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ON ARCHIMEDEAN MV-ALGEBRAS

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In this paper we generalize results of Cignoli [1] and the author [4] concerning complete MV-algebras to the case of archimedean MV-algebras.

1. Preliminaries and main results

For MV-algebras we apply the same terminology and notation as in [3] and [4]. We suppose the reader to be acquainted with [3], Section 1 or [4], Section 1.

Cignoli [1] studied the structure of MV-algebras which are complete nad atomic. His main result is the following theorem:

Theorem 1. ([1], Theorem 2.6.) *Let $\mathcal{A}$ be an MV-algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is complete and atomic.*

(ii) *$\mathcal{A}$ is a direct product of finite linearly ordered MV-algebras.*

A direct product $\prod\limits_{i \in I} \mathcal{A}_i$ of MV-algebras $\mathcal{A}_i$ is complete if and only if all $\mathcal{A}_i$ are complete. Further, a complete linearly ordered MV-algebra is finite if and only if it is atomic (cf. [4], 1.3). Thus Theorem 1 can be equivalently expressed as follows:

Theorem 1'. *Let $\mathcal{A}$ be an MV-algebra. Then the condition* (i) *from* Theorem 1 *is equivalent with the following condition:*

(ii$'$) *$\mathcal{A}$ is a direct product of MV-algebras which are linearly ordered, complete and atomic.*

Let α be a cardinal, $\alpha > 1$. An element a of an MV-algebra $\mathcal{A}$ is said to be an α-atom if the interval $[0, a]$ of $\mathcal{A}$ is a chain with $\operatorname{card}[0, a] = \alpha$. Hence the notion of the 2-atom coincides with the notion of the atom.

In [4] the following result was established:

Theorem 2. *Let $\mathcal{A}$ be an MV-algebra and let α be a cardinal, $\alpha > 1$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is complete and α-atomic.*

(ii) *$\mathcal{A}$ is a direct product of algebras which are linearly ordered, complete and α-atomic.*

Next, if (i) *is valid, then either $\alpha = 2$ or $\alpha = c$ (the cardinality of continuum).*

A nonempty subset $\{a_j\}_{j\in J}$ of an MV-algebra $\mathcal{A}$ is said to be orthogonal if $a_{j(1)} \wedge a_{j(2)} = 0$ whenever $j(1)$ and $j(2)$ are distinct elements of J. The MV-algebra $\mathcal{A}$ will be called orthogonally complete if each orthogonal subset of $\mathcal{A}$ possesses the supremum in $\mathcal{A}$.

In an analogous way the notions of orthogonality and orthogonal completeness in lattice ordered groups are defined. If G is a lattice ordered group, then neither the completeness of G implies the orthogonal completeness, nor conversely.

Since each MV-algebra $\mathcal{A}$ has a greatest element we infer that if $\mathcal{A}$ is complete then it must be orthogonally complete. On the other hand, an orthogonally complete MV-algebra need not be complete.

If $\mathcal{A}$ is an MV-algebra, then it can be constructed by means of an abelian lattice ordered group G with a strong unit u; G is uniquely determined (cf. Mundici [5]; cf. also [3], Section 1).

$\mathcal{A}$ will be said to be archimedean if G is archimedean (for an internal characterization of this notion cf. 2.1 below). Each complete MV-algebra is archimedean, but not conversely. A direct product $\prod\limits_{i\in I} \mathcal{A}_i$ is archimedean if and only if all $\mathcal{A}_i$ are archimedean.

In the present paper the following results will be proved:

Theorem 3. *Let $\mathcal{A}$ be an MV-algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is orthogonally complete, archimedean and atomic.*

(ii) *$\mathcal{A}$ is complete and atomic.*

In the following theorem (and also in Theorem 6) we assume that Continuum Hypothesis is valid.

Theorem 4. *Let $\mathcal{A}$ be an MV-algebra. Let α be a cardinal, $\alpha > 1$. If $\mathcal{A}$ is archimedean and α-atomic, then $\alpha \in \{2, \aleph_0, c\}$.*

Theorem 5. *Let $\mathcal{A}$ be an MV-algebra. Suppose that $\mathcal{A}$ is archimedean and α-atomic for some $\alpha > 1$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is orthogonally complete.*

(ii) *$\mathcal{A}$ is a direct product of MV-algebras which are linearly ordered and α-atomic.*

Theorem 6. *Let $\mathcal{A}$ be an archimedean orthogonally complete MV-algebra. Then $\mathcal{A}$ can be expressed as a direct product $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \mathcal{A}_4$ such that*

(i) *$\mathcal{A}_1$ is atomic;*

(ii) *$\mathcal{A}_2$ is $\aleph_0$-atomic;*

(iii) *$\mathcal{A}_3$ is c-atomic;*

(iv) *if $\alpha > 1$, then there is no α-atom in $\mathcal{A}_4$.*

2. Proofs of Theorems 3–6

Let us begin with a more detailed investigation of the archimedean property. Let $\mathcal{A}$ be an MV-algebra and let G be the corresponding lattice ordered group with a strong unit u (cf. [3]). The underlying set A of $\mathcal{A}$ is the interval $[0, u]$ of G. Therefore the group operation $+$ on G can be viewed as a partial binary operation on A; namely, for $a_1, a_2 \in A$ the operation $a_1 + a_2$ is defined in A iff $a_1 + a_2 \in A$. Let n be a positive integer, $a_i = a \in A$ for $i = 1, 2, \ldots, n$. If $na = a_1 + a_2 + \ldots + a_n$ belongs to A, then na is said to be defined in $\mathcal{A}$.

Consider the following condition for $\mathcal{A}$:

(A′) There exists $a \in A$ such that (i) $0 < a$, and (ii) for each positive integer n, na is defined in $\mathcal{A}$ and $na < u$.

2.1. Lemma. *The following conditions are equivalent:*

(i) $=$ (A′)

(ii) *$\mathcal{A}$ fails to be archimedean.*

P r o o f. Let (A′) hold. Hence G is not archimedean and thus $\mathcal{A}$ is not archimedean. Conversely, assume that (ii) is valid. Hence G is not archimedean. Thus there are elements b and d in G such that $0 < nb < d$ holds for each positive integer n. Put $b_1 = u \wedge b$. Then $0 < b_1$. There exists a positive integer m such that $d \leqslant mu$.

By way of contradiction, suppose that (A′) is not valid. We have $nmb_1 < mu$ for each positive integer n. Hence $0 < nb_1 < u$ for each positive n, which is a contradiction. $\square$

From 2.1 we obtain an internal characterization of the archimedean property for MV-algebras.

2.2. Lemma. *If $\mathcal{A}$ is complete, then it is archimedean.*

Proof. Let $\mathcal{A}$ be complete. Then by [4], 1.1, G is complete as well. It is well-known that each complete lattice ordered group is archimedean. Thus $\mathcal{A}$ is archimedean. □

2.3. Lemma. *Let $\mathcal{A}$ be an archimedean MV-algebra. Let a be an α-atom in $\mathcal{A}$, $\alpha > 1$. Then $\mathcal{A}$ can be represented as a direct product $\mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1$ is linearly ordered and $a \in A_1$.*

Proof. Let G be as above. Hence G is archimedean and the interval $[0, a]$ of $\mathcal{A}$ is a chain in G. Then in view of [5], Theorem 1, there is a direct product decomposition $G = G_1 \times G_2$ such that $[0, a] \subset G_1$; moreover, G_1 is linearly ordered. Put $X_i = [0, u] \cap G_i (i = 1, 2)$. For $x \in A$ let x_i be the component of x in G_i $(i = 1, 2)$. The mapping $x \longrightarrow (x_1, x_2)$ defines a direct product decomposition of the lattice $[0, u]$ with the factors X_1 and X_2. We have $X_i = [0, u_i]$ $(i = 1, 2)$ and u_i is a strong unit in G_i. Hence we can consider the MV-algebra $\mathcal{A}_i$ on $[0, u_i]$, where the corresponding lattice ordered group is $G_i (i = 1, 2)$. In view of [3], 3.5 we obtain that $\mathcal{A}$ is a direct product $\mathcal{A}_1 \times \mathcal{A}_2$. Clearly $a \in A_1 = [0, u_1]$.

Each direct factor of an archimedean lattice ordered group is archimedean. Thus both G_1 and G_2 are archimedean. This yields that $\mathcal{A}_1$ and $\mathcal{A}_2$ are archimedean as well. □

If $\mathcal{A}$ and a are as in 2.3, then we denote $\mathcal{A}_1 = \mathcal{A}_1(a)$; we also put $G_1 = G_1(a)$, where G_1 is as in the proof of 2.3.

Let R and Z be the additive group of all reals or all integers, respectively, with the natural linear order.

2.4. Proposition. *Let $\mathcal{A}, a$ and α be as in* 2.3. *Then $\alpha \in \{2, \aleph_0, c\}$.*

Proof. Let $G_1 = G_1(a)$. We have already remarked above that G_1 is archimedean. It is well-known that then G_1 must be isomorphic to an ℓ-subgroup R' of R. Hence the interval $[0, a]$ of G_1 is isomorphic to an interval $[0, a']$ of R'. If $[0, a']$ is finite, then clearly $\alpha = 2$. If $[0, a']$ is infinite, then $\alpha \geqslant \aleph_0$. Since $[0, a'] \subset R$ we infer that $\alpha \leqslant c$. Now Continuum Hypothesis yields that either $\alpha = \aleph_0$ or $\alpha = c$. □

Theorem 4 above is a corollary of 2.4.

2.5. Proposition. *Let $\mathcal{A}$ be an archimedean orthogonally complete MV-algebra. Suppose that for each $x \in A$ there is $a \in A$ such that $0 < a \leqslant x$ and $[0, a]$ is a chain. Then $\mathcal{A}$ is a direct product of linearly ordered MV-algebras.*

Proof. The case $A = \{0\}$ is trivial; suppose that $A \neq \{0\}$. For each $a \in A$ such that $0 < a$ and $[0, a]$ is a chain we construct the linearly ordered MV-algebra $\mathcal{A}_1(a)$. Let $\{\mathcal{A}_i\}_{i\in I}$ be the system of all MV-algebras that can be constructed in this way; next, let $\{G_i\}_{i\in I}$ be the system of the corresponding ℓ-subgroups of G. All G_i are direct factors of G, thus they are polars of G. Also, all G_i are linearly ordered, hence $G_{i(1)} \cap G_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. If $b \in A$ and b_i is the component of b in G_i, then the system $\{b_i\}_{i\in I}$ is orthogonal. Clearly $b_i \in A$ for each $i \in I$. By applying the orthogonal completeness of $\mathcal{A}$ and using the same method as in [4], proof of (A), part (b), then we obtain that our assertion is valid. □

Proof of Theorem 3. Let (ii) be true. Then $\mathcal{A}$ is orthogonally complete. In view of 2.2, $\mathcal{A}$ is archimedean, thus (i) holds.

Conversely, suppose that (i) is satisfied. Similarly as in the proof of 2.5 we can suppose that $A \neq \{0\}$. In view of 2.5, $\mathcal{A}$ is a direct product of a system $\{\mathcal{A}_i\}_{i\in I}$ where each $\mathcal{A}_i$ is linearly ordered. All $\mathcal{A}_i$ are archimedean and atomic. Thus all $\mathcal{A}_i$ are finite (cf. the proof of 2.4). Hence all $\mathcal{A}_i$ are complete. This yields that $\mathcal{A}$ is complete as well. □

Proof of Theorem 5. We suppose that $\mathcal{A}$ is an archimedean MV-algebra which is α-atomic for some $\alpha > 1$.

Let (i) be valid. We apply 2.5 and construct the system $\{\mathcal{A}_i\}_{i\in I}$. Again, we can suppose that $A \neq \{0\}$, whence $I \neq \emptyset$. All $\mathcal{A}_i$ (being direct factors of $\mathcal{A}$) must be α-atomic. Hence (ii) holds.

Let (ii) be true. Let $\{x_j\}_{j\in J}$ be an orthogonal subset of A. We have to prove that $\sup\{x_j\}_{j\in J}$ exists in $\mathcal{A}$. It suffices to consider the case when $x_j \neq 0$ for each $j \in J$. For $j \in J$ and $i \in I$ let x_{ji} be the component of x_j in $\mathcal{A}_i$. Then the system $\{x_{ji}\}_{(ji)\in J\times I}$ is orthogonal and hence there exists $x = \bigvee\limits_{j,i} x_{ji}$ in $\mathcal{A}$. Since $x_j = \bigvee\limits_{i\in I} x_{ji}$ is valid for each $j \in J$, we have $x \geqslant x_j$ for each $j \in J$. Next, let $y \in A$, $y \geqslant x_j$ for each $j \in J$. Let y_i be the component of y in the direct factor A_i. If $i \in I$ and $x_{ji} \neq 0$ for some $j \in J$, then $y_i \geqslant x_{ji}$, whence $y \geqslant x$. Thus $x = \bigvee\limits_{j\in J} x_j$. □

Proof of Theorem 6. We suppose that $\mathcal{A}$ is archimedean and orthogonally complete. We apply 2.3. If $a \in A$ is an α-atom for some $\alpha > 1$, then we construct $G_1 = G_1(a)$ as in the proof of 2.3; next, we construct $\mathcal{A}_1 = \mathcal{A}_1(a)$. Let $\{G_i\}_{i\in I}$ and $\{\mathcal{A}_i\}_{i\in I}$ be the set of all lattice ordered groups or all MV-algebras, respectively, that can be constructed in this way.

If $i(1)$ and $i(2)$ are distinct elements of I, then $\mathcal{A}_{i(1)} \cap \mathcal{A}_{i(2)} = \{0\}$. For each $i \in I$ let u_i be the component of u in $\mathcal{A}_i$. The system $\{u_i\}_{i \in I}$ is orthogonal, hence there exists $u^0 = \bigvee_{i \in I} u_i$ in A. We have $u^0 \leqslant u$, thus there is u_4 in A such that $u^0 + u_4 = u$.

Assume there is an α-atom a in A with $a \leqslant u_4, \alpha > 1$. Hence there exists $i(1) \in I$ with $G_{i(1)} = G_1(a)$. Then

$$(u_4)_{i(1)} \geqslant a > 0 .$$

A simple calculation (analogous to that performed in [4], proof of (A)) yields that

$$u_{i(1)} = (u^0)_{i(1)} + (u_4)_{i(1)} \geqslant u_{i(1)} + a ,$$

which is a contradiction. Therefore $u_4 \wedge a = 0$ for each α-atom a with $\alpha > 1$. Hence $u_4 \wedge u_i = 0$ for each $i \in I$ and thus $u_4 \wedge u^0 = 0$. Hence

$$u = u_4 + u^0 = u_4 \vee u_0 .$$

Since the lattice $[0, u]$ is distributive, it is a direct product $[0, u^0] \times [0, u_4]$ (we consider the mapping $x \longrightarrow (x \wedge u^0, x \wedge u_4)$ for each $x \in [0, u]$). We can construct the MV-algebras $\mathcal{A}_0$ and $\mathcal{A}_4$ with $A_0 = [0, u^0]$ and $A_4 = [0, u_4]$. In view of [3], 3.5, $\mathcal{A}$ is a direct product $\mathcal{A}_0 \times \mathcal{A}_4$. We have verified that if $\alpha > 1$, then $\mathcal{A}_4$ has no α-atom.

Let $0 < x \in A_0$. Then

$$x = x \wedge u^0 = x \wedge (\bigvee_{i \in I} u_i) = \bigvee_{i \in I} (x \wedge u_i) .$$

Hence there is $i(1) \in I$ such that $x \wedge u^0 > 0$. It is clear that $x \wedge u_{i(1)}$ is an α-atom in $\mathcal{A}_0$ for some $\alpha > 1$. Thus we can apply Proposition 2.5 for $\mathcal{A}_0$. In view of the construction in the proof of 2.5, $\mathcal{A}_0$ is a direct product of the system $\{\mathcal{A}_i\}_{i \in I}$.

Each $\mathcal{A}_i$ is α-atomic for some $\alpha \in \{2, \aleph_0, c\}$ (cf. Theorem 4). Put

$I_1 = \{i \in I : \mathcal{A}_i \text{ is 2-atomic}\}$,
$I_2 = \{i \in I : \mathcal{A}_i \text{ is } \aleph_0\text{-atomic}\}$,
$I_3 = \{i \in I : \mathcal{A}_i \text{ is } c\text{-atomic}\}$.

In view of the direct product decomposition under consideration there are MV-algebras $\mathcal{A}_j (j = 1, 2, 3)$ such that

(a) $\mathcal{A}_j$ is a direct product $\prod_{i \in I_j} \mathcal{A}_i$ for $j = 1, 2, 3$;
(b) $\mathcal{A}_0$ is a direct product $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$.

According to our construction, $\mathcal{A}_1$ is 2-atomic, $\mathcal{A}_2$ is $\aleph_0$-atomic and $\mathcal{A}_3$ is c-atomic. We also obtain that $\mathcal{A}$ is a direct product $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \mathcal{A}_4$. The proof is complete. $\square$

Let us remark that if $u^0 = u$, then $\mathcal{A}_4$ is a trivial direct factor, i.e., $A_4 = \{0\}$. Next, some of the sets I_1, I_2 or I_3 can be empty. E.g., if $I_1 = \emptyset$, then $A_1 = \{0\}$, and analogously in the case $I_2 = \emptyset$ or $I_3 = \emptyset$.

3. Examples and counterexamples

3.1. Let α be an infinite cardinal. In [4] a linearly ordered MV-algebra $\mathcal{A}_\alpha$ was constructed such that whenever $0 < a \in A_\alpha$, then a is an α-atom in $\mathcal{A}_\alpha$. The algebra $\mathcal{A}_\alpha$ is not archimedean.

3.2. Let R and Z be as above. Next, let Q be the additive group of all rationals with the natural linear order. We choose $0 < r_0 \in R$, $0 < z_0 \in Z$ and $0 < q_0 \in Q$. We can construct MV-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that A_1 is the interval $[0, r_0]$ of R, A_2 is the interval $[0, z_0]$ of Z, A_3 is the interval $[0, q_0]$ of Q and the corresponding lattice ordered groups are R, Z and Q, respectively. Then all $\mathcal{A}_i$ are archimedean and linearly ordered, $\mathcal{A}_1$ is c-atomic, $\mathcal{A}_2$ is 2-atomic and $\mathcal{A}_3$ is $\aleph_0$-atomic. By forming direct products of replicas of $\mathcal{A}_1, \mathcal{A}_2$ or $\mathcal{A}_3$, respectively, we obtain MV-algebras of arbitrarily large cardinalities which are atomic, $\aleph_0$-atomic or c-atomic.

3.3. Let $\mathcal{A}_3$ be as in 3.2. Then $\mathcal{A}_3$ is orthogonally complete but fails to be complete.

3.4. There exists a Boolean algebra B such that B is complete, $\operatorname{card} B > 1$ and B has no atom. Let $\mathcal{A}$ be an MV-algebra which is constructed from B as in the concluding part of [3]. Then $\mathcal{A}$ is complete, $\operatorname{card} A > 1$ and for each cardinal α with $\alpha > 1$, there are no α-atoms in $\mathcal{A}$.

3.5. Put $G = Z \circ (Z \times Z)$, where the symbol Z denotes the operation of lexicographic product. Put $u = (1, 0, 0)$. Then u is a strong unit in G, hence we can construct the corresponding MV-algebra $\mathcal{A}$ (by means of [3], 1.3); the underlying set of $\mathcal{A}$ is the interval $[0, u]$ of G. The MV-algebra $\mathcal{A}$ is orthogonally complete and atomic, but it is not a direct product of linearly ordered MV-algebras.

Added in proof: In the forthcoming monograph R. Cignoli, I. M. L. D'Ottaviano, D. Mundici: Algebraic Foundations of Many-valued Reasoning a different terminology for MV-algebras is used; instead of the above term "archimedean", the term "semisimple" is applied.

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