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L^p-DISCREPANCY AND STATISTICAL INDEPENDENCE OF SEQUENCES

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Dedicated to Prof. Tibor Šalát on the occasion of his 70th birthday

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Abstract. We characterize statistical independence of sequences by the L^p-discrepancy and the Wiener L^p-discrepancy. Furthermore, we find asymptotic information on the distribution of the L^2-discrepancy of sequences.

Keywords: sequences, statistical independence, discrepancy, distribution functions

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1. Introduction

Let x_n and y_n be two infinite sequences in the unit interval $[0,1)$. The pair of sequences (x_n, y_n) is called *statistically independent* if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f(x_n)g(y_n) - \frac{1}{N^2}\sum_{n=1}^{N} f(x_n)\sum_{n=1}^{N} g(y_n)\right) = 0$$

for all continuous real functions f, g defined on $[0,1]$, cf. [11]. In other words, the double sequence (x_n, y_n) is called statistically independent if it has statistically independent coordinate sequences x_n and y_n.

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For (x_n, y_n) and any $p > 0$ we define the L^p statistical independence discrepancy ${}_S D_N^{(p)}$, the Wiener L^p statistical independence discrepancy ${}_S W_N^{(p)}$, and the statistical independence *star* discrepancy ${}_S D_N^*$ by the following: denote

$$F_N(x,y) := \frac{1}{N}\sum_{n=1}^{N} \chi_{[0,x)}(x_n)\chi_{[0,y)}(y_n),$$

where $\chi_{[0,x)}(t)$ is the *characteristic function* of the interval $[0,x)$. Then

$$\begin{aligned} {}_S D_N^{(p)} &:= \int_0^1 \int_0^1 \left|F_N(x,y) - F_N(x,1)F_N(1,y)\right|^p \mathrm{d}x\,\mathrm{d}y, \\ {}_S W_N^{(p)} &:= \int_{\mathcal{C}_0}\int_{\mathcal{C}_0} \left| \frac{1}{N}\sum_{n=1}^{N} f(x_n)g(y_n) - \frac{1}{N^2}\sum_{n=1}^{N} f(x_n)\sum_{n=1}^{N} g(y_n)\right|^p \mathrm{d}f\,\mathrm{d}g, \\ {}_S D_N^* &:= \sup_{x,y\in[0,1]} \left|F_N(x,y) - F_N(x,1)F_N(1,y)\right|, \end{aligned} \tag{1.1}$$

where $\mathrm{d}f$ is the Wiener measure on the set $\mathcal{C}_0$ of all continuous functions defined on $[0,1]$ satisfying $f(0) = 0$. Furthermore, we write ${}_S D_N^{(p)} = {}_S D_N^{(p)}(x_n, y_n)$ and similarly for ${}_S W_N^{(p)}$ and ${}_S D_N^*$.

These definitions of discrepancy originate from the theory of uniform distribution of sequences, where the star discrepancy, the L^p-discrepancy and the Wiener discrepancy are given by

$$\begin{aligned} D_N^*(x_n) &= \sup_{x\in[0,1]} \left|F_N(x) - x\right|, \\ D_N^{(p)} &= \int_0^1 \left|F_N(x) - x\right|^p \,\mathrm{d}x, \\ W_N^{(p)} &= \int_{\mathcal{C}_0} \left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right|^p \mathrm{d}f, \end{aligned} \tag{1.2}$$

where $F_N(x) := \frac{1}{N}\sum_{n=1}^{N} \chi_{[0,x)}(x_n)$. Again, a sequence x_n is called uniformly distributed, if $D_N^*(x_n)$ tends to 0 for $N \to \infty$. This is equivalent to $\lim_{N\to\infty} D_N^{(p)} = 0$ and $\lim_{N\to\infty} W_N^{(p)} = 0$ (cf. [9]).

The following explicit formulæ for statistical independence discrepancies are known. In [5] the following formula is given:

$$ {}_S D_N^{(2)} = \frac{1}{16\pi^4} \sum_{\substack{k,l=-\infty \\ k,l\neq 0}}^{\infty} \frac{1}{k^2 l^2} \left| \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\pi\mathrm{i}(kx_n + ly_n)} - \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} \mathrm{e}^{2\pi\mathrm{i}(kx_n + ly_m)} \right|^2. \tag{1.3}$$

Furthermore, in [13] an alternative expression is presented:

$$
\begin{aligned}
(1.4)\qquad {}_S D_N^{(2)} &= \frac{1}{N^2}\sum_{m,n}^{N}\bigl(1-\max(x_m,x_n)\bigr)\bigl(1-\max(y_m,y_n)\bigr)\\
&+\frac{1}{N^4}\sum_{m,n,k,l=1}^{N}\bigl(1-\max(x_m,x_k)\bigr)\bigl(1-\max(y_n,y_l)\bigr)\\
&-\frac{2}{N^3}\sum_{m,k,l=1}^{N}\bigl(1-\max(x_m,x_k)\bigr)\bigl(1-\max(y_m,y_l)\bigr).
\end{aligned}
$$

For the Wiener L^2 statistical independence discrepancy in [13] we have

$$
\begin{aligned}
(1.5)\qquad {}_S W_N^{(2)} &= \frac{1}{N^2}\sum_{m,n}^{N}\frac{\min(x_m,x_n)}{2}\frac{\min(y_m,y_n)}{2}\\
&+\frac{1}{N^4}\sum_{m,n,k,l=1}^{N}\frac{\min(x_m,x_n)}{2}\frac{\min(y_k,y_l)}{2}\\
&-\frac{2}{N^3}\sum_{m,k,l=1}^{N}\frac{\min(x_m,x_k)}{2}\frac{\min(y_m,y_l)}{2}.
\end{aligned}
$$

These are extensions of classical formulæ, which can be found in [9]. The notion of Wiener discrepancy was introduced in [13].

In [5] it is proved that $\lim_{N\to\infty} {}_S D_N^* = 0$ does not characterize the statistical independence of (x_n, y_n). On the other hand, $\lim_{N\to\infty} {}_S D_N^{(p)} = 0$ for $p = 2$ is a characterization and it has been conjectured that the same is true also for any $p > 0$. In Section 2 we will prove this conjecture and we will also prove the same for the Wiener discrepancy ${}_S W_N^{(p)}$. Moreover, we will see that the statistical independence is fully described by the set of distribution functions of a given sequence (x_n, y_n).

In [13] it is proved that ${}_S W_N^{(2)} = \frac{1}{4} {}_S D_N^{(2)}$, but a similar relation for ${}_S W_N^{(p)}$, $p > 0$ is not valid, which we will demonstrate in Section 4.

In Section 3 of this paper we will discuss the asymptotical distribution of L^2-discrepancy. This continues investigations of the star discrepancy due to Kolmogorov [8]. It is now well-known that

$$
(1.6)\qquad \lim_{N\to\infty}\mathbb{P}\left(\sqrt{N}D_N^*(x_n) < t\right) = \sum_{k=-\infty}^{\infty}(-1)^k \mathrm{e}^{-2k^2t^2}.
$$

We will make use of a heuristic approach to this result due to Doob [4], which has been justified by Donsker [3]. The heuristics states that the discrepancy function

$F_N(x) - x$ behaves like a trajectory of the Wiener process. Especially this behaviour holds for continuous functionals of the discrepancy function, as the supremum or the L^p-norm.

2. Statistical independence

As we have mentioned in the introduction, the equivalence

$$(x_n, y_n) \text{ is statistically independent } \iff \lim_{N\to\infty} {}_S D_N^{(2)} = 0$$

was proved in [5]. We shall extend this characterization of statistical independence to any $p > 0$. To do this we need the following notation:

For a given infinite sequence (x_n, y_n) in $[0,1)^2$, let $G(x_n, y_n)$ be the set of all distribution functions of (x_n, y_n).

Here $g\colon [0,1]^2 \to [0,1]$ is a distribution function of (x_n, y_n) if there exists an increasing sequence of indices $N_1 < N_2 < \dots$ such that $\lim_{k\to\infty} F_{N_k}(x,y) = g(x,y)$ for every point $(x,y) \in [0,1]^2$. Following [9, p. 54] two distribution functions g_1 and g_2 are considered to be equivalent, if $g_1(x,y) = g_2(x,y)$ a.e. on $[0,1]^2$ or equivalently, $g_1(x,y) = g_2(x,y)$ for every $(x,y) \in [0,1]^2$ if both g_1 and g_2 are continuous.

Theorem 1. *For any sequence (x_n, y_n) in $[0,1)^2$ and any $p > 0$ we have*

$$(x_n, y_n) \textit{ is statistically independent} \iff \lim_{N\to\infty} {}_S D_N^{(p)} = 0.$$

P r o o f. By the well known first Helly lemma and the Lebesgue theorem of dominated convergence we have

$$\lim_{N\to\infty} \int_0^1\!\!\int_0^1 \bigl|F_N(x,y) - F_N(x,1)F_N(1,y)\bigr|^p \,\mathrm{d}x\,\mathrm{d}y = 0 \iff$$
$$\forall\bigl(g \in G(x_n, y_n)\bigr) \int_0^1\!\!\int_0^1 |g(x,y) - g(x,1)g(1,y)|^p \,\mathrm{d}x\,\mathrm{d}y = 0.$$

The right hand side is true for all $p > 0$, and for $p = 2$, the left hand side characterizes the statistical independence. Thus the proof is complete. □

The following is an immediate consequence of the above proof:

Theorem 2. *For every $(x_n, y_n) \in [0,1)^2$,*

$$(x_n, y_n) \textit{ is statistically independent} \iff$$
$$\forall\bigl(g \in G(x_n, y_n)\bigr) g(x,y) = g(x,1)g(1,y) \textit{ a.e. on } [0,1]^2.$$

Using the proof of Theorem 1 with Remark 1 in [13] and observing that any neighbourhood in the supremum topology in $\mathcal{C}_0$ has a positive Wiener measure, we have a condition for statistical independence in terms of the Wiener statistical independence discrepancy.

Theorem 3. *For any $p > 0$ the sequence (x_n, y_n) is statistically independent, if and only if*

$$\lim_{N\to\infty} {}_S W_N^{(p)} = 0.$$

Using Theorem 2 we can describe the case when the star discrepancy ${}_S D_N^*$ tends to 0.

Theorem 4. *If $G(x_n, y_n)$ contains only continuous distribution functions, then*

$$(x_n, y_n) \text{ is statistically independent} \Longleftrightarrow \lim_{N\to\infty} {}_S D_N^* = 0.$$

P r o o f. The case $\Longleftarrow$ follows immediately. The implication $\Longrightarrow$ follows from Theorem 2 and the fact that, for continuous $g \in G(x_n, y_n)$, the convergence

$$\lim_{k\to\infty} F_{N_k}(x, y) = g(x, y)$$

is uniform in $[0,1]^2$. Hence we have $\lim_{k\to\infty} {}_S D_{N_k}^* = 0$ and this leads to $\lim_{N\to\infty} {}_S D_N^* = 0$. $\square$

In [14] it is shown that one can use the Wiener-Schoenberg theorem for the proof of continuity of $g \in G(x_n)$ (cf. the monograph of L. Kuipers and H. Niederreiter [9, Th. 7.5, p. 55]). The same method can be used for $G(x_n, y_n)$.

3 Uniform distribution

In order to describe the asymptotic distribution function of the L^2-discrepancy, we use a theorem due to Donsker [3] and the well-known Feynman-Kac formula (cf. [7]). Donsker's theorem states that for a functional F, which is continuous in the uniform topology on the space of sample paths of the Wiener process, the following limit relation holds:

$$\text{(3.1)} \qquad \lim_{N\to\infty} \mathbb{P}\left(F\left(\sqrt{N}\left(F_N(x) - x\right)\right) \leqslant \alpha\right) = \mathbb{P}\left(F\left(x(.)\right) \leqslant \alpha\right),$$

where $x(t)$ is a trajectory of the Wiener process with $x(0) = x(1) = 0$.

The Feynman-Kac formula relates the Laplace transform of the distribution function of the integral $\int_0^t V(x(\tau))\,\mathrm{d}\tau$ (V is a positive function) to the solutions of the eigenvalue problem

$$\frac{1}{2}\psi''(x) - V(x)\psi(x) = -\lambda\psi(x), \quad \psi \in L^2(-\infty,\infty). \tag{3.2}$$

The relation is given by the formula

$$\mathbb{E}\left(\exp\left(-\int_0^t V(x(\tau))\,\mathrm{d}\tau\right) \,\middle|\, x(t)=0\right) = \sqrt{2\pi t}\sum_n \mathrm{e}^{-\lambda_n t}\psi_n(0)^2, \tag{3.3}$$

where λ_n are the eigenvalues and ψ_n are the corresponding normalized eigenfunctions of (3.2).

In order to get information on the distribution function of L^2-discrepancy we have to study equation (3.2) for $V(x) = x^2$. Clearly, this procedure could also be applied for $V(x) = |x|^p$ to study the distribution of L^p-discrepancy, but it is not enough known to get as precise information as in the L^2-case. We will write

$$\Phi(T) = \lim_{N\to\infty} \mathbb{P}\left(\sqrt{N}D_N^{(2)} < T\right) \tag{3.4}$$

for the limit distribution of the L^2-discrepancy.

First, we notice that by the rescaling property of the Wiener process we have

$$\mathbb{E}\left(\exp\left(-\int_0^t x(\tau)^2\,\mathrm{d}\tau\right) \,\middle|\, x(t)=0\right) = \mathbb{E}\left(\exp\left(-t^2\int_0^1 x(\tau)^2\,\mathrm{d}\tau\right) \,\middle|\, x(1)=0\right). \tag{3.5}$$

For the case studied here equation (3.2) has the form

$$\frac{1}{2}\psi''(x) - x^2\psi(x) = -\lambda\psi(x),$$

which is the differential equation for the Hermite functions (cf. [10,p. 253]). Thus we have $\lambda_n = \frac{2n+1}{\sqrt{2}}$ and

$$\psi_n(x) = \frac{\sqrt[8]{2}}{\sqrt[4]{\pi}}\frac{1}{2^n\sqrt{(2n)!}}\mathrm{e}^{-\frac{x^2}{\sqrt{2}}}H_n\left(\sqrt[4]{2}x\right),$$

where H_n are the Hermite polynomials as defined in [10,p. 249]. Hence we derive

$$\begin{aligned}\mathbb{E}\left(\exp\left(-\int_0^t x(\tau)^2\,\mathrm{d}\tau\right) \,\middle|\, x(t)=0\right) &= \sqrt{2\sqrt{2}t}\sum_{n=0}^{\infty}\exp\left(-\frac{4n+1}{\sqrt{2}}t\right)\frac{1}{4^n}\binom{2n}{n} = \\ &= \sqrt{\frac{\sqrt{2}t}{\sinh\sqrt{2}t}}.\end{aligned}$$

Using (3.5) we obtain

$$\mathbb{E}\left(\exp\left(-s\int_0^1 x(\tau)^2\,\mathrm{d}\tau\right)\,\middle|\,x(1)=0\right)=\sqrt{\frac{\sqrt{2s}}{\sinh\sqrt{2s}}}$$

for the Laplace transform of the distribution function of the limit distribution of $N(D_N^{(2)})^2$. Notice that this function is holomorphic in the region $\Re s > -\frac{\pi^2}{2}$. Furthermore, it has a branch cut of the square-root type at the point $s=-\frac{\pi^2}{2}$. Thus using the Laplace inversion theorem and asymptotic techniques for the Laplace transform (cf. [2]) we obtain

$$\Phi(T)=1-\frac{1}{\sqrt{\pi T}}\mathrm{e}^{-\frac{\pi^2}{2}T}+O\left(\frac{1}{T^{\frac{3}{2}}}\mathrm{e}^{-\frac{\pi^2}{2}T}\right). \tag{3.6}$$

We remark here that for the case of L^p-discrepancy the whole procedure also works. Again the Laplace transform of the distribution function is holomorphic in a region $\Re s > -\varepsilon$ for some $\varepsilon > 0$, but this is a consequence of (1.6). We could not derive this analytic information from the knowledge of the asymptotics of the eigenvalues and eigenfunctions (cf. [15], [12]), nor could we find the location of the singularity of the largest real part, whose type would yield asymptotic information on the limiting distribution of the L^p-discrepancy.

4. Relation between Wiener and classical L^2 discrepancy

We start with the Paley-Wiener formula (cf. [1]):

$$\int_{\mathcal{C}_0} F\left[\int_0^1 f(x)\,\mathrm{d}m(x)\right]\mathrm{d}f=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2}F(bu)\,\mathrm{d}u,\quad b^2=\int_0^1 m^2(t)\,\mathrm{d}t,$$

where $F(u)$ is a (real or complex-valued) measurable function defined on $(-\infty,\infty)$ such that $\mathrm{e}^{-u^2}F(bu)$ is of class L_1 and $m(1)=0$. Thus, putting $F(u)=|u|^p$ and $m(x)=F_N(x)-x$, in the classical case we have

$$W_N^{(p)}=\frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{p+1}{2}\Big)\left(D_N^{(2)}\right)^{\frac{p}{2}}.$$

Assuming $m(x,y)=m_1(x)m_2(y)$ on $[0,1]^2$ and $m_1(1)=m_2(1)=0$, the Paley-Wiener formula can also be used for computing the two-dimensional integral

$$\int_{\mathcal{C}_0}\int_{\mathcal{C}_0} F\left[\int_0^1\int_0^1 f(x)g(y)\,\mathrm{d}m(x,y)\right]\mathrm{d}f\,\mathrm{d}g.$$

For any x_1, x_2 and y_1, y_2 in $[0,1)$, there exist $m_1(x)$ and $m_2(y)$, $m_1(1) = m_2(1) = 0$, such that $F_2(x,y) - F_2(x,1)F_2(1,y) = m_1(x)m_2(y)$ $(x, y \in [0,1])$. Hence

$$_SW_2^{(p)} = \frac{1}{\pi}\Gamma^2\Big(\frac{p+1}{2}\Big)\left({}_SD_2^{(2)}\right)^{\frac{p}{2}}$$

for every $p > 0$.

The proof of $_SW_N^{(2)} = \frac{1}{4}{}_SD_N^{(2)}$ in [13] is also extremely simple: Using (1.3) we have

$$_SD_N^{(2)}(x_n, y_n) = {}_SD_N^{(2)}(1 - x_n, 1 - y_n)$$

and using $1 - \max(x_m, x_n) = \min(1 - x_m, 1 - x_n)$ and (1.5) we have the result.

These results give rise to the question whether there is a relation of the type

$$_SW_N^{(p)} = c_p\left({}_SD_N^{(2)}\right)^{\frac{p}{2}} \tag{4.1}$$

between the different notions of statistical independence discrepancy. In the following we give explicit formulae for these discrepancies which lead to the negative answer.

The Paley-Wiener formula is equivalent to

$$\int_{\mathcal{C}_0}\left(\int_0^1 f(x)\,\mathrm{d}m(x)\right)^{2k}\mathrm{d}f = \frac{(2k-1)!!}{2^k}\left(\int_0^1 \mathrm{d}t\left(\int_0^1 \chi_{[t,1]}(x)\,\mathrm{d}m(x)\right)^2\right)^k,$$

where $k = 1, 2, \ldots$, and $(2k-1)!! = (2k-1)(2k-3)\ldots 3\cdot 1$ and for the exponent $2k+1$ the left hand integral is zero. (For this formula the assumption $m(1) = 0$ is superfluous.) The formal two-dimensional analogue is the relation $A = cB$, where

$$A := \int_{\mathcal{C}_0}\int_{\mathcal{C}_0}\left(\int_0^1\int_0^1 f(x)g(y)\,\mathrm{d}m(x,y)\right)^{2k}\mathrm{d}f\,\mathrm{d}g,$$
$$B := \left(\int_0^1\int_0^1\left(\int_0^1\int_0^1 \chi_{[t_1,1]}(x)\chi_{[t_2,1]}\,\mathrm{d}m(x,y)\right)^2\mathrm{d}t_1\,\mathrm{d}t_2\right)^k,$$

and c is independent of $m(x,y)$. These integrals can be expressed as

$$A = \int_0^1\cdots\int_0^1\left(\int_{\mathcal{C}_0} f(u_1)\ldots f(u_{2k})\,\mathrm{d}f\right)\left(\int_{\mathcal{C}_0} g(v_1)\ldots g(v_{2k})\,\mathrm{d}g\right)$$
$$\mathrm{d}m(u_1,v_1)\ldots\,\mathrm{d}m(u_{2k},v_{2k}),$$
$$B = \int_0^1\cdots\int_0^1 (\min(u_1,u_2)\ldots\min(u_{2k-1},u_{2k}))\ (\min(v_1,v_2)\ldots\min(v_{2k-1},v_{2k}))$$
$$\mathrm{d}m(u_1,v_1)\ldots\,\mathrm{d}m(u_{2k},v_{2k}).$$

Furthermore, by the well known formula (which can also be proved by applying the above Paley-Wiener formula)

$$\int_{\mathcal{C}_0} f(u_1)\dots f(u_{2k})\,\mathrm{d}f = \frac{(2k-1)!!}{2^k(2k)!}\sum_{\pi}\min(u_{\pi(1)},u_{\pi(2)})\dots\min(u_{\pi(2k-1)},u_{\pi(2k)}),$$

where the summation $\sum_{\pi}$ ranges over all permutations π of $(1,\dots,2k)$. For the odd case $2k+1$ the integral vanishes. Next we choose $m(x,y)$ such that $\mathrm{d}m(a_i,b_i)=z_i$ for $i=1,\dots 2k$, and $\mathrm{d}m(x,y)=0$ otherwise. Here we shall view z_i as independent variables. Assuming $A=cB$ and comparing the coefficients at $z_1\dots z_{2k}$, we have $C=c'D$, where

$$\begin{aligned} C := {} & \sum_{\pi}\bigl(\min(a_{\pi(1)},a_{\pi(2)})\dots\min(a_{\pi(2k-1)},a_{\pi(2k)})\bigr)\times \\ & \times\sum_{\pi}\bigl(\min(b_{\pi(1)},b_{\pi(2)})\dots\min(b_{\pi(2k-1)},b_{\pi(2k)})\bigr), \\ D := {} & \sum_{\pi}\bigl(\min(a_{\pi(1)},a_{\pi(2)})\dots\min(a_{\pi(2k-1)},a_{\pi(2k)})\bigr)\times \\ & \times\bigl(\min(b_{\pi(1)},b_{\pi(2)})\dots\min(b_{\pi(2k-1)},b_{\pi(2k)})\bigr). \end{aligned}$$

Putting $a_i=b_i$, $i=1,\dots,2k$, we have

$$\begin{aligned} & \Bigl(\sum_{\pi}\bigl(\min(a_{\pi(1)},a_{\pi(2)})\dots\min(a_{\pi(2k-1)},a_{\pi(2k)})\bigr)\Bigr)^2 \\ & = c'\sum_{\pi}\bigl(\min(a_{\pi(1)},a_{\pi(2)})\dots\min(a_{\pi(2k-1)},a_{\pi(2k)})\bigr)^2, \end{aligned}$$

which is impossible, for $k>1$ and general a_i.

The proof of impossibility of (4.1) is more difficult. First, we have mentioned that for

$$m(x,y)=F_N(x,y)-F_N(x,1)F_N(1,y)$$

we have $A={}_SW_N^{(2k)}$ and $B=({}_SD_N^{(2)})^k$. Moreover, $\mathrm{d}m(x,y)\neq 0$ only for $x=x_m$ and $y=y_n$, where $1\leqslant m,n\leqslant N$. Precisely, assuming that $x_1,\dots,x_N$ and $y_1,\dots,y_N$ are one-to-one we have

$$\mathrm{d}m(x_m,y_n)=\begin{cases}\frac{1}{N}-\frac{1}{N^2} & \text{if } m=n,\\ -\frac{1}{N^2} & \text{in other cases.}\end{cases}$$

For brevity, we shall use the following notations:

$$\begin{aligned}
&\mathbf{m} := (m_1, \ldots, m_{2k}),\\
&\pi(\mathbf{m}) := (m_{\pi(1)}, \ldots, m_{\pi(2k)}),\\
&\mathbf{x_m} := (x_{m_1}, \ldots, x_{m_{2k}}),\\
&\mathbf{1} \leqslant \mathbf{m} \leqslant \mathbf{N} \iff 1 \leqslant m_1 \leqslant N \wedge \ldots \wedge 1 \leqslant m_{2k} \leqslant N,\\
&l(\mathbf{m}, \mathbf{n}) := \#\{1 \leqslant i \leqslant 2k; m_i = n_i\},\\
&\mu(\mathbf{x_m}) := \prod_{i=1}^{k} \min(x_{m_{2i-1}}, x_{m_{2i}}).
\end{aligned}$$

Computing the integrals A and B for such $m(x, y)$ we can find

$$\begin{aligned}
{}_SW_N^{(2k)} &= \frac{1}{N^{4k}} \left(\frac{1}{2^{2k}k!}\right)^2 \sum_{\substack{\mathbf{1} \leqslant \mathbf{m} \leqslant \mathbf{N}\\ \mathbf{1} \leqslant \mathbf{n} \leqslant \mathbf{N}}} \mu(\mathbf{x_m})\mu(\mathbf{y_n}) \times\\
&\quad \times \sum_{\pi_1, \pi_2} (N-1)^{l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))} \cdot (-1)^{2k - l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))},\\
\left({}_SD_N^{(2)}\right)^k &= \frac{1}{N^{4k}} \sum_{\substack{\mathbf{1} \leqslant \mathbf{m} \leqslant \mathbf{N}\\ \mathbf{1} \leqslant \mathbf{n} \leqslant \mathbf{N}}} \mu(\mathbf{x_m})\mu(\mathbf{y_n}) \times\\
&\quad \times (N-1)^{l(\mathbf{m}, \mathbf{n})} \cdot (-1)^{2k - l(\mathbf{m}, \mathbf{n})}.
\end{aligned}$$

We can regard $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ as independent variables. Then we see that ${}_SW_N^{(2k)}$ and $\left({}_SD_N^{(2)}\right)^k$ are homogeneous polynomials of the degree k in $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$, respectively.

In the following denote

$$x_a = \max_{1 \leqslant i \leqslant N} x_i, \; x_b = \max_{1 \leqslant i \leqslant N, i \neq a} x_i, \; y_c = \max_{1 \leqslant i \leqslant N} y_i, \; y_d = \max_{1 \leqslant i \leqslant N, i \neq c} y_i,$$

and let $a \neq c$ and $b = d$. Next we shall find coefficients of $x_a^{k-1} x_b y_c^{k-1} y_d$ in ${}_SW_N^{(2k)}$ and $\left({}_SD_N^{(2)}\right)^k$, respectively.

First, $\mu(\mathbf{x_m}) = x_a^{k-1} x_b$ only for

$$\mathbf{m} = \begin{cases} (a, \ldots, a, b, a, \ldots, a) \text{ (type I)},\\ (a, \ldots, a, b, b, a, \ldots, a) \text{ (type II)}, \end{cases}$$

where the couple (b, b) lies at the place with indices $(2i-1, 2i)$. We have $2k$ vectors of type I and $k(2k-1)$ vectors of type II. If $\mathbf{m}$ is of type I and π ranges over all permutations of $(1,\ldots,2k)$, then all vectors of type I occur in $\pi(\mathbf{m})$ $(2k-1)!$ times. If $\mathbf{m}$ is of type II, then all vectors of the form

$$(a,\ldots,a,b,a,\ldots,a,b,a,\ldots,a) \text{ (type II}'\text{)}$$

occur in $\pi(\mathbf{m})$ with multiplicity $2.(2k-2)!$. For $(\mathbf{m},\mathbf{n})$ of type (I,I) we have $l(\mathbf{m},\mathbf{n}) = 1$ in $2k$ cases and $l(\mathbf{m},\mathbf{n}) = 0$ in $(2k)^2 - k$ cases. For $(\mathbf{m},\mathbf{n})$ of type (I,II) we have $l(\mathbf{m},\mathbf{n}) = 1$ in $2k$ cases and $l(\mathbf{m},\mathbf{n}) = 0$ in $2k^2 - 2k$ cases. For $(\mathbf{m},\mathbf{n})$ of type (II,II) we have only $l(\mathbf{m},\mathbf{n}) = 2$ in k cases and $l(\mathbf{m},\mathbf{n}) = 0$ in $k^2 - k$ cases. Similarly, for type (I,II$'$) we have

$$l(\mathbf{m},\mathbf{n}) = \begin{cases} 1 \text{ in } 2k(2k-1) \text{ cases,} \\ 0 \text{ in } k(2k-1)(2k-2) \text{ cases,} \end{cases}$$

and for (II$'$,II$'$) we have

$$l(\mathbf{m},\mathbf{n}) = \begin{cases} 2 \text{ in } k(2k-1) \text{ cases,} \\ 1 \text{ in } 2k(2k-1)(2k-2) \text{ cases,} \\ 0 \text{ in } k(2k-1)(k-1)(2k-3) \text{ cases.} \end{cases}$$

Summing up all of the above we have

$$\begin{aligned} \sum_{\substack{\mathbf{1}\leqslant\mathbf{m}\leqslant\mathbf{N} \\ \mathbf{1}\leqslant\mathbf{n}\leqslant\mathbf{N} \\ \mu(\mathbf{x_m})=x_a^{k-1}x_b \\ \mu(\mathbf{y_n})=y_c^{k-1}y_d}} & (N-1)^{l(\mathbf{m},\mathbf{n})} \cdot (-1)^{2k-l(\mathbf{m},\mathbf{n})} \\ & = k(N-1)^2 - 6k(N-1) + 9k^2 - 7k, \end{aligned}$$

$$\sum_{\substack{\mathbf{1}\leqslant\mathbf{m}\leqslant\mathbf{N} \\ \mathbf{1}\leqslant\mathbf{n}\leqslant\mathbf{N} \\ \mu(\mathbf{x_m})=x_a^{k-1}x_b \\ \mu(\mathbf{y_n})=y_c^{k-1}y_d}} \sum_{\pi_1,\pi_2} (N-1)^{l(\pi_1(\mathbf{m}),\pi_2(\mathbf{n}))} \cdot (-1)^{2k-l(\pi_1(\mathbf{m}),\pi_2(\mathbf{n}))}$$
$$= ((2k)!)^2 \left((2k^2-k)(N-1)^2 - (8k^3-4k^2+2k)(N-1) + (4k^4-4k^3+3k^2-k)\right),$$

which is a contradiction to

$${}_SW_N^{(2k)} = c_{2k}\left({}_SD_N^{(2)}\right)^k.$$

Using the expressions (1.3), (1.4) and (1.5) we immediately have:

Theorem 5.

(i) *The sequences* (x_n, y_n), (y_n, x_n), $(1-x_n, y_n)$, $(1-x_n, 1-y_n)$ *and* $(t_1 x_n, t_2 x_n)$ *are simultaneously statistically independent. Here* $t_1, t_2 \in (0,1]$, *and in the case* $x_n = 0$ *we reduce* $1 - x_n \bmod 1$.

(ii) (c, y_n) *is statistically independent with any* $y_n, c \in [0,1)$, *where c is a constant.*

Using an example given in [5] we will generalize (ii) in the following way. Define, for $\alpha \in [0,1]$, the *one-jump* distribution function $c_\alpha(x)$ as

$$c_\alpha(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < \alpha, \\ 1, & \text{for } \alpha < x \leqslant 1. \end{cases}$$

Theorem 6. *Assume that the sequence* x_n *in* $[0,1)$ *has the limit law* c_α, *i.e.* $\lim_{N\to\infty} F_N(x) = c_\alpha(x)$ *a.e. Then for any sequence* y_n *in* $[0,1)$ (x_n, y_n) *is statistically independent.*

Proof. For a continuous $g\colon [0,1] \to \mathbb{R}$ we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) g(y_n) - \frac{1}{N^2} \sum_{n=1}^{N} f(x_n) \sum_{n=1}^{N} g(y_n) \right| \leqslant 2 \sup_{x \in [0,1]} |g(x)| \frac{1}{N} \sum_{n=1}^{N} |f(x_n) - f(\alpha)|,$$

and for a continuous $f\colon [0,1] \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f(x_n) - f(\alpha)| = \int_0^1 |f(x) - f(\alpha)| \, \mathrm{d}c_\alpha(x) = 0.$$

□

Theorem 7. *For sequences* x_n, y_n, x'_n *and* y'_n *in* $[0,1)$ *we assume that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (|x_n - x'_n| + |y_n - y'_n|) = 0.$$

Then the sequences (x_n, y_n) *and* (x'_n, y'_n) *are simultaneously statistically independent.*

Proof. This follows from the expression (1.5) and from the fact that

$$\big| |x - y||u - v| - |x' - y'||u' - v'| \big| \leqslant |x - x'| + |y - y'| + |u - u'| + |v - v'|$$

for $x, y, u, v, x', y', u', v' \in [0,1]$. □

Motivated by Theorem 2, a trivial example of statistical independence is given by a sequence (x_n, y_n) which is uniformly distributed in the square. Another example is any sequence (x_n, y_n) which has only one-jump distribution functions. A more general example:

Let G_1 and G_2 be any nonempty closed and connected sets of one-dimensional distribution functions. Denote

$$G_1 \cdot G_2 := \{g_1(x)g_2(y); g_1 \in G_1, g_2 \in G_2\}.$$

Again $G_1 \cdot G_2$ is nonempty closed and connected and thus by R. Winkler [16] there exists a sequence (x_n, y_n) in $[0,1)^2$ such that $G(x_n, y_n) = G_1 \cdot G_2$. By Theorem 2, this sequence is statistically independent.

Furthermore, Theorem 2 may be used for a generalization of the notion of statistical independence to the multidimensional sequence $(x_n, y_n, z_n, \dots)$ in $[0,1)^s$ (precisely, the statistical independence of its coordinate sequences $x_n, y_n, z_n, \dots$) as follows:

$(x_n, y_n, z_n, \dots)$ is statistically independent if, for every distribution function $g \in G(x_n, y_n, z_n, \dots)$ we have

$$g(x, y, z, \dots) = g(x, 1, 1, \dots)g(1, y, 1, \dots)g(1, 1, z, \dots)\dots$$

a.e. on $[0,1]^s$. As an example we give the following sequences described in [6]:

Let $\mathbf{x}_n$ be defined by

$$\mathbf{x}_n = \left((-1)^{[[\log^{(j)} n]^{1/p_1}]} [\log^{(j)} n]^{1/p_1}, \dots, (-1)^{[[\log^{(j)} n]^{1/p_s}]} [\log^{(j)} n]^{1/p_s} \right) \bmod 1,$$

where $\log^{(j)} n$ denotes the jth iterated logarithm $\log \dots \log n$, and $p_1, \dots, p_s$ are coprime positive integers. Then, for $j > 1$, the set of all distribution functions of $\mathbf{x}_n$ coincides (under equivalence) with the set of all one-jump distribution functions on $[0,1]^s$, and thus the sequence $\mathbf{x}_n$ is statistically independent.

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