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RADICAL CLASSES OF MV-ALGEBRAS

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The fundamental paper on MV-algebras is [1] containing the basic results on the algebraic aspects of Łukasiewicz multi-valued logics.

It is well-known that there exist relations between

(i) MV-algebras and abelian lattice ordered groups (cf. [14]),

and

(ii) MV-algebras and cyclically ordered groups (cf. [5]).

In the present paper we modify the method from (i) by applying certain partial algebras. Namely, we represent an MV-algebra by a bounded distributive lattice with a partial binary operation (partial addition).

In this context the notion of substructure of an MV-algebra is defined in a natural way.

Radical classes of lattice ordered groups were investigated in [3], [4], [6], [7], [8], [13] and [15]; for the case of generalized Boolean algebras cf. [12].

We recall that a nonempty class X of lattice ordered groups which is closed with respect to isomorphisms is defined to be a radical class if it satisfies the following conditions:

1) If $G_1 \in X$ and G_2 is a convex ℓ-subgroup of G_1, then $G_2 \in X$.
2) If H is a lattice ordered group and G_i $(i \in I)$ are convex ℓ-subgroups of H such that $G_i \in X$ for each $i \in I$, then $\bigvee_{i\in I} G_i$ belongs to X.

If, moreover, all lattice ordered groups belonging to X are abelian, then X is called abelian.

A nonempty class Y of MV-algebras which is closed with respect to isomorphisms will be called a radical class if the following conditions are satisfied:

1′) Whenever $\mathcal{A}_1 \in Y$ and $\mathcal{A}_2$ is a substructure of A_1, then $A_2 \in Y$.

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2′) If $\mathcal{B}$ is an MV-algebra and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are substructures of $\mathcal{B}$ such that $\mathcal{A}_i \in Y$ for $i = 1, 2, \ldots, n$, then $\bigvee_{i=1}^{n} \mathcal{A}_i$ belongs to Y.

Let $\mathcal{R}_a$ and $\mathcal{R}_m$ be the collection of all abelian radical classes of lattice ordered groups or the collection of all radical classes of MV-algebras, respectively. Both these collections are partially ordered by the class-theoretical inclusion.

We prove that the partially ordered collections $\mathcal{R}_a$ and $\mathcal{R}_m$ are isomorphic.

1. Preliminaries

For MV-algebras we apply the same definitions and the same notation as in [5]; cf. also [9].

Hence an MV-algebra is an algebraic structure $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$, where $\oplus, *$ are binary operations, $\neg$ is a unary operation and $0, 1$ are unary operations on A such that the identities (m_1)–(m_9) from Definition 11 in [5] are satisfied.

For the sake of completeness and for applications below we recall these conditions in detail:

(m_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(m_2) $x \oplus 0 = x$;
(m_3) $x \oplus y = y \oplus x$;
(m_4) $x \oplus 1 = 1$;
(m_5) $\neg\neg x = x$;
(m_6) $\neg 0 = 1$;
(m_7) $x \oplus \neg x = 1$;
(m_8) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;
(m_9) $x * y = \neg(\neg x \oplus \neg y)$.

If $\mathcal{A}$ is an MV-algebra and if we put, for any $x, y \in A$,

$$x \vee y = (x * \neg y) \oplus y, \tag{1}$$

$$x \wedge y = \neg(\neg x \vee \neg y), \tag{2}$$

then $(A; \wedge, \vee)$ is a distributive lattice with the least element 0 and the greatest element 1 (cf. [14]). We denote $(A; \wedge, \vee) = \mathcal{L}(\mathcal{A})$.

The relations between MV-algebras and lattice ordered groups are described in the following two fundamental theorems (cf. [14], Theorems 2.5 and 3.8).

1.1. Theorem. *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0,u]$ of G. For each a and b in A we put*

$$a \oplus b = (a+b) \wedge u, \quad \neg a = u - a, \tag{3}$$

$$a * b = \neg(\neg a \oplus \neg b). \tag{4}$$

Further we set $u = 1$. Then $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-aglebra.

If G and $\mathcal{A}$ are as in 1.1, then we denote $\mathcal{A} = \mathcal{A}_0(G; u)$.

1.2. Theorem. *Let $\mathcal{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G; u)$.*

If G_1 is another abelian lattice ordered group with a strong unit u_1 such that $\mathcal{A} = \mathcal{A}_0(G_1, u_1)$, then there exists an isomorphism φ of G onto G_1 such that $\varphi(a) = a$ for each $a \in A$. Hence, up to isomorphism, G is uniquely determined by $\mathcal{A}$.

2. Partial operation $+_A$

Again, let $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ be an MV-algebra. We consider the operations $\wedge$ and $\vee$ on A defined by (1) and (2) in Section 1. Thus $(A; \wedge, \vee)$ is a lattice; the corresponding partial order on A is denoted by $\leqslant$. Further, let G be as in 1.2; the symbol $+$ denotes the group operation on G. Then A is the interval $[0,u]$ of G.

Theorem 1.2 shows that the basic algebraic operations of $\mathcal{A}$ can be defined by applying the lattice operations $\wedge, \vee$ on $[0,u]$ and the group operation $+$ on G.

Let us remark that if $a_1, a_2 \in [0,u]$ then, in general, $a_1 + a_2$ need not belong to $[0,u]$; hence in applying the construction from 1.2 (cf. (3)) we deal also with elements of G which do not belong to A.

We will verify that for defining the basic algebraic operations of $\mathcal{A}$ it suffices to use

(i) the lattice operations on $[0,u]$, and
(ii) a partial binary operation $+_A$ defined on $[0,u] = A$;

this partial operation is defined as follows.

Let $a_1, a_2 \in A$. If $a_1 + a_2 \in A$, then we put

$$a_1 +_A a_2 = a_1 + a_2;$$

if $a_1 + a_2 \notin A$, then $a_1 +_A a_2$ is not defined.

2.1. Lemma. *Let $\mathcal{A}$ and $+_A$ be as above. Then the following conditions are satisfied:*

(a_1) *If $a_1, a_2 \in A$ and $a_1 +_A a_2$ is defined, then $a_2 +_A a_1$ is defined and $a_1 +_A a_2 = a_2 +_A a_1$.*

(a_2) *If $a_1, a_2, a_3 \in A$ and $a_1 +_A (a_2 +_A a_3)$ is defined, then $(a_1 +_A a_2) +_A a_3$ is defined and $a_1 +_A (a_2 +_A a_3) = (a_1 +_A a_2) +_A a_3$.*

(a_3) *If $a_1, a_2, a_3 \in A$, $a_1 +_A a_2$ and $a_1 +_A a_3$ are defined, then*

$$a_2 \leqslant a_3 \Leftrightarrow a_1 +_A a_2 \leqslant a_1 +_A a_3.$$

(a_4) *$a +_A 0 = a$ for each $a \in A$.*

(a_5) *If $a_1, a_2, a_3 \in A$, $a_1 < a_2$ and if $a_2 +_A a_3$ is defined , then $a_1 +_A a_3$ is defined.*

(a_6) *If $a_1, a_2 \in A$, $a_1 \leqslant a_2$, then there exists $x \in A$ such that $a_1 +_A x = a_2$. If, moreover, $a_1', a_2' \in A$, $a_1 \leqslant a_1' \leqslant a_2' \leqslant a_2$, $a_1' +_A x' = a_2'$, then $x' \leqslant x$.*

(a_7) *If $a_1, a_2, a_3 \in A$, $a_1 \wedge a_2 = 0 = a_1 \wedge a_3$ and if $a_2 +_A a_3$ is defined, then $a_1 \wedge (a_2 +_A a_3) = 0$.*

(a_8) *If $a_1, a_2, \in A$, then $a_1 +_A a_2$ is defined if and only if $(a_1 \wedge a_2) +_A (a_1 \vee a_2)$ is defined, and if this is the case, then $a_1 +_A a_2 = (a_1 \wedge a_2) +_A (a_1 \vee a_3)$.*

P r o o f. The validity of (a_1) - (a_8) is an immediate consequence of the definition of the operation $+_A$ in A and of the well-known properties of lattice ordered groups. □

The following result is easy to verify:

2.1.1. *Let $(A; \wedge, \vee)$ be a distributive lattice with the least element 0 and with a partial binary operation $+_A$ satisfying the conditions from* 2.1. *If $a_1, a_2 \in A$ and $a_1 +_A a_2$ is defined, then the mapping $\varphi(t) = t +_A a_2$ is an isomorphism of the lattice $[0, a_1]$ onto $[a_1, a_1 +_A a_2]$.*

2.2. Definition. Let A be a bounded lattice with the least element 0. Suppose that a partial binary operation $+_A$ on A is defined such that the conditions (a_1)-(a_8) from 2.1 are satisfied. Then $(A; +_A, \wedge, \vee, 0)$ is said to be an m-algebra.

2.3. Definition. Let $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ be an MV-algebra. Let the operations $\vee, \wedge$ be as in (1),(2) and let the partial operation $+_A$ be as in 2.1, where $\mathcal{A} = \mathcal{A}_0(G; u)$. Then the m-algebra $\mathcal{A}_1 = (A; +_A, \wedge, \vee, 0)$ will be said to be generated by $\mathcal{A}$ and it will be denoted by $\mathcal{A}^0$ or by $\mathcal{A}^0(G, u)$.

Now suppose that $(A; +_A, \wedge, \vee, 0)$ is an m-algebra. Let u be the greatest element of the lattice $(A; \wedge, \vee)$. Put $u = 1$.

Let $a \in A$. In view of (a_3), (a_6) and (a_4) there exists a uniquely determined element x in A such that $a +_A x = u$. We put

$$x = \neg a.$$

2.4. Lemma. *The unary operation $\neg$ on A satisfies the conditions* (m_5) *and* (m_6).

P r o o f. (m_5) is a consequence of (a_1). From (a_4) we infer that (m_6) is valid. □

Let $a_1, a_2 \in A$. We put

$$x = (a_1 \wedge a_2) \wedge (\neg(a_1 \vee a_2)).$$

Then

$$(a_1 \vee a_2) +_A (\neg(a_1 \vee a_2)) = u,$$

hence in view of (a_5), the element $(a_1 \vee a_2) +_A x$ is defined in A; we denote

$$(a_1 \vee a_2) +_A x = a_1 \oplus a_2.$$

2.5. Lemma. *The operation $\oplus$ on A satisfies the conditions* (m_2), (m_3) *and* (m_4).

P r o o f. Let a_1, a_2 and x be as above.

a) Let $a_2 = 0$. Then $x = 0 \wedge (\neg a_1) = 0$, whence in view of ($a_4$),

$$a_1 \oplus a_2 = a_1 +_A 0 = a_1.$$

Thus (m_2) holds.

b) Since the definition of $a_1 \oplus a_2$ is symmetric with respect to a_1 and a_2, we infer that (m_3) is valid.

c) Now suppose that $a_2 = u$. Then

$$x = a_1 \wedge (\neg u) = a_1 \wedge 0 = 0,$$

whence $a_1 \oplus u = u +_A 0 = u$. Therefore ($m_4$) is satisfied. □

2.6. Lemma. *If $a_1, a_2 \in A$ and $a_1 +_A a_2$ is defined, then $a_1 \oplus a_2 = a_1 +_A a_2$.*

P r o o f. Suppose that $a_1 +_A a_2$ is defined. Then in view of (a_8),

$$a_1 +_A a_2 = (a_1 \wedge a_2) +_A (a_1 \vee a_2).$$

Since

$$u = (\neg(a_1 \vee a_2)) +_A (a_1 \vee a_2),$$

according to (a_3) we have

$$a_1 \wedge a_2 \leqslant \neg(a_1 \vee a_2),$$

thus $x = a_1 \wedge a_2$. Therefore $a_1 \oplus a_2 = a_1 +_A a_2$. □

2.7. Lemma. *The operations* $\neg$ *and* $\oplus$ *on* A *satisfy the identity* (m_7).

Proof. This is a consequence of the definition of the operation $\neg$ on A and of 2.6. □

2.8. Lemma. *Let* A *be as above and suppose that the lattice* $(A; \wedge, \vee)$ *is linearly ordered. Let* $a_1, a_2 \in A$. *Then either* $a_1 \oplus a_2 = a_1 +_A a_2$ *or* $a_1 \oplus a_2 = u$.

Proof. Without loss of generality we can suppose that $a_1 \leqslant a_2$. Then

$$x = a_1 \wedge \neg a_2$$

and thus

$$a_1 \oplus a_2 = a_2 +_A (a_1 \wedge \neg a_2).$$

If $a_1 \leqslant \neg a_2$, then $a_1 \oplus a_2 = a_1 +_A a_2$. In the case $a_1 > \neg a_2$ we have

$$a_1 \oplus a_2 = a_2 +_A \neg a_2 = u.$$

□

2.9. Lemma. *Let* A *be as above and suppose that the lattice* $(A; \wedge, \vee)$ *is linearly ordered. Then the condition* (m_1) *is satisfied.*

Proof. If $a_1 +_A (a_2 +_A a_3)$ is defined in A, then in view of (a_2) also $(a_1 +_A a_2) +_A a_3$ is defined in A and the two elements under consideration are equal. Then according to 2.6 the relation $a_1 \oplus (a_2 \oplus a_3) = (a_1 \oplus a_2) \oplus a_3$ is valid.

Next suppose that $a_1 +_A (a_2 +_A a_3)$ is not defined in A. Then in view of (a_2) and (a_3), $(a_1 +_A a_2) +_A a_3$ is not defined in A, either. Thus 2.8 yields that $a_1 \oplus (a_2 \oplus a_3) = u = (a_1 \oplus a_2) \oplus a_3$. □

2.10. Lemma. *Let* A *be as in* 2.9. *Then the condition* (m_8) *is satisfied.*

Proof. Let $x, y \in A$ and suppose that $x \leqslant y$. Put

$$v_1 = \neg(\neg x \oplus y) \oplus y,$$
$$v_2 = \neg(x \oplus \neg y) \oplus x.$$

If $x = y$, then $\neg x \oplus y = \neg y \oplus x = u$, whence

$$\neg(\neg x \oplus y) = \neg(x \oplus \neg y) = 0$$

and thus $v_1 = v_2$.

If $x < y$ and if $\neg x +_A y$ is defined, then $u = \neg x +_A x < \neg x +_A y$, which is a contradiction. Thus $\neg x +_A y$ is not defined. Hence $\neg x \oplus y = u$ and $\neg(\neg x \oplus y) = 0$. Therefore $v_1 = y$. Now we calculate v_2 in the present case.

There exists $z \in A$ with $x +_A z = y$. Then

$$(x +_A z) +_A \neg y = y +_A \neg y = u.$$

Hence according to 2.9,

$$(x +_A \neg y) +_A z = u,$$
$$\neg z = x +_A \neg y = x \oplus \neg y.$$

Then in view of (a_3) and (a_6),

$$z = \neg(x \oplus \neg y),$$
$$y = x +_A z = x \oplus z = x \oplus \neg(x \oplus \neg y) = \neg(x \oplus \neg y) \oplus x = v_1.$$

Therefore in the case $x \leqslant y$ we have $v_1 = v_2$. The case $y \leqslant x$ can be treated amalogously. □

In view of the above results we obtain

2.11. Proposition. *Let $(A; +_A, \wedge, \vee, 0)$ be an m-algebra with the greatest element u. Suppose that $(A; \wedge, \vee)$ is a chain. Let $\neg, \oplus$ be as above, and let $(*)$ be defined by* (m_9). *Finaly, let $u = 1$. Then $(A; \oplus, *, \neg, 0, 1)$ is an MV-algebra.*

3. Congruence relations

In this section we suppose that $\mathcal{A} = (A; \wedge, \vee, +_A, 0)$ is an m-algebra. We apply the conditions (a_1)–(a_8) from 2.1.

We denote by $E(A)$ the system of all equivalence relations on the set A. The system $E(A)$ is partially ordered in the usual way. Then $E(A)$ is a complete lattice. The least element of $E(A)$ will be denoted by ϱ_0. The lattice operations on $E(A)$ are denoted by $\wedge$ and $\vee$.

3.1. Definition. Let $\varrho \in E(A)$. Suppose that the following conditions are satisfied:

(i) ϱ is a congruence relation of the lattice $(A; , \wedge, \vee)$.
(ii) If a_i, b_i $(i = 1,2)$ are elements of A such that $a_i \varrho b_i$ $(i = 1,2)$ and both $a_1 +_A a_2$, $b_1 +_A b_2$ are defined in A, then $(a_1 +_A a_2)\varrho(b_1 +_A b_2)$.
(iii) If a_i, b_i, x_i $(i = 1,2)$ are elements of A such that $a_i +_A x_i = b_i$ for $i = 1,2$, $a_1 \varrho a_2$ and $b_1 \varrho b_2$, then $x_1 \varrho x_2$.

Under these conditions ϱ is called a congruence of the m-algebra $\mathcal{A}$.

The system of all congruences of $\mathcal{A}$ will be denoted by $\operatorname{Con}\mathcal{A}$.

3.2. Lemma. *Let I be a nonempty set and let $\{\varrho_i\}_{i\in I}$ be a subset of* $\operatorname{Con}\mathcal{A}$. *Then both $\bigvee\limits_{i\in I} \varrho_i$ and $\bigwedge\limits_{i\in I} \varrho_i$ belong to* $\operatorname{Con}\mathcal{A}$.

P r o o f. This is an immediate consequence of 3.1. □

For $x \in A$ and $\varrho \in E(A)$ we denote

$$x(\varrho) = \{y \in A \colon\ x\varrho y\};$$

further, for $X \subseteq A$ we put

$$X(\varrho) = \{x(\varrho) \colon\ x \in X\}.$$

If ϱ is fixed and if no misunderstanding can occur, then we write

$$x(\varrho) = \overline{x}, \quad X(\varrho) = \overline{X}.$$

For $\overline{x}, \overline{y} \in \overline{A}$ we put $\overline{x} \leqslant \overline{y}$ if there exist $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ such that $x_1 \leqslant y_1$. Then $\overline{A}$ turns out to be a distributive lattice with the least element $\overline{0}$.

Let $\overline{x}, \overline{y}, \overline{z} \in \overline{A}$. We put $\overline{x} +_{\overline{A}} \overline{y} = \overline{z}$ if there exist elements $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ such that $x_1 +_A y_1 \in \overline{z}$. Then $+_{\overline{A}}$ is a partial binary operation on $\overline{A}$.

3.3. Lemma. $\overline{\mathcal{A}} = (\overline{A}; +_{\overline{A}}, \wedge, \vee, \overline{0})$ *is an m-algebra.*

P r o o f. It is a routine to verify that the conditions (a_1)–(a_8) are satisfied in $\overline{\mathcal{A}}$. □

We denote $\overline{\mathcal{A}} = \mathcal{A}/\varrho$. The m-algebra $\mathcal{A}$ is called simple if $\operatorname{card}\operatorname{Con}\mathcal{A} \leqslant 2$. By the procedure analogous to the well-known method for general algebras (and, in fact, by using Axiom of Choice) we obtain

3.4. Lemma. *Let x and y be distinct elements of A. Then there exists $\varrho \in \operatorname{Con} \mathcal{A}$ such that $\mathcal{A}/\varrho$ is simple and $x(\varrho) \neq y(\varrho)$.*

From 3.1 we immediately obtain

3.5. Lemma. *Let $\varrho \in \operatorname{Con} \mathcal{A}$, $X = 0(\varrho)$. Then*

(i) *X is a convex sublattice of the lattice $(A; \wedge, \vee)$ containing the element 0;*
(ii) *if $a_1, a_2 \in X$ and if $a_1 +_A a_2$ is defined, then $a_1 +_A a_2 \in X$.*

Let Y be a subset of A satisfying the conditions (i) and (ii) from 3.5. If $a_1, a_2 \in A$, $a_1 \leqslant a_2$ and if x is as in (a_6), then we denote $x = a_2 -_A a_1$. For $a, b \in A$ we put $a \varrho_Y b$ if

$$(a \vee b) -_A (a \wedge b) \in Y.$$

3.6. Lemma. *ϱ_Y is an equivalence relation on A.*

P r o o f. From the definition of ϱ_Y we obtain that ϱ_Y is reflexive and symmetric. Let $a, b, c \in A$, $a \varrho_Y b$, $b \varrho_Y c$. Denote (cf. Fig. 1)

$$p_1 = a \wedge b, \quad q_1 = a \vee b,$$
$$p_2 = b \wedge c, \quad q_2 = b \vee c.$$

Then $q_1 -_A p_1, q_2 -_A p_2 \in Y$ and in view of (a_3),

$$b -_A p_1 \leqslant q_1 -_A p_1, \quad q_1 -_A b \leqslant q_1 -_A p_1,$$
$$b -_A p_2 \leqslant q_2 -_A p_2, \quad q_2 -_A b \leqslant q_2 -_A p_1.$$

Hence all the elements $b -_A p_1$, $q_1 -_A b$, $b -_A p_2$ and $q_2 -_A b$ belong to Y.

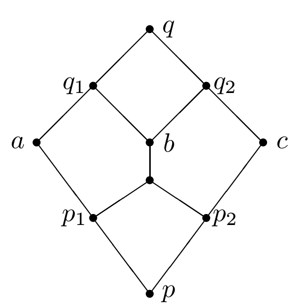

Fig. 1

Put $p_1 \wedge p_2 = p$, $q_1 \vee q_2 = q$. Then

$$p_1 \vee p_2 \leqslant b,$$

whence

$$(p_1 \vee p_2) -_A p_1 \leqslant b -_A p_1, \quad (p_1 \vee p_2) -_A p_2 \leqslant b -_A p_2.$$

Then $(p_1 \vee p_2) -_A p_1$ and $(p_1 \vee p_2) -_A p_2$ belong to Y.

In view of (a_1) and (a_8),

$$(p_1 \vee p_2) -_A p_1 = p_2 -_A p,$$
$$(p_1 \vee p_2) -_A p_2 = p_1 -_A p.$$

Thus $p_1 -_A p$ and $p_2 -_A p$ belong to Y. Analogously we obtain that $q -_A q_1$ and $q -_A q_2$ belong to Y. Since $p \leqslant p_1 \leqslant q_1 \leqslant q$ we get

$$q -_A p = (q -_A q_1) +_A (q_1 -_A p_1) +_A (p_1 -_A p).$$

The set Y satisfies the condition (ii) from 3.5, hence $q -_A p$ belongs to Y. According to (a_3),

$$(a \vee c) -_A (a \wedge c) \leqslant q -_A p,$$

thus $a \varrho_Y c$. Therefore ϱ_Y is transitive. □

3.7. Lemma. *ϱ_Y is a congruence with respect to the operations $\wedge$ and $\vee$ on A.*

P r o o f. In view of 3.6, it suffices to verify that if $a, b, c \in A$, $a \varrho_Y b$, then $(a \wedge c) \varrho_Y (b \wedge c)$ and $(a \vee c) \varrho_Y (b \vee c)$.

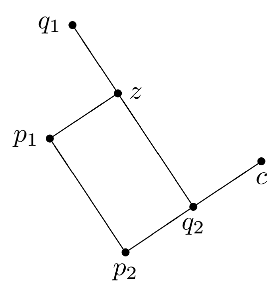

Fig. 2

Let p_1 and q_1 be as in the proof of 3.6. Put (cf. Fig. 2)

$$p_2 = p_1 \wedge c, \quad q_2 = q_1 \wedge c, \quad p_1 \vee q_2 = z.$$

Then, since $(A; \wedge, \vee)$ is a distributive lattice, we have

$$(a \wedge c) \wedge (b \wedge c) = p_2, \quad (a \wedge c) \vee (b \wedge c) = q_2.$$

Further

$$z \geqslant p_1, \quad z -_A p_1 \leqslant q_1 -_A p_1,$$

whence $z -_A p_1 \in Y$. Also,

$$z -_A p_1 = q -_A p_2.$$

Therefore $q -_A p_2 \in Y$ and so $(a \wedge c) \varrho_Y (b \wedge c)$. Similarly we obtain that $(a \vee c) \varrho_Y (b \vee c)$.

□

3.8. Lemma. *The relation* ϱ_Y *satisfies the condition* (ii) *from* 3.1.

Proof. a) Let $a, b, x \in A$. Suppose that $a \varrho_Y b$ and that both elements $a +_A x$ and $b +_A x$ are defined in A. We prove that $(a +_A x) \varrho (b +_A x)$.

Let p_1 and q_1 be as in the proof of 3.6. There exists $y \in Y$ with $p_1 +_A y = q_1$. In view of 2.1.1 we have

$$(a +_A x) \wedge (b +_A x) = (a \wedge b) +_A x = p_1 +_A x,$$
$$(a +_A x) \vee (b +_A x) = (a \vee b) +_A x = q_1 +_A x,$$
$$q_1 +_A x = (p_1 +_A y) +_A x = (p_1 +_A x) +_A y,$$

whence according to (a_8),

$$((a +_A x) \vee (b +_A x)) -_A ((a +_A x) \wedge ((b +_A x)) = y.$$

Since $y \in Y$, we get $(a +_A x) \varrho (b +_A x)$.

b) Now let $a_i, b_i \in A$, $a_i \varrho b_i$ $(i = 1, 2)$ and suppose that both $a_1 +_A a_2$, $b_1 +_A b_2$ are defined in A. Put

$$z_1 = a_1 \wedge b_1, \quad z_2 = a_2 \wedge b_2.$$

Then $z_1 +_A a_2$, $z_1 +_A z_2$, $z_1 +_A b_2$ are defined in A and

$$a_1 \varrho_Y z_1 \varrho_Y b_1, \quad a_2 \varrho_Y z_2 \varrho_Y b_2.$$

Hence the result proved in part a) yields

$$(a_1 +_A a_2) \varrho_Y (z_1 +_A a_2) \varrho_Y (z_1 +_A z_2) \varrho_Y (z_1 +_A b_2) \varrho_Y (b_1 +_A b_2).$$

In view of 3.6, the relation ϱ_Y is transitive, hence the condition (ii) from 3.1 is valid.

□

3.9. Lemma. *The relation* ϱ_Y *satisfies the condition* (iii) *from* 3.1.

P r o o f. Let $a_i, b_i, x_i \in A$, $a_i +_A x_i = b_i$ $(i = 1, 2)$, $a_1 \varrho_Y a_2$ and $b_1 \varrho_Y b_2$. (Cf. Fig. 3.) Denote

$$a_1 \vee a_2 = a_3, \quad b_1 \vee b_2 = b_3.$$

Then $a_1 \varrho_Y a_3$, $b_1 \varrho_Y b_3$. In view of (a_8) we have

$$b_1 -_A (b_1 \wedge a_3) = (b_1 \vee a_3) -_A a_3.$$

We set

$$z_1 = (b_1 \wedge a_3) -_A a_1, \quad z_2 = b_3 -_A (b_1 \vee a_3).$$

Then $z_1 \leqslant a_3 -_A a_1$, $z_2 \leqslant a_3 -_A b_1$, whence $z_1, z_2 \in Y$. Thus $z_1 \varrho_Y z_2$. Denote

$$z = b_1 -_A (b_1 \wedge a_3), \quad x_3 = b_3 -_A a_3.$$

Then

$$x_1 = z_1 +_A z, \quad x_3 = z_2 +_A z.$$

Thus in view of 3.8 we have $x_1 \varrho_Y x_3$. Analogously we obtain $x_2 \varrho_Y x_3$. Therefore $x_1 \varrho_Y x_2$. □

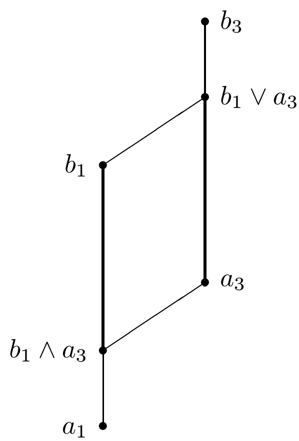

Fig. 3

3.10. Lemma. *The relation* ϱ_Y *is a congruence of the* m*-algebra* $\mathcal{A}$.

P r o o f. This is a consequence of 3.6, 3.7, 3.8 and 3.9. □

The following result is easy to verify.

3.11. Lemma. *Let Y and ϱ_Y be as above and let $a \in A$. Then $a\varrho_Y 0$ if and only if $a \in Y$.*

4. Polars and direct products

Again, let $\mathcal{A}$ be an m-algebra; we apply the notation as above.

For $X \subseteq A$ we put

$$X^{\perp} = \{y \in A\colon\ y \wedge x = 0 \quad \text{for each } x \in X\};$$

$X^{\perp}$ is said to be a *polar* in $\mathcal{A}$.

4.1. Lemma. *Let $X \subseteq A$. Then*

(i) *$X^{\perp}$ is a convex sublattice of the lattice $(A; \wedge, \vee)$ and $0 \in X^{\perp}$;*

(ii) *if $y_1, y_2 \in X^{\perp}$ and if $y_1 +_A y_2$ is defined in A, then $y_1 +_A y_2$ belongs to $X^{\perp}$.*

Proof. It is obvious that 0 belongs to $X^{\perp}$. Further, if $y \in X^{\perp}$ and $y_1 \in A$, $y_1 \leqslant y$, then $y_1 \in X^{\perp}$. In view of the distributivity of $(A; \wedge, \vee)$, the set $X^{\perp}$ is closed with respect to the operation $\vee$. Hence $X^{\perp}$ is an ideal of the lattice $(A; \wedge, \vee)$.

Let $y_1, y_2 \in X^{\perp}$ and suppose that $y_1 +_A y_2$ is defined in A. Then according to (a_7) the element $y_1 +_A y_2$ belongs to $X^{\perp}$. □

4.2. Lemma. *Let $X \subseteq A$. Then there exists a congruence relation ϱ of $\mathcal{A}$ such that $0(\varrho) = X^{\perp}$.*

Proof. This is a consequence of 3.10, 3.11 and 4.1. □

A polar $X^{\perp}$ is called nontrivial if $\{0\} \neq X^{\perp} \neq A$.

4.3. Lemma. *The following conditions are equivalent:*

(i) *Each polar of $\mathcal{A}$ is trivial.*

(ii) *The lattice $(A; \wedge, \vee)$ is a chain.*

Proof. It is clear that (ii) implies (i). Suppose that the lattice $(A; \wedge, \vee)$ is not linearly ordered. Hence there are $a_1, a_2 \in A$ such that a_1 and a_2 are incomparable. Denote

$$a_1 \wedge a_2 = a_3, \quad a_1 \vee a_2 = a_4,$$
$$a_1 -_A a_3 = a_1', \quad a_2 -_A a_4 = a_2'.$$

Put $\varphi(t) = t +_A a_3$ for each $t \in [0, a_4 -_A a_3]$. Then according to 2.1.1, φ is an isomorphism of the lattice $[0, a_4 -_A a_3]$ onto the lattice $[a_3, a_4]$. Hence

$$\varphi^{-1}(a_1) \wedge \varphi^{-1}(a_2) = 0, \quad \varphi^{-1}(a_1) \neq 0 \neq \varphi^{-1}(a_2).$$

Denote $X = \{\varphi^{-1}(a_1)\}$. Then $\varphi^{-1}(a_2) \in X^{\perp}$, hence $X^{\perp} \neq \{0\}$. On the other hand, $\varphi^{-1}(a_2) \notin X^{\perp}$, thus $X^{\perp} \neq A$. Therefore (i) fails to hold. □

4.4. Lemma. *If the m-algebra $\mathcal{A}$ is simple, then $(A; \wedge, \vee)$ is linearly ordered.*

Proof. Let $\mathcal{A}$ be simple. Thus in view of 4.2, each polar of $\mathcal{A}$ is trivial. Hence according to 4.3, $(A; \wedge, \vee)$ is linearly ordered. □

Let I be a nonempty set and for each $i \in I$ let $\mathcal{A}_i = (A_i; +_A, \wedge, \vee, 0)$ be an m-algebra; let u_i be the greatest element of $(A_i; \wedge, \vee)$.

We denote by A the cartesian product of the sets A_i $(i \in I)$. The partial order on A is defined coordinate-wise. Then $(A; \wedge, \vee)$ is a bounded distributive lattice. For $a \in A$ we denote by a_i the i-th component of a.

Let $a, b \in A$. If for each $i \in I$ the element $a_i +_{A_i} b_i = c^i$ is defined in A_i, then we put $a + b = c$, where $c_i = c^i$ for each $i \in I$. If there is $i \in I$ such that $x_i +_{A_i} b_i$ is not defined in A_i, then we consider $a +_A b$ to be not defined in A. In this way we obtain an m-algebra $\mathcal{A} = (A; +_A, \wedge, \vee, 0)$ which will be denoted by

$$\mathcal{A} = \prod_{i \in I} \mathcal{A}_i;$$

it is said to be the direct product of m-algebras $\mathcal{A}_i$.

For $X \subseteq A$ and $i \in I$ we put

$$X(\mathcal{A}_i) = \{x_i \colon x \in X\}.$$

If $B \subseteq A$, then we define a partial binary operation $+_B$ on B as follows: if $b_1, b_2 \in B$ and $b_1 +_A b_2$ is defined in A and belongs to B, then we put $b_1 +_B b_2 = b_1 +_A b_2$; otherwise $b_1 +_B b_2$ is not defined.

4.5. Definition. Let $\emptyset \neq B \subseteq A$ be such that the following conditions are satisfied:

(i) $B(\mathcal{A}_i) = A_i$ for each $i \in I$;
(ii) B is a sublattice of the lattice $(A; \wedge, \vee)$ with the least element 0 and the greatest element u such that $0(\mathcal{A}_i) = 0_i$ and $u(\mathcal{A}_i) = u_i$ for each $i \in I$.
(iii) If $b_1, b_2 \in B$ and if the element $b_1 +_A b_2$ is defined in A, then this element belongs to B.
(iv) If $b_1, b_2 \in B$, $b_1 \leqslant b_2$, then there exists $b_3 \in B$ such that $b_1 +_A b_3 = b_2$.

Under these conditions the structure $\mathcal{B} = (B; +_B, \wedge, \vee, 0, u)$ is called a subdirect product of m-algebras $\mathcal{A}_i$.

We denote this fact by writing

$$\mathcal{B} = \operatorname{sub} \prod_{i \in I} \mathcal{A}_i.$$

It is clear that $\mathcal{B}$ is an m-algebra.

By the standard method analogous to that from the theory of general algebras we obtain the following result:

4.6. Lemma. *Let ϱ_i $(i \in I)$ be elements of* $\operatorname{Con} \mathcal{A}$ *such that* $\bigwedge_{i \in I} \varrho_i = \varrho_0$. *Then $\mathcal{A}$ is a subdirect product of m-algebras $\mathcal{A}/\varrho_i$.*

Now, 3.4 and 4.6 yield

4.7. Lemma. *Each m-algebra is a subdirect product of simple m-algebras.*

4.8. Proposition. *Each m-algebra is a subdirect product of linearly ordered m-algebras.*

P r o o f. Let $\mathcal{A}$ be an m-algebra. In view of 4.7, $\mathcal{A}$ is a subdirect product of simple m-algebras. Now it suffices to apply 4.4. □

4.9. Theorem. *Let $\mathcal{A}$ be an m-algebra. Suppose that the operations $\neg$ and $\oplus$ are defined as in* Section 3 *and that the operation $*$ is defined by means of* (m_8). *Put $1 = u$, where u is the greatest element of $\mathcal{A}$. Then $\mathcal{A}' = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra.*

P r o o f. This is a consequence of 4.8 and 2.11. □

Our present situation is as follows. To each MV-algebra $\mathcal{A}$ we can assign an m-algebra $\mathcal{A}^1 = f_1(\mathcal{A})$ by the construction described in Section 2. Further, to each m-algebra $\mathcal{A}^m$ we can assign an MV-algebra $f_2(\mathcal{A}^m)$ by the construction from Sections 3, 4.

By considering these constructions we immediately obtain that for each MV-algebra $\mathcal{A}$ and each m-algebra $\mathcal{A}^m$ the relations

$$f_2(f_1(\mathcal{A})) = \mathcal{A}, \quad f_1(f_2(\mathcal{A}^m)) = \mathcal{A}^m$$

are valid. Moreover, if $f_1(\mathcal{A}) = \mathcal{A}^m$, then both $\mathcal{A}$ and $\mathcal{A}^m$ are defined on the same underlying set A. Thus we conclude that the algebraic structures $\mathcal{A}$ and $\mathcal{A}^m$ do not essentially differ.

In view of the consideration at the end of the previous section we often will not distinguish between the MV-algebra $\mathcal{A}$ and the corresponding m-algebra $f_1(\mathcal{A})$ (under the notation as above).

Let $\mathcal{A}$ be as in Section 4 and let $b \in A$, $B = [0, b]$. We consider the partial binary operation $+_B$ on B as in Section 4.

5.1. Definition. Let $\mathcal{A}$ and B be as above. Then the algebraic structure $\mathcal{B} = (B; +_B, \wedge, \vee, 0)$ will be called a substructure of $\mathcal{A}$.

From 5.1 we immediately obtain

5.2. Lemma. *Let $\mathcal{A}$ be an m-algebra and let $\mathcal{B}$ be a substructure of $\mathcal{A}$. Then $\mathcal{B}$ is an m-algebra as well.*

We denote by $\mathcal{S}(\mathcal{A})$ the system of all substructures of $\mathcal{A}$. This system is partially ordered by the set-theoretical inclusion; i.e., if $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{S}(\mathcal{A})$, $\mathcal{B}_i = (B_i; +_{B_i}, \wedge, \vee, 0)$ $(i \in I)$, then we put $\mathcal{B}_1 \leqslant \mathcal{B}_2$ if $B_1 \subseteq B_2$.

If $\mathcal{B}_1 \leqslant \mathcal{B}_2$, then clearly $\mathcal{B}_1 \in \mathcal{S}(\mathcal{B}_2)$. Further, the mapping $\varphi(\mathcal{B}_1) = b_1$, where b_1 is the greatest element of B_1, is an isomorphism of $\mathcal{S}(\mathcal{A})$ onto the lattice $(A; \wedge, \vee)$. Hence $\mathcal{S}(\mathcal{A})$ is a distributive lattice. From this we obtain

5.3. Lemma. *Let $\mathcal{B}_i \in \mathcal{S}(\mathcal{A})$, $B_i = [0, b_i]$ $(i = 1, 2, \ldots, m)$, and let*

$$b^1 = b_1 \wedge b_2 \wedge \ldots \wedge b_n, \quad b^2 = b_1 \vee b_2 \vee \ldots \vee b_n.$$

Then

$$\mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \ldots \wedge = \mathcal{B}^1, \quad \mathcal{B}_1 \vee \mathcal{B}_2 \vee \ldots \vee \mathcal{B}_n = \mathcal{B}^2,$$

where $B^1 = [0, b^1]$ and $B^2 = [0, b^2]$.

Now let $\mathcal{R}_a$ and $\mathcal{R}_m$ be as in the introduction above. (Recall that, as we have remarked above, for an MV-algebra $\mathcal{A}$ we identify $\mathcal{A}$ with $f_1(\mathcal{A})$.)

Let $X \in \mathcal{R}_a$. We denote by $\varphi_1(X)$ the class of all m-algebras $\mathcal{A}$ such that the following condition is satisfied:

(α) There exists a lattice ordered group G having a strong unit u such that $G \in X$ and $\mathcal{A} = \mathcal{A}_0(G, u)$.

5.4. Lemma. *Let $X \in \mathcal{R}_a$. Then $\varphi_1(X) \in \mathcal{R}_m$.*

P r o o f. Put $\varphi_1(X) = Y$. We have to verify that Y satisfies the conditions 1′) and 2′) from the definition of the radical class of m-algebras.

a) Let $\mathcal{A} \in Y$ and let $\mathcal{B}$ be a substructure of $\mathcal{A}$. We apply the notation as above. Hence for the underlying set B of $\mathcal{B}$ we have $B \subseteq A$; moreover B is an interval $[0, u_1]$ of $(A; \wedge, \vee)$. Thus B is an interval of the lattice ordered group G. Let G_1 be the convex ℓ-subgroup of G which is generated by the element u_1. Then $G_1 \in X$ and u_1 is a strong unit of G_1. Therefore we have $\mathcal{B} = \mathcal{A}_0\ (G_1, u_1)$. We obtain $\mathcal{B} \in Y$ and hence Y is closed with respect to substructures.

b) Let $\mathcal{A}$ be an m-algebra and let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ be substructures of $\mathcal{A}$ such that all $\mathcal{B}_i$ $(i = 1, 2, \ldots, n)$ belong to Y. Under analogous notation as in a) we assume that for $i \in \{1, 2, \ldots, n\}$ the m-algebra $\mathcal{B}_i$ has as the underlying set an interval $[0, u_i]$ of G, where $\mathcal{A} = \mathcal{A}_0\ (G, u)$. Let G_i be the convex ℓ-subgroup of G generated by the element u_i. Then $G_i \in X$. Put $u^0 = u_1 \vee u_2 \vee \ldots \vee u_n$ and $\mathcal{B} = \mathcal{A}_0\ (G_0, u^0)$, where G^0 is the convex ℓ-subgroup of G generated by u^0. Then

$$G^0 = G_1 \vee G_2 \vee \ldots \vee G_n,$$

whence $G^0 \in X$ and thus $\mathcal{B} \in Y$. According to 5.3,

$$\mathcal{B} = \mathcal{B}_1 \vee \mathcal{B}_2 \vee \ldots \vee \mathcal{B}_n.$$

Therefore Y satisfies the condition 2'). □

Let X_1 be a nonempty class of lattice ordered groups. We denote by $\overline{X}_1$ the class of all lattice ordered groups G that have the following property:

There exist a set $\{G_i\}_{i \in I}$ of convex ℓ-subgroups of G and a set $\{H_i\}_{i \in I}$ of lattice ordered groups belonging to X_1 such that

(i) $G = \bigvee_{i \in I} G_i$, and
(ii) for each $i \in I$, G_i is isomorphic to a convex ℓ-subgroup of H_i.

5.5. Lemma. (Cf. [8], Lemma 2.1.) *Let X_1 be a nonempty class of lattice ordered groups. Then*

(i) *$\overline{X}_1$ is a radical class of lattice ordered groups;*
(ii) *if X_2 is a radical class of lattice ordered groups with $X_1 \subseteq X_2$, then $\overline{X}_1 \subseteq X_2$.*

$\overline{X}_1$ will be said to be the radical class generated by X_1.

Let $Y \in \mathcal{R}_m$. We denote by $\varphi^0(Y)$ the class of all lattice ordered groups G such that G has a strong unit u and $\mathcal{A}_0(G, u) \in Y$. Further, let $\varphi_2(Y) = \overline{\varphi^0(Y)}$ (under the notation as above). Hence φ_2 is a mapping of the collection $\mathcal{R}_m$ into $\mathcal{R}_a$.

5.6. Lemma. *Let $X \in \mathcal{R}_a$. Then $\varphi_2(\varphi_1(X)) = X$.*

P r o o f. Let $G\in X$ and $\varphi_1(X)=Y$. For $0\leqslant u\in G$ we denote by G_u the convex ℓ-subgroup of G generated by u. All m-algebras $\mathcal{A}_0(G_u,u)$ belong to Y. Thus all G_u belong to $\varphi^0(Y)$. Further we have

$$\bigvee_{u\in G^+} G_u = G,$$

hence in view of 5.5, G is an element of $\varphi_2(Y)$. Thus $X\subseteq\varphi_2(Y)$.

For proving the inverse inclusion we first observe that we clearly have $\varphi^0(Y)\subseteq X$. Then according to 5.5 we obtain

$$\varphi_2(Y)=\overline{\varphi^0(Y)}\subseteq X,$$

completing the proof. □

The following result is well-known.

5.7.1. Lemma. *Let $H_i (i\in I)$ be convex ℓ-subgroups of a lattice ordered group G and let $0\leqslant h\in\bigvee_{i\in I} H_i$. Then there exist $i(1),i(2),\dots,i(n)\in I$ and $h_{i(1)}\in H^+_{i(1)},\dots,h_{i(n)}\in H^+_{i(n)}$ such that $h=h_{i(1)}+\ldots+h_{i(n)}$.*

5.7.2. Lemma. *Let H be an abelian lattice ordered group and let H_i $(i=0,1,2,\dots,n)$ be convex ℓ-subgroups of H, $0\leqslant h_i\in H_i$, $h=h_0+h_1+\ldots+h_n$. Then there are elements $0\leqslant t_i\in H_i$ $(i=0,1,2,\dots,n)$ such that $h=t_0\vee t_1\vee\ldots\vee t_n$.*

P r o o f. a) Consider the case $n=1$. Put

$$x=h_0\wedge h_1,\quad y=h_0\vee h_1.$$
$$t_0=h_0+x,\quad t_1=h_1+x.$$

Then $t_0\in H_0^+$, $t_1\in H_1^+$ and

$$t_0\vee t_1=(h_0+x)\vee(t_0\vee x)=(h_1\vee t_0)+x=y+x=h_0+h_1.$$

b) Suppose that $n>1$ and that the assertion holds for $n-1$. Hence there are $t'_0\in H_0^+$, $t'_1\in H_1^+$, …, $t'_{n-1}\in H^+_{n-1}$ such that

$$h_0+h_1+\ldots+h_{n-1}=t'_0\vee t'_1\vee\ldots t'_{n-1}.$$

Thus

$$\begin{aligned}h_0+h_1+\ldots+h_{n-1}+h_n&=(t'_0\vee t'_1\vee\ldots\vee t'_{n-1})+h_n\\&=(t'_0+h_n)\vee(t'_1+h_n)\vee\ldots\vee(t'_{n-1}+h_n).\end{aligned}$$

Then according to a) there are $t_0 \in H_0^+$, $h'_0 \in H_n^+$, …, $t_{n-1} \in H_{n-1}^+$, $h'_{n-1} \in H_n^+$ such that

$$t'_0 + h_n = t_0 \vee h'_0, \ldots, t'_{n-1} + h_n = t_{n-1} \vee h'_{n-1}.$$

Therefore we have

$$h = t_0 \vee t_1 \vee \ldots \vee t_{n-1} \vee t_n,$$

where $t_n = h'_0 \vee h'_1 \vee \ldots \vee h'_{n-1}$. Clearly $t_n \in H_n^+$. □

5.7. Lemma. *Let $Y \in \mathcal{R}_m$. Then $\varphi_1(\varphi_2(Y)) = Y$.*

P r o o f. Put $\varphi_2(Y) = X$. Let $\mathcal{A} \in Y$. Hence there is $G \in \varphi^0(Y)$ such that G has a strong unit u and $\mathcal{A} = \mathcal{A}_0(G, u)$. Thus $G \in \overline{\varphi^0(Y)} = X$; therefore $\mathcal{A} \in \varphi_1(X)$. We obtain $Y \subseteq \varphi_1(\varphi_2(Y))$.

Conversely, let $\mathcal{A} \in \varphi_1(\varphi_2(Y))$. Hence there exists $G \in \varphi_2(Y)$ and $0 \leqslant u \in G$ such that $\mathcal{A} = \mathcal{A}_0(G_1, u)$, where G_1 is the convex ℓ-subgroup of G that is generated by u. Then $G_1 \in \varphi_2(Y)$, because $\varphi_2(Y) \in \mathcal{R}_a$.

Since $\varphi_2(Y) = \overline{\varphi^0(Y)}$, according to 5.5 there exist convex ℓ-subgroups H_i $(i \in I)$ of G such that

(i) $G_1 = \bigvee_{i \in I} H_i$,

(ii) for each $i \in I$, H_i is isomorphic to a convex ℓ-subgroup H'_i of some lattice ordered group H_i^* belonging to $\varphi^0(Y)$.

In view of 5.7.1 there exists a finite subset I_1 of I and there are elements $0 \leqslant a_i \in H_i$ $(i \in I_1)$ such that

$$u = \sum_{i \in I_1} a_i.$$

Then according to 5.7.2 there are elements $0 \leqslant a'_i \in H_i$ $(i \in I_1)$ with

$$u = \bigvee_{i \in I_1} a'_i.$$

Let H_i^0 $(i \in I_1)$ be the convex ℓ-subgroup of G_1 generated by the element a'_i and denote $\mathcal{A}_i = \mathcal{A}_0(H_i^0, a'_i)$. Then for each $i \in I_1$, $\mathcal{A}_i$ is a substructure of $\mathcal{A}$. According to 5.3,

$$\mathcal{A} = \bigvee_{i \in I_1} \mathcal{A}_i.$$

Further, in view of the definition of $\varphi^0(Y)$ we obtain that for each $i \in I_1$, $\mathcal{A}_i$ belongs to Y. Thus $\mathcal{A} \in Y$ and then $\varphi_1(\varphi_2(Y)) \subseteq Y$, completing the proof. □

5.8. Lemma. (i) *If* $X_1, X_2 \in \mathcal{R}_a$, $X_1 \leqslant X_2$, *then* $\varphi_1(X_1) \leqslant \varphi_1(X_2)$. (ii) *If* $Y_1, Y_2 \in \mathcal{R}_m$, $Y_1 \leqslant Y_2$, *then* $\varphi_2(Y_1) \leqslant \varphi_2(Y_2)$.

Proof. This is an immediate consequence of the definitions of φ_1 and φ_2. □

5.9. Theorem. φ_1 *is an isomorphism of* $\mathcal{R}_a$ *onto* $\mathcal{R}_m$.

Proof. This is implied by 5.4, 5.6, 5.7 and 5.8. □

Results on the properties of partial order in $\mathcal{R}_a$ (e.g., on the existence of infima and suprema, distributive laws, existence of atoms and antiatoms, covering properties) were proved in [6]. In view of 5.9, analogous results are valid for $\mathcal{R}_m$.

6. Examples and concluding remarks

By applying 5.6, 5.7 and 5.9 we can construct examples of radical classes of MV-algebras from the examples of radical classes of lattice ordered groups which were treated in papers quoted in references above.

Let us mention the following examples.

1) The class of all finite MV-algebras.

2) The class of all complete MV-algebras.

3) The class of all archimedean MV-algebras.

4) The class of all MV-algebras $\mathcal{A}$ such that the lattice $(A; \wedge, \vee)$ is completely distributive.

5) The class of all MV-algebras $\mathcal{A}$ such that the lattice $(A; \wedge, \vee)$ is α-distributive, where α is a given cardinal.

We remark that the system $C(G)$ of all convex ℓ-subgroups of a lattice ordered group G is a complete lattice.

On the other hand, the system $S(\mathcal{A})$ of all substructures of $\mathcal{A}$ is a lattice, but it need not be complete (because the lattice $(A; \wedge, \vee)$ need not be complete).

The definitions of the radical class of lattice ordered groups and of the radical class of MV-algebras essentially differ with respect to the conditions 2) and 2′): in the condition 2) the power of the set I can be arbitrary, in 2′) we deal with a finite set of substructures.

We could consider a strenghtened version of 2′), namely

2″) If $\mathcal{B}$ is an MV-algebra and $\{\mathcal{A}_i\}_{i\in I}$ are substructures of $\mathcal{B}$ belonging to Y such that $\bigvee_{i\in I} \mathcal{A}_i$ does exist in $S(\mathcal{B})$, then $\bigvee_{i\in I} \mathcal{A}_i$ also belongs to Y.

If we modify the definition of $\mathcal{R}_m$ in such a way that a radical class of MV-algebras is a nonempty class Y of MV-algebras which is closed with respect to isomorphisms and satisfies the conditions 1′) and 2″) then the construction from Section 5 (concerning φ_1 and φ_2 and giving a one-to-one correspondence between radical classes of lattice ordered groups and radical classes of MV-algebras) would not be valid.

For example, if X is the class of all lattice ordered groups G such that each interval of G is finite, then X is a radical class. However, $\varphi_1(X)$ does not satisfy the condition 2″).

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