connection $F_d$ on $J^{r-1}E$ by*

$$\mathrm{Im}\, h_{F_d} = H'(J^{r-1}E). \tag{3.8}$$

*Locally, $F_d$ is given by*

$$F^\alpha_L = \sigma_L(H^\beta_{J,i}) + \frac{1}{2}\sum_{0\leqslant|I|\leqslant r-2}\sigma_L(U^\alpha_{iI}H^{\beta I}_{J\alpha}),\quad |L| = 1+|J|, \tag{3.9}$$
$$F^{\beta I}_{J\alpha} = \frac{1}{2}H^{\beta I}_{J\alpha};\quad 0\leqslant|I|\leqslant r-2,\ |J| = r-1.$$

P r o o f. The relation (3.9) follows from (3.6) and (2.10). □

An $f(3,-1)$-structure $H$ on $J^{r-1}E$ defined by (3.1) is called *symmetric* if

$$\sigma_L(H^\beta_{J,i}) = H^\beta_{Ji}, \quad \forall L \text{ with } |L| = r,\ |J| = r-1.$$

**Theorem 3.4.** *The set of the dynamical connections on $J^{r-1}E$ and the set of the symmetric $f(3,-1)$-structures defined by* (3.1) *have the same cardinality.*

P r o o f. A bijection is given by

$$\begin{aligned} F^\alpha_L &= H^\beta_L + \frac{1}{2} \sum_{0 \leqslant |I| \leqslant r-2} \sigma_L(u^\alpha_{iI} H^{\beta I}_{J\alpha}), \quad |L| = 1 + |J|,\ |J| = r-1, \\ F^{\beta I}_{J\alpha} &= \frac{1}{2} H^{\beta I}_{J\alpha}, \quad 0 \leqslant |I| \leqslant r-2,\ |J| = r-1, \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} H^\beta_L &= F^\beta_L - \sum_{0 \leqslant |I| \leqslant r-2} \sigma_L(u^\alpha_{iI} F^{\beta I}_{J\alpha}), \quad |L| = 1 + |J|,\ |J| = r-1, \\ H^{\beta I}_{J\alpha} &= 2F^{\beta I}_{J\alpha}, \quad 0 \leqslant |I| \leqslant r-2,\ |J| = r-1. \end{aligned} \tag{3.11}$$

□

**Theorem 3.5.** *Each connection of order $r$ defines a symmetric $f(3,-1)$-structure.*

P r o o f. Let $h = \theta_1 \circ \Lambda = dx^i \otimes \Gamma_i$, where $\Gamma_i$ is given by (2.2), be the horizontal 1-form of a connection $\Lambda$ of order $r$. Consider the tensor field

$$A = \sum_{p=1}^{r-1} [h, \overset{p}{J}{}^i]_{FN} \otimes d_i^{p+1}, \tag{3.12}$$

where $\overset{p}{J}{}^i$ is given by (1.17) and $d_i^{p+1}$ is given by (1.16). Using the definition of the bracket $[\ ,\ ]_{FN}$ we deduce

$$A = \sum_{p=1}^{r-1} dx^k \wedge \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i \otimes d_i^{p+1}.$$

For the Lie derivation $\mathscr{L}_{\Gamma_k}$ we have

$$\begin{gathered} \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i = \sum_{|I|=p-1} (\mathscr{L}_{\Gamma_k} \theta^\alpha_I \otimes \partial^{Ii}_\alpha + \theta^\alpha_I \otimes \mathscr{L}_{\Gamma_k} \partial^{Ii}_\alpha),\ 1 \leqslant p \leqslant r-1; \\ \mathscr{L}_{\Gamma_k} \theta^\alpha_I = \theta^\alpha_{Ik},\ 0 \leqslant |I| \leqslant r-2;\ \mathscr{L}_{\Gamma_k} \theta^\alpha_I = du^\alpha_{Ik} - \Lambda^\alpha_{Ikh} dx^h,\ |I| = r-2; \\ \mathscr{L}_{\Gamma_k} \partial^{Ii}_\alpha = [\Gamma_k, \partial^{Ii}_\alpha] = -\delta^i_k \partial^I_\alpha - \sum_{|J|=r-1} \partial^{Ii}_\alpha(\Lambda^\beta_{kJ}) \partial^J_\beta,\ 0 \leqslant |I| \leqslant r-2. \end{gathered}$$

Then we can write

$$
\begin{aligned}
A &= \sum_{p=1}^{r-1} \sum_{|I|=p-1} dx^k \wedge (\mathscr{L}_{\Gamma_k} \theta^\alpha_I \otimes \partial^{Ii}_\alpha + \theta^\alpha_I \otimes \mathscr{L}_{\Gamma_k} \partial^{Ii}_\alpha) \otimes d^{p+1}_i \\
&= \sum_{0\leqslant |I| < r-2} dx^k \wedge (\theta^\alpha_{Ik} \otimes \partial^{Ii}_\alpha - \delta^i_k \theta^\alpha_I \otimes \partial^I_\alpha) \otimes d^r_i \\
&\quad + \sum_{|I|=r-2} [(du^\alpha_{Ik} - \Lambda^\alpha_{Ikh} dx^h) \otimes \partial^{Ii}_\alpha - \delta^i_k \theta^\alpha_I \otimes \partial^I_\alpha)] \otimes d^r_i \\
&\quad - \sum_{0\leqslant |I| \leqslant r-2} \sum_{|J|=r-1} \partial^{Ii}_\alpha (\Lambda^\beta_{kJ}) dx^k \wedge \theta^\alpha_I \otimes \partial^J_\beta \otimes d^r_i .
\end{aligned}
$$

Let $\operatorname{tr} A = \sum\limits_{p=1}^{r-1} \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i dx^k (d^{p+1}_i) = \sum\limits_{p=1}^{r-1} \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^k$. Then

$$
\begin{aligned}
\operatorname{tr} A = &\sum_{0\leqslant |I| < r-2} (\theta^\alpha_{Ik} \otimes \partial^{Ik} - n\theta^\alpha_I \otimes \partial^I_\alpha) \\
&+ \sum_{|I|=r-2} [(du^\alpha_{Ik} - \Lambda^\alpha_{Ikh} dx^h) \otimes \partial^{Ik}_\alpha - n\theta^\alpha_I \otimes \partial^I_\alpha] \\
&- \sum_{0\leqslant |I| \leqslant r-2} \sum_{|J|=r-1} \partial^{Ii}_\alpha (\Lambda^\beta_{iJ}) \theta^\alpha_I \otimes \partial^J_\beta \\
= &\sum_{0\leqslant |I| \leqslant r-2} \theta^\alpha_I \otimes \partial^I_\alpha - n \sum_{0\leqslant |I| \leqslant r-2} \theta^\alpha_I \otimes \partial^I_\alpha \\
&+ \sum_{|J|=r-1} du^\alpha_J \otimes \partial^J_\alpha - \sum_{|J|=r-1} \Lambda_{Jh} dx^h \otimes \partial^J_\alpha - \sum_{|J|=r-1} \sum_{0\leqslant |I| \leqslant r-2} \partial^{Ii}_\beta (\Lambda^\alpha_{iJ}) \theta^\beta_I \otimes \partial^J_\alpha \\
= &(1-n)\theta_2 + \sum_{|J|=r-1} \left( du^\alpha_J - \Lambda^\alpha_{Jh} dx^h - \sum_{0\leqslant |I| \leqslant r-2} \partial^{Ii}_\beta (\Lambda^\alpha_{iJ}) \theta^\beta_I \right) \otimes \partial^J_\alpha .
\end{aligned}
$$

Now we put

$$
\begin{aligned}
(3.13) \qquad & H = -(n-2)\theta_2 - \operatorname{tr} A, \text{ i.e.} \\
& H = \theta_2 + \sum_{|J|=r-1} \left( \Lambda^\alpha_{Ji} dx^i + \sum_{0\leqslant |I| \leqslant r-2} \partial^{Ii}_\beta (\Lambda^\alpha_{iJ}) \theta^\beta_I - du^\alpha_J \right) \otimes \partial^I_\alpha .
\end{aligned}
$$

$H$ is a symmetric $f(3,-1)$-structure on $J^{r-1}E$, satisfying the condition (3.1). □

It is easy to establish the following theorems.

**Theorem 3.6.** *Each connection of order $r$ defines a dynamical connection. Conversely, each dynamical connection determines a connection of order $r$.*

*If $\Lambda$ is a connection of order $r$ then the associated dynamical connection $F_d$ is given by*

$$(3.14)\quad F^{\alpha}_{L} = \Lambda^{\alpha}_{L} + \frac{1}{2}\sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^{\beta}_{iI}\partial^{Ik}_{\beta}(\Lambda^{\alpha}_{kJ})), \quad |L| = 1+|J|,\ |J| = r-1;$$

$$F^{\beta I}_{J\alpha} = \frac{1}{2}\partial^{Ii}_{\alpha}(\Lambda^{\beta}_{iJ}), \quad 0\leqslant |I| \leqslant r-2,\ |J| = r-1.$$

*A dynamical connection $F_d$ determines a connection of order $r$ given by*

$$(3.15)\qquad \Lambda^{\alpha}_{L} = F^{\alpha}_{L} - \sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^{\beta}_{iI}F^{\alpha I}_{J\beta}),\ |L| = 1+|J|,\ |J| = r-1.$$

**Theorem 3.7.** *Let $\omega\colon J^1(J^{r-1}E)\to J^rE$ be the bundle morphism*

$$\omega\colon (x^i, u^{\alpha}_{I}, u^{\alpha}_{J}, u^{\alpha}_{L}, u^{\alpha I}_{J\beta}) \mapsto (x^i, u^{\alpha}_{I}, \widetilde{u}^{\alpha}_{L}),\ 0\leqslant|I|\leqslant r-2,\ |J| = r-2,\ |L| = r,$$

*where*

$$\widetilde{u}^{\alpha}_{L} = u^{\alpha}_{L} - \sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^{\beta}_{iI}u^{\alpha I}_{J\beta}),\ |L| = 1+|J|,$$

*and $F_d$ is a dynamical connection on $J^{r-1}E$. The associated connection of order $r$ is given by*

$$(3.16)\qquad \Lambda = \omega\circ F_d.$$

## 4. A geometric study of systems of partial differential equations of second order

A dynamical connection $F_d$ on $J^1E$ is locally characterized by the vector fields $\{\Gamma_i, H_\alpha, V^i_\alpha\}$, where

$$(4.1)\qquad \Gamma_i = \partial_i + u^{\alpha}_{i}\partial_\alpha + F^{\alpha}_{ij}V^j_\alpha,\ H_\alpha = \partial_\alpha + F^{\beta}_{i\alpha}V^i_\beta,\ V^i_\alpha = \partial^i_\alpha,$$

with $F^{\alpha}_{ij} = F^{\alpha}_{ji}$. The 1-forms associated with (4.1) are $\{dx^i, \theta^\alpha, \Psi^\alpha_i\}$, where

$$(4.2)\qquad \theta^\alpha = du^\alpha - u^\alpha_i dx^i;$$

$$\Psi^\alpha_i = du^\alpha_i - F^\alpha_{i\beta}du^\beta - (F^\alpha_{ij} + u^\beta_i F^\alpha_{j\beta})dx^j = du^\alpha_i - F^\alpha_{i\beta}\theta^\beta - F^\beta_{i\beta}dx^j.$$

For the vector fields (4.1) the following relations are satisfied:

$$
\begin{aligned}
(4.3)\qquad [\Gamma_i,\Gamma_j] &= T^\alpha_{ijk}V^k_\alpha,\ T^\alpha_{ijk}=\Gamma_i(F^\alpha_{jk})-\Gamma_j(F^\alpha_{ik}),\\
[\Gamma_i,H_\alpha] &= -F^\beta_{i\alpha}H_\beta+T^\gamma_{ik\alpha}V^k_\gamma,\ T^\gamma_{ik\alpha}=\Gamma_i(F^\gamma_{k\alpha})+F^\beta_{i\alpha}F^\gamma_{k\beta}-H_\alpha(F^\gamma_{ik}),\\
[\Gamma_i,V^j_\alpha] &= -\delta^j_i H_\alpha+T^{j\gamma}_{ik\alpha}V^k_\gamma,\ T^{j\gamma}_{ik\alpha}=\delta^j_i F^\gamma_{k\alpha}-\partial^j_\alpha(F^\gamma_{ik}),\\
[H_\alpha,H_\beta] &= T^\gamma_{\alpha\beta k}V^k_\gamma,\ T^\gamma_{\alpha\beta k}=H_\alpha(F^\gamma_{k\beta})-H_\beta(F^\gamma_{k\alpha}),\\
[V^i_\alpha,V^j_\beta] &= 0.
\end{aligned}
$$

For the forms (4.2) we have

$$
\begin{aligned}
d\theta^\alpha &= -\Psi^\alpha_i\wedge dx^i-F^\alpha_{i\beta}\theta^\beta\wedge dx^i,\\
d\Psi^\alpha_i &= \frac{1}{2}T^\alpha_{jki}dx^j\wedge dx^k+T^\alpha_{ki\beta}\theta^\beta\wedge dx^k-\frac{1}{2}T^\alpha_{\beta\gamma i}\theta^\beta\wedge\theta^\gamma\\
&\quad -\partial^k_\gamma(F^\alpha_{i\beta})\Psi^\gamma_k\wedge\theta^\beta+T^{k\alpha}_{ji\beta}\Psi^\beta_k\wedge dx^j.
\end{aligned}
$$

The tensor field of type (1,1) associated with $F_d$ (see 3.11) is given by

$$(4.5)\qquad H=\theta^\alpha\otimes\partial_\alpha+(H^\alpha_{ij}dx^j+H^\alpha_{i\beta}du^\beta-du^\alpha_i)\otimes V^i_\alpha,$$

where

$$(4.6)\qquad H^\alpha_{ij}=F^\alpha_{ij}-(u^\beta_i F^\alpha_{j\beta}+u^\beta_j F^\alpha_{i\beta}),\ H^\alpha_{i\beta}=2F^\alpha_{i\beta}.$$

With respect to the basis $\{\Gamma_i,H_\alpha,V^i_\alpha\}$ and the co-basis $\{dx^i,\theta^\alpha,\Psi^\alpha_i\}$ the tensor field $H$ has the form

$$(4.7)\qquad H=\theta^\alpha\otimes\partial_\alpha+[(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)dx^j+F^\alpha_{i\beta}\theta^\beta-\Psi^\alpha_i]\otimes V^i_\alpha.$$

From (4.7) we obtain

$$(4.8)\qquad H(\Gamma_i)=(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)V^i_\alpha, H(H_\alpha)=H_\alpha+F^\beta_{i\alpha}V^i_\beta, H(V^i_\alpha)=-V^i_\alpha$$

and

$$
\begin{aligned}
(4.9)\qquad {}^tH(dx^i) &= dx^i(H)=0,\ t_H(\theta^\alpha)=\theta^\alpha(H)=\theta^\alpha,\\
{}^tH(\Psi^\alpha_i) &= -\Psi^\alpha_i+(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)dx^j+F^\alpha_{i\beta}\theta^\beta.
\end{aligned}
$$

Let now $\omega=f(x)dx^1\wedge\ldots\wedge dx^n$ be a volume form on $B$ and $\omega_i=\iota_{\partial_i}\omega$ (the interior product with respect to $\partial_i$). Then

$$d\omega_i=f^{-1}(\partial_i f)\omega,\ dx^j\wedge\omega_i=\delta^j_i\omega.$$

Consider $\widetilde{J}\colon\ J^1E \to T^*(J^1E) \wedge \Lambda^{n-1}(B) \otimes VT(J^1E)$ defined by

$$\widetilde{J} = \theta^\alpha \wedge \omega_i \otimes V^i_\alpha; \tag{4.10}$$

then

$$\operatorname{Im} \widetilde{J} = \Lambda^{n-1}(B) \otimes VT(J^1E),\ \ \widetilde{J} \circ \widetilde{J} = 0.$$

We call *the Poincaré-Cartan form* of a function $L \in \mathscr{F}(J^1E)$ the $n$-form $\theta_L$ defined by

$$\theta_L = \widetilde{J}(L) + L\omega, \tag{4.11}$$

where $\widetilde{J}(L) +^t \widetilde{J}(dL) = dL(\widetilde{J})$. In a local fibered chart we have

$$\theta_L = \partial^i_\alpha(L)\theta^\alpha \wedge \omega_i + L\omega. \tag{4.12}$$

Now we consider the $(n+1)$-form

$$\Omega_L = d\theta_L. \tag{4.13}$$

Using a dynamical connection $F_d$ on $J^1E$, the relations (4.4) and the fact that

$$df = \Gamma_i(f)dx^i + H_\alpha(f)\theta^\alpha + \partial^i_\alpha(f)\Psi^\alpha_i,\ \forall\ f \in \mathscr{F}(J^1E)$$

we obtain

$$\begin{aligned}\Omega_L = {} & \partial^j_\beta(\partial^i_\alpha L)\Psi^\beta_j \wedge \theta^\alpha \wedge \omega_i - \frac{1}{2}[H_\beta(\partial^i_\alpha L) \\ & - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i - [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_i f)\delta^i_\alpha L]\theta^\alpha \wedge \omega.\end{aligned} \tag{4.14}$$

Denoting $A^{ij}_{\alpha\beta} = \partial^i_\alpha(\partial^j_\beta L)$ we have the relations

$$A^{ij}_{\alpha\beta} = A^{ji}_{\alpha\beta} = A^{ij}_{\beta\alpha}. \tag{4.15}$$

We now make a general remark.

**Remark.** Let $T$ be a tensor field of type (1,1) on a differential manifold $M$ and let $\Omega$ be a 3-form on $M$. We can define in terms of $T$ the following 3-forms on $M$:

$$\begin{aligned}(T^{(1)}\Omega)(X,Y,Z) &= \Omega(TX,Y,Z) + \Omega(X,TY,Z) + \Omega(X,YTZ),\\ (T^{(2)}\Omega)(X,Y,Z) &= \Omega(TX,TY,Z) + \Omega(TX,Y,TZ) + \Omega(X,TY,TZ).\end{aligned} \tag{4.16}$$

On the other hand, we can associate with $T$ an antiderivation $\delta_T$ of degree zero on the algebra of forms on $M$. $\delta_T$ is uniquely determined by the conditions

$$\delta_T f = 0,\ \forall f \in \mathscr{F}(M); \quad \delta_T \theta = {}^t T\theta,\ \forall\, \theta \in \Lambda^1(M).$$

For a $k$-form $\omega \in \Lambda^k(M)$ we have

$$(\delta_T \omega)(X_1, \dots, X_k) = (T^{(1)}\omega)(X_1, \dots, X_k). \tag{4.17}$$

If we consider the operator $d_T$ given by

$$d_T = \delta_T \circ d - d \circ \delta_T \tag{4.18}$$

then we have

$$\begin{gathered} d \circ d_T = -d_T \circ d,\ d_T^2 \circ d = d \circ d_T^2, \\ \iota_X \circ d_T + d_T \circ \iota_X = \mathscr{L}_{TX} + [\delta_T, \mathscr{L}_X]. \end{gathered} \tag{4.19}$$

**Theorem 4.1.** *The $(n+1)$-form $\Omega_L$ from (4.13) has the decomposition*

$$\Omega_L = \Omega_L^c + H^{(2)}\Omega_L - H^{(1)}\Omega_L, \tag{4.20}$$

*where*

$$\Omega_L^c = A_{\alpha\beta}^{ij}\Psi_i^\alpha \wedge \theta^\beta \wedge \omega_j, \tag{4.21}$$

$$\begin{aligned} H^{(1)}\Omega_L = &- A_{\alpha\beta}^{ij} F_{i\gamma}^\alpha \theta^\gamma \wedge \theta^\beta \wedge \omega_j - [H_\beta(\partial_\alpha^i L) - H_\alpha(\partial_\beta^i L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i \\ &- [\Gamma_k(\partial_\alpha^k L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial_\alpha^k L]\theta^\alpha \wedge \omega, \end{aligned} \tag{4.22}$$

$$H^{(2)}\Omega_L = - A_{\alpha\beta}^{ij} F_{i\gamma}^\alpha \theta^\gamma \wedge \theta^\beta \wedge \omega_j - \frac{1}{2}[H_\beta(\partial_\alpha^i L) - H_\alpha(\partial_\beta^i L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i. \tag{4.23}$$

P r o o f. By using the above remark and (4.9) we have

$$\begin{aligned} H^{(1)}\Omega_L &= A_{\alpha\beta}^{ij}[{}^t H(\Psi_i^\alpha) \wedge \theta^\beta \wedge \omega_j + \Psi_i^\alpha \wedge^t H(\theta^\beta) \wedge \omega_j + \Psi_i^\alpha \wedge \theta^\beta \wedge H^{(1)}(\omega_j)] \\ &\quad - \frac{1}{2}[H_\beta(\partial_\alpha^i L) - H_\alpha(\partial_\beta^i L)][{}^t H(\theta^\alpha) \wedge \theta^\beta \wedge \omega_i + \theta^\alpha \wedge^t H(\theta^\beta) \wedge \omega_i \\ &\quad + \theta^\alpha \wedge \theta^\beta \wedge H^{(1)}(\omega_i)] - [\Gamma_k(\partial_\alpha^k L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial_\alpha^k][{}^t H(\theta^\alpha) \wedge \omega \\ &\quad + \theta^\alpha \wedge H^{(1)}(\omega)] \\ &= A_{\alpha\beta}^{ij}[-\Psi_i^\alpha + (F_{i\gamma}^\alpha u_k^\gamma - F_{k\gamma}^\alpha u_k^\gamma)dx^k + F_{i\gamma}^\alpha \theta^\gamma + \Psi_i^\alpha] \wedge \theta^\beta \wedge \omega_j \\ &\quad - \frac{1}{2}[H_\beta(\partial_\alpha^i L) - H_\alpha(\partial_\beta^i L)](\theta^\alpha \wedge \theta^\beta \wedge \omega_i + \theta^\alpha \wedge \theta^\beta \wedge \omega_i) \\ &\quad - [\Gamma_k(\partial_\alpha^k L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial_\alpha^k] \wedge \theta^\alpha \wedge \omega \\ &= -(A_{\alpha\beta}^{ij} F_{i\gamma}^\alpha u_j^\gamma - A_{\alpha\beta}^{ij} F_{i\gamma}^\alpha u_j^\gamma)\theta^\beta \wedge \omega_j - A_{\alpha\beta}^{ij} F_{i\gamma}^\alpha \theta^\gamma \wedge \theta^\beta \wedge \omega_j \\ &\quad - [H_\beta(\partial_\alpha^i L) - H_\alpha(\partial_\beta^i L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i - [\Gamma_k(\partial_\alpha^k L) - \partial_\alpha L \\ &\quad - f^{-1}(\partial_k f)\partial_\alpha^k L]\theta^\alpha \wedge \omega. \end{aligned}$$

Similarly,

$$H^{(2)}\Omega_L = A^{ij}_{\alpha\beta}\,{}^tH(\Psi^\alpha_i) \wedge^t H(\theta^\beta) \wedge \omega_j - \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i$$
$$= -A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}\theta^\gamma \wedge \theta^\beta \wedge \omega_j - \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i.$$

Then

$$H^{(2)}\Omega_L - H^{(1)}\Omega_L = \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i$$
$$+ [\Gamma_K(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial^k_\alpha L]\theta^\alpha \wedge \omega.$$

If $\Omega^c_L$ is given by (4.21) then (4.20) is verified. □

The above theorem suggests the following definition:

A dynamical connection $F_d$ is said to be *compatible with* $L$ if $H^{(1)}\Omega_L = H^{(2)}\Omega_L$.

**Theorem 4.2.** *A dynamical connection $F_d$ is compatible with $L$ iff the following conditions are satisfied:*

$$A^{ij}_{\alpha\beta}F^\alpha_{ij} + B_\beta = 0,\ A^{ij}_{\alpha\beta}F^\alpha_{j\gamma} = \frac{1}{2}\partial^i_\beta B_\gamma + R^i_{\beta\gamma}, \tag{4.24}$$

*where*

$$B_\alpha = \partial_k\partial^k_\alpha L + u^\beta_k\partial_\beta u^k_\alpha L - \partial_\alpha L + f^{-1}(\partial_k f)\partial^k_\alpha L \tag{4.25}$$

*and*

$$R^i_{\beta\gamma} = R^i_{\gamma\beta}.$$

P r o o f. The definition yields

$$\partial_k(\partial^k_\alpha L) + u^\beta_k\partial_\beta\partial^k_\alpha L - \partial_\alpha L + f^{-1}(\partial_k f)\partial^k_\alpha L + A^{ij}_{\alpha\beta}F^\beta_{ij} = 0,$$
$$\partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta sL + F^\gamma_{k\beta}A^{ki}_{\gamma\alpha} - F^\gamma_{k\alpha}A^{ki}_{\gamma\beta} = 0; \tag{4.26}$$

by (4.25) we obtain

$$A^{ij}_{\alpha\beta}F^\alpha_{ij} + B_\beta = 0$$

and

$$\partial^i_\beta B_\alpha = \partial_k A^{ik}_{\beta\alpha} + \partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta L + u^\gamma_k\partial_\gamma A^{ik}_{\beta\alpha} + f^{-1}(\partial_k f)A^{ik}_{\beta\alpha}. \tag{4.27}$$

From (4.27) we obtain

$$\partial_\beta \partial^i_\alpha L - \partial_\alpha \partial^i_\beta L = \partial^i_\beta B_\alpha - \partial_k A^{ik}_{\alpha\beta} - u^\gamma_k \partial_\gamma A^{ik}_{\alpha\beta} + f^{-1}(\partial_k f) A^{ik}_{\alpha\beta},$$
$$\partial_\alpha \partial^i_\beta L - \partial_\beta \partial^i_\alpha L = \partial^i_\alpha B_\beta - \partial_k A^{ik}_{\alpha\beta} - u^\gamma_k \partial_\gamma A^{ik}_{\alpha\beta} + f^{-1}(\partial_k f) A^{ik}_{\alpha\beta}$$

and

$$\partial_\beta \partial^i_\alpha L - \partial_\alpha \partial^i_\beta L = \frac{1}{2}(\partial^i_\beta B_\alpha - \partial^i_\alpha B_\beta). \tag{4.28}$$

(4.26) and (4.28) imply

$$\frac{1}{2}(\partial^i_\beta B_\alpha - \partial^i_\alpha B_\beta) + A^{ki}_{\gamma\alpha} F^\gamma_{k\beta} - A^{ki}_{\gamma\beta} F^\gamma_{k\alpha} = 0$$

or

$$\Big(\frac{1}{2}\partial^i_\beta B_\alpha - A^{ki}_{\gamma\beta} F^\gamma_{k\alpha}\Big) - \Big(\frac{1}{2}\partial^i_\beta B_\alpha - A^{ki}_{\gamma\alpha} F^\gamma_{k\beta}\Big) = 0.$$

Therefore

$$A^{ij}_{\alpha\beta} F^\alpha_{j\gamma} = \frac{1}{2}\partial^i_\beta B_\alpha + R^i_{\beta\gamma},\ \ R^i_{\beta\gamma} = R^i_{\gamma\beta}.$$

A function $L \in \mathscr{F}(J^1 E)$ is called *regular* is $\det \|A^{ij}_{\alpha\beta}\| \neq 0$. Let us note that $\|\widetilde{A}^{\alpha\beta}_{ij}\| = \|A^{ij}_{\alpha\beta}\|^{-1}$. □

**Theorem 4.3.** *If $L$ is regular then the connections $F_d$ compatible with $L$ are given by*

$$F^\alpha_{ij} = \widetilde{A}^{\alpha\beta}_{ih}\Big(P^h_{\beta j} - \frac{1}{n}\delta^h_j B_\beta\Big), \tag{4.29}$$
$$F^\alpha_{i\beta} = \widetilde{A}^{\alpha\gamma}_{ij}\Big(R^j_{\gamma\beta} + \frac{1}{2}\partial^j_\gamma B_\beta\Big),$$

*where $P(P^h_{\beta j})$ is a tensor field of type $(1,2)$ with Trace $P_\alpha = 0$ and $(\delta^h_k \delta^j_i - \delta^h_i \delta^j_k) P^l_{\alpha j} = 0$; $R = (R^i_{\alpha\beta})$ is a symmetric tensor field of type $(1,2)$.*

P r o o f. We consider the system of linear equations

$$A^{ij}_{\alpha\beta} F^\alpha_{ik} + \frac{1}{n}\delta^j_k B_\beta = P^j_{\beta k}. \tag{4.30}$$

Setting $j = k$ and summing one obtains the first relation (4.24) if Trace $P_\alpha = 0$. From (4.30) we deduce the first relation (4.29). The symmetry of $F_{ij}$ implies

$$\widetilde{A}^{\alpha\beta}_{ij}(P^j_{\beta k} - \frac{1}{n}\delta^j_k B_\beta) = \widetilde{A}^{\alpha\beta}_{kj}\Big(P^j_{\beta i} - \frac{1}{n}\delta^j_i B_\beta\Big),$$

which leads to $(\delta^h_k \delta^j_i - \delta^h_i \delta^j_k) P^l_{\alpha j} = 0$. The second relation (4.29) results from (4.24). □

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