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ON A PROBLEM CONCERNING STRATIFIED GRAPHS

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The concept of a stratified graph was introduced by G. Chartrand, L. Holley, R. Rashidi and N. Sherwani in [1]. A stratified graph may be considered as an ordered pair $(G, \mathcal{S})$, where G is a connected undirected graph without loops and multiple edges and $\mathcal{S}$ is a partition of its vertex set $V(G)$. The classes of $\mathcal{S}$ are called strata. If their number is k, we denote them usually by $X_1, \ldots, X_k$ and speak about a k-stratified graph.

By the symbol $d(x, y)$ we denote the distance in a graph between two its vertices x, y; this is the minimum length of a path connecting the vertices x and y in G. By $\delta(i, j)$ for two numbers i, j we denote the Kronecker delta defined so that $\delta(i, j) = 1$ for $i = j$ and $\delta(i, j) = 0$ for $i \neq j$.

If $u \in V(G)$, $X \in \mathcal{S}$, then the X-proximity of u, denoted by $\delta_X(u)$, is the minimum of $d(u, x)$ for $x \in X$. The maximum X-proximity of G, denoted by $\Delta_X(G)$, is the maximum of $\delta_X(u)$ for $u \in V(G)$.

In [1] the following problem has been suggested:

Determine for which integers $k \geqslant 3$ and positive integers $a_1, a_2, \ldots, a_k$ there exists a k-stratified graph $(G, \mathcal{S})$ with strata $X_1, X_2, \ldots, X_k$ such that $\Delta_{X_i}(G) = a_i$ for $i = 1, \ldots, k$.

The solution of this problem is given by the following theorem.

Theorem 1. *Let $k \geqslant 2$ be an integer, let $a_1, a_2, \ldots, a_k$ be positive integers. Then there exists a k-stratified graph $(G, \mathcal{S})$ with strata $X_1, X_2, \ldots, X_k$ such that $\Delta_{X_i}(G) = a_i$ for $i = 1, \ldots, k$.*

Proof. We construct pairwise vertex-disjoint graphs $H_0, H_1, \ldots, H_k$. The graph H_0 is the complete graph with k vertices $u_1, \ldots, u_k$. For $i = 1, \ldots, k$ the graph H_i is the Cartesian product of a path having a_i vertices and a complete graph with $k - 1$ vertices. Its vertices are $v_i(p, q)$ for all $p \in \{1, \ldots, a_i\}$ and all $q \in \{1, \ldots, k\} - \{i\}$. Two vertices $v_i(p_1, q_1)$, $v_i(p_2, q_2)$ are adjacent if and only if either $p_1 = p_2$ and

$q_1 \neq q_2$, or $|p_1 - p_2| = 1$ and $q_1 = q_2$. Now for $i = 1, \dots, k$ we join the vertex u_i of H_0 by edges with all vertices $v_i(1, q)$ of H_i. The resulting graph will be denoted by G. Now we construct the partition $\mathcal{S}$ of $V(G)$. We have $\mathcal{S} = \{X_1, \dots, X_k\}$, where the strata $X_1, \dots, X_k$ are defined so that $u_i \in X_i$ and $v_i(p, q) \in X_q$ for any i, p, q.

Consider the stratum X_i for some $i \in \{1, \dots, k\}$. For a vertex $v_i(p, q)$ of H_i we have $\delta_{X_i}\big(v_i(p, q)\big) = d\big(v_i(p, q), u_i\big) = p \leqslant a_i$ and in particular, $\delta_{X_i}\big(v_i(a_i, q)\big) = a_i$. For a vertex u_j of H_0 we have $\delta_{X_i}(u_j) = d(u_j, u_i) = 1 - \delta(i, j) \leqslant 1 \leqslant a_i$. If $j \neq i$, then for a vertex $v_j(p, q)$ of H_j we have $\delta_{X_i}\big(v_j(p, q)\big) = d\big(v_j(p, q), v_j(p, i)\big) = 1 - \delta(i, q) \leqslant 1 \leqslant a_i$. Hence $\Delta_{X_i}(G) = a_i$. □

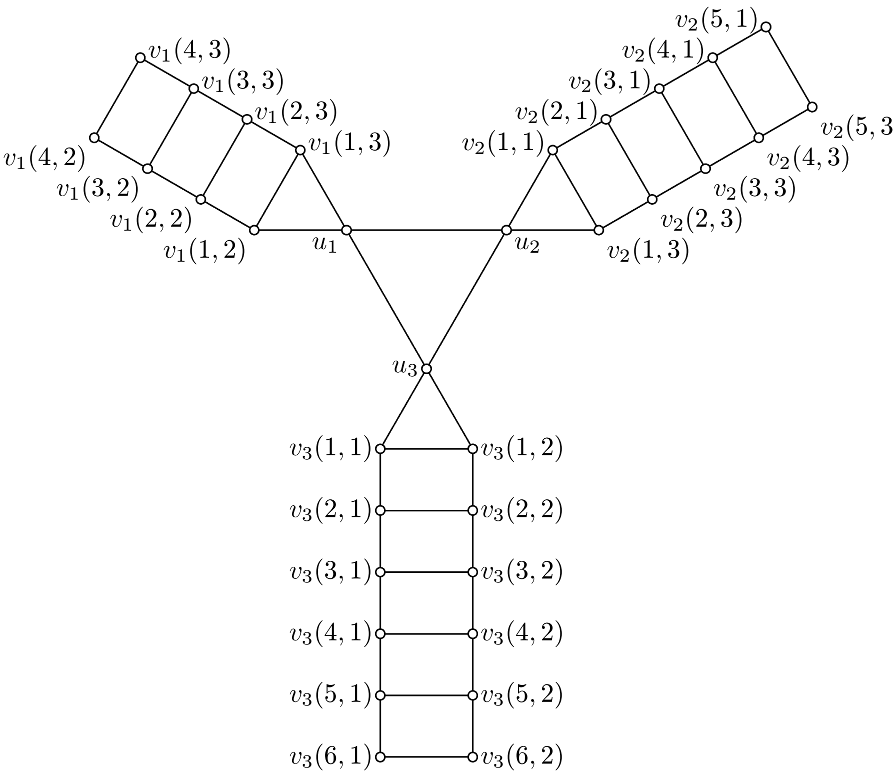

Fig. 1 shows the graph G for $k = 3$, $a_1 = 4$, $a_2 = 5$, $a_3 = 6$.

We will add a result concerning stratified trees. If $u \in V(G)$, $X \in \mathcal{S}$, then the X-eccentricity $e_X(u)$ of u is the maximum of $d(u, x)$ for $x \in X$. The minimum of $e_X(u)$ for all vertices $u \in V(G)$ is the X-radius of G, denoted by $\operatorname{rad}_X G$, and the maximum is the X-diameter of G, denoted by $\operatorname{diam}_X G$. By $\operatorname{rad} G$ and $\operatorname{diam} G$ we denote the usual radius and diameter of G, respectively.

We will consider a stratified tree $(T, \mathcal{S})$. If $X \in \mathcal{S}$, then by $T(X)$ we denote the least subtree of T which contains the set X. The tree $T(X)$ is the union of all paths connecting pairs of vertices of X in T.

Theorem 2. *Let $(T, \mathcal{S})$ be a stratified tree, let $X \in \mathcal{S}$. Then*

$$\operatorname{rad}_X T = \operatorname{rad} T(X),$$
$$\operatorname{diam}_X T \leqslant 2 \operatorname{rad}_X T - 1.$$

Proof. Suppose that there exists a vertex $u \in V(T) - V\big(T(X)\big)$ such that $e_X(u) = \operatorname{rad}_X T$. As T is a tree, there exists a unique vertex v of $T(X)$ whose distance from u is minimum. Now let $x \in X$. The path connecting v and x is in $T(X)$, while the path connecting u and v has only the vertex v in common with $T(X)$. Therefore the path connecting u and x is the union of these two paths, which implies $d(u, x) = d(u, v) + d(v, x)$ and thus $d(u, x) > d(v, x)$. As x was chosen arbitrarily, also $e_X(u) > e_X(v)$, which is a contradiction. Therefore all vertices v for which $e_X(v) = \operatorname{rad}_X T$ are in $T(X)$. Now consider a vertex $w \in V\big(T(X)\big)$. The paths connecting w with vertices of X are in $T(X)$; therefore $e(w) \geqslant e_X(w)$ where $e(w)$ denotes the (usual) eccentricity of w in $T(X)$. The eccentricity $e(w)$ is in fact the maximum of $d(w, z)$ taken over all terminal vertices of $T(X)$. Evidently all terminal vertices of $T(X)$ are in X and thus $e(w) \leqslant e_X(w)$ and consequently $e(w) = e_X(w)$. This implies $\operatorname{rad}_x T = \operatorname{rad} T(X)$. As $T(X)$ is a tree, we have

$$\operatorname{diam} T(X) \geqslant 2 \operatorname{rad} T(X) - 1 = 2 \operatorname{rad}_X T - 1.$$

The X-diameter $\operatorname{diam}_X T$ is the maximum of $d(u, x)$ for $u \in V(T)$ and $x \in X$. The diameter $\operatorname{diam} T(X)$ is in fact the maximum of $d(x, y)$, where x, y are terminal vertices of $T(X)$; evidently all terminal vertices of $T(X)$ belong to X. Hence

$$\operatorname{diam}_X T \geqslant \operatorname{diam} T(X) \geqslant 2 \operatorname{rad} T(X) - 1 = 2 \operatorname{rad}_X T - 1.$$

□

References

[1] *G. Chartrand, L. Hansen, R. Rashidi, N. Sherwani*: Distance in stratified graphs. Czechoslovak Math. J. *50(125)* (2000), 35—46.

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