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LEXICOGRAPHIC PRODUCTS OF HALF LINEARLY ORDERED GROUPS

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Abstract. The notion of the half linearly ordered group (and, more generally, of the half lattice ordered group) was introduced by Giraudet and Lucas [2].

In the present paper we define the lexicographic product of half linearly ordered groups. This definition includes as a particular case the lexicographic product of linearly ordered groups.

We investigate the problem of the existence of isomorphic refinements of two lexicographic product decompositions of a half linearly ordered group.

The analogous problem for linearly ordered groups was dealt with by Maltsev [5]; his result was generalized by Fuchs [1] and the author [3].

The isomorphic refinements of small direct product decompositions of half lattice ordered groups were studied in [4].

Keywords: half linearly ordered group, lexicographic product, isomorphic refinements

MSC 2000: 06F15

1. Preliminaries

Let G be a group and suppose that it is, at the same time, a partially ordered set.

We denote by $G\uparrow$ (or $G\downarrow$) the set of all $x \in G$ such that, whenever $y, z \in G$ and $y \leqslant z$, then $xy \leqslant xz$ (or $xy \geqslant xz$, respectively).

1.1. Definition. (Cf. [2].) G is said to be a half linearly ordered group if the following conditions are satisfied:

1) the partial order $\leqslant$ on G is non-trivial;
2) if $x, y, z \in G$ and $y \leqslant z$, then $yx \leqslant zx$;
3) $G = G\uparrow \cup G\downarrow$;
4) $G\uparrow$ is a linearly ordered set.

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The neutral element of G will be denoted by e. In view of 1), $G \neq \{e\}$. It is obvious that the following conditions are equivalent:

(i) $G{\downarrow} = \emptyset$;

(ii) G is a linearly ordered group with more than one element.

We denote by $\mathcal{HL}$ the class of all half linearly ordered groups. Next, let $\mathcal{HL}_1$ be the class of all elements of $\mathcal{HL}$ which fail to be linearly ordered.

We will apply the following results (cf. [2]):

1.2. Proposition. *Let $G \in \mathcal{HL}_1$. Then*

(i) *$G{\uparrow}$ is a subgroup of the group G having the index 2;*

(ii) *the partially ordered set $G{\downarrow}$ is isomorphic to $G{\uparrow}$;*

(iii) *if $x \in G{\uparrow}$ and $y \in G{\downarrow}$, then x and y are incomparable.*

1.3. Proposition. *Let $G \in \mathcal{HL}_1$. Then*

(i) *for each $x \in G$ with $x \neq e$ the relation $x^2 = e \Longleftrightarrow x \in G{\downarrow}$ is valid;*

(ii) *if $x \in G{\downarrow}$ and $y \in G{\uparrow}$, then $xyx = y^{-1}$;*

(iii) *the group $G{\uparrow}$ is abelian.*

2. Lexicographic products

Let I be a nonempty set and for each $i \in I$ let $G_i \in \mathcal{HL}$. We denote by G^1 the cartesian product of the groups G_i $(i \in I)$. The elements of G^1 will be expressed as $g = (\ldots, g_i, \ldots)_{i\in I}$ or $g = (g_i)_{i\in I}$; g_i is the component of g in G_i. We put

$$I(g) = \{i \in I\colon\ g_i \neq e\}.$$

Now let us suppose that I is a linearly ordered set and that either

(i_0) $G_i \in \mathcal{HL}_1$ for each $i \in I$, or

(ii_0) $G_i \notin \mathcal{HL}_1$ for each $i \in I$.

If (i_0) is valid then we choose an element $g^{(1)} \in G^1$ such that $g_i^{(1)} \in G_i{\downarrow}$ for each $i \in I$.

We denote by $G^0(g^{(1)})$ the set of all $g \in G^1$ such that either

(i_1) $g_i \in G_i{\uparrow}$ for each $i \in I$ and the set $I(g)$ is well-ordered, or

(ii_1) $g_i \in G_i{\downarrow}$ for each $i \in I$ and the set $I(g^{(1)}g^{-1})$ is well-ordered.

Let A and $B(g^{(1)})$ be the sets of all elements of $G^0(g^{(1)})$ which satisfy the condition (i_1) or (ii_1), respectively. Hence $G^0(g^{(1)})$ is a disjoint union of A and $B(g^{(1)})$.

2.1. Lemma. *$G^0(g^{(1)})$ is a subgroup of the group G^1.*

P r o o f. If $a, a' \in A$, then clearly $aa' \in A$ and $a^{-1} \in A$. If $b, b' \in B(g^{(1)})$, then in view of 1.2 (i) we have $bb' \in A$ and $b^{-1} \in B(g^{(1)})$. For $a \in A$ and $b \in B(g^{(1)})$ the relations $ab \in B(g^{(1)})$ and $ba \in B(g^{(1)})$ are valid. $\square$

We define a binary relation $\leqslant$ on $G^0(g^{(1)})$ as follows:

for $a, a' \in A$ we put $a \leqslant a'$ if either $a = a'$, or $a \neq a'$ and $a_{i(0)} < a'_{i(0)}$, where $i(0)$ is the least element of I $(a'a^{-1})$;

for $b, b' \in B(g^{(1)})$ we define the relation $b \leqslant b'$ analogously;

if $a \in A$ and $b \in B(g^{(1)})$, then we consider a and b to be incomparable (i.e., neither $a \leqslant b$ nor $b \leqslant a$).

It is easy to verify that the relation $\leqslant$ is a partial order on $G^0(g^{(1)})$.

2.2. Lemma. *$G^0(g^{(1)})$ is a half linearly ordered group.*

Proof. In view of 2.1, $G^0(g^{(1)})$ is a group. We consider the above defined partial order on $G^0(g^{(1)})$. We have to verify that the conditions 1)–4) from 1.1 are satisfied.

Choose $i(0) \in I$. We have $G_{i(0)} \in \mathcal{HL}$, thus the partial order on $G_{i(0)}$ is nontrivial. Hence there exists $g^{(i(0))} \in G_{i(0)}$ such that $e < g^{(i(0))}$. In view of the definition of $G^0(g^{(1)})$ there exists $g \in G^0(g^{(1)})$ such that $g_{i(0)} = g^{(i(0))}$ and $g_i = e$ for each $i \in I \setminus \{i(0)\}$. Then $e < g$ and hence 1) holds.

The validity of 2) is obvious. Next, $G^0(g^{(1)})\uparrow = A$ and $G^0(g^{(1)})\downarrow = B(g^{(1)})$, whence 3) is valid. From the definition of the partial order $\leqslant$ on $G^0(g^{(1)})$ we conclude that the condition 4) is satisfied as well. $\square$

2.3. Proposition. *Let G^1 be as above and let $g^{(1)}, g^{(2)} \in G^1$ be such that $g_i^{(1)}, g_i^{(2)} \in G_i\downarrow$ for each $i \in I$. Then there exists an isomorphism φ of $G^0(g^{(1)})$ onto $G^0(g^{(2)})$ such that $\varphi(g^{(1)}) = g^{(2)}$ and $\varphi(a) = a$ for each $a \in A$.*

Proof. Let A be as above. Let us now write B^1 instead of $B^0(g^{(1)})$. Analogously we write B^2 instead of $B^0(g^{(2)})$.

Let $b_1 \in B^1$. We have $Ag^{(1)} = B^1$. Hence there exists a uniquely determined element $a \in A$ with $ag^{(1)} = b_1$. We put $\varphi(b_1) = ag^{(2)}$. In particular, we obtain $\varphi(g^{(1)}) = g^{(2)}$. For each $a \in A$ we set $\varphi(a) = a$. Hence φ is a mapping of $B^0(g^{(1)})$ into $B^0(g^{(2)})$.

For each $a_1, a_2 \in A$ we have $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$. Let $b_1, b_1' \in B^1$. There exist $a, a' \in A$ with $b_1 = ag^{(1)}$, $b_1' = a'g^{(1)}$. Then $\varphi(b_1) = ag^{(2)}$, $\varphi(b_1') = a'g^{(2)}$. Next, $b_1 b_1' \in A$ and hence

$$\varphi(b_1 b_1') = b_1 b_1' = ag^{(1)}a'g^{(1)} = a(a')^{-1}$$

(in the last step we have applied 1.3 (ii)). Similarly,

$$\varphi(b_1)\varphi(b_1') = ag^{(2)}a'g^{(2)} = a(a')^{-1},$$

thus $\varphi(b_1 b_1') = \varphi(b_1)\varphi(b_1')$.

Let a, b_1 be as above and let $a_1 \in A$. Then

$$\varphi(a_1 b_1) = \varphi(a_1 a g^{(1)}) = a_1 a g^{(2)},$$
$$\varphi(a_1)\varphi(b_1) = a_1 a g^{(2)} = \varphi(a_1 b_1).$$

Consider the element $g^{(1)} a_1$. Clearly $g^{(1)} a_1 \in B^1$. There exists $x \in G^0(g^{(1)})$ with $g^{(1)} a_1 = x g^{(1)}$. If $x \in B^1$, then $x g^{(1)} \in A$, which is impossible. Hence $x \in A$ and thus

$$\varphi(g^{(1)} a_1) = \varphi(x g^{(1)}) = x g^{(2)}.$$

Moreover, in view of 1.2 and 1.3 we have

$$x = g^{(1)} a_1 g^{(1)} = a_1^{-1},$$

hence

$$\varphi(g^{(1)} a_1) = a_1^{-1} g^{(2)}.$$

Next, in a similar way we obtain

$$\varphi(g^{(1)})\varphi(a_1) = g^{(2)} a_1 = a_1^{-1} g^{(2)} = \varphi(g^{(1)} a_1).$$

It is obvious that φ is a monomorphism. If $y \in G^0(g^{(2)})$, then either $y \in A$ and hence $\varphi(y) = y$, or there is $a \in A$ with $y = a g^{(2)}$ and in this case $\varphi(a g^{(1)}) = y$. Thus φ is a bijection. By summarizing, φ is an isomorphism of the group $G^0(g^{(1)})$ onto the group $G^0(g^{(2)})$.

Let $x_1, x_2 \in G^0(g^{(1)})$, $y_i = \varphi(x_i)$ $(i = 1, 2)$. Suppose that x_1 and x_2 are comparable. Then either (i) $x_1, x_2 \in A$, or (ii) $x_1, x_2 \in B^1$. If (i) holds, then we have trivially

$$x_1 \leqslant x_2 \Longleftrightarrow y_1 \leqslant y_2.$$

Let (ii) be valid. There are $a_1, a_2 \in A$ with $x_i = a_i g^{(1)}$ $(i = 1, 2)$. We have

$$x_1 \leqslant x_2 \Longleftrightarrow a_1 \leqslant a_2 \Longleftrightarrow y_1 \leqslant y_2,$$

completing the proof. □

Proposition 2.3 shows that if we consider our construction up to isomorphism, then the choice of $g^{(1)}$ is not essential. Let us write G^0 instead of $G^0(g^{(1)})$.

2.4. Definition. Let G_i $(i \in I)$, $g^{(1)}$ and G^0 be as above. Then G^0 is said to be the lexicographic product of half linearly ordered groups G_i and we express this fact by writing

(1) $$G^0 = \Gamma_{i \in I} G_i;$$

G_i are called lexicographic factors of G^0.

It is clear that if G_i are linearly ordered groups, then the above definition coincides with the usual notion of the lexicographic product of linearly ordered groups (cf., e.g., [1], [5]).

In what follows we assume that all G_i belong to $\mathcal{HL}_1$.

Let (1) be valid and let φ be an isomorphism of a half linearly ordered group G onto G^0. Then φ is called a lexicographic product decomposition of G.

3. Congruence relations

In this section some auxiliary results on congruence relations will be obtained. Next we prove that to each lexicographic product decomposition of $G{\uparrow}$ there corresponds a lexicographic product decomposition of G.

Congruence relations on half lattice ordered groups were investigated in [4]. In the particular case of half linearly ordered groups stronger results than those in [4] can be proved.

Let $G \in \mathcal{HL}$ and $a, b \in G$, $a \leqslant b$. Then we write $a \vee b = b$ and $a \wedge b = a$. Hence $\vee$ and $\wedge$ are partial binary operations on G.

Let ϱ be an equivalence relation on G. Consider the following conditions for this relation:

(i) ϱ is a congruence relation with respect to the group operation.

(ii) If $\circ \in \{\vee, \wedge\}$, $x, y, z \in G$, $x\varrho y$ and if $x \circ z$ exists in G, then $y \circ z$ exists in G and $(x \circ z)\varrho(y \circ z)$.

3.1. Definition. An equivalence relation ϱ on G is said to be a congruence relation on G if it satisfies the conditions (i) and (ii).

The set of all congruence relations on G will be denoted by $\operatorname{Con} G$. The symbol $\operatorname{con} G$ denotes the set of all equivalence relations on G which satisfy the condition (i). Both the sets $\operatorname{Con} G$ and $\operatorname{con} G$ are partially ordered in the usual way; then they are complete lattices.

The symbols $\operatorname{Con} G{\uparrow}$ and $\operatorname{con} G{\uparrow}$ have analogous meanings.

For $x \in G$ and $\varrho \in \operatorname{con} G$ we put $\overline{x}(\varrho) = \{y \in G\colon\ y\varrho x\}$. For $x \in G{\uparrow}$ and $\tau \in \operatorname{con} G{\uparrow}$ the meaning of $\overline{x}(\tau)$ is analogous.

3.2. Lemma. *Let $G \in \mathcal{HL}_1$ and let X be a subgroup of $G{\uparrow}$. Then X is normal in G.*

Proof. Let $x \in X$ and $g \in G$. If $g \in G{\uparrow}$, then in view of 1.3 (iii) we have $g^{-1}xg = x$. Next, let $g \in G{\downarrow}$. Then according to 1.3 (i), $g^{-1} = g$. Thus 1.3 (ii) yields that $g^{-1}xg = x^{-1}$ and hence $g^{-1}xg \in X$. $\square$

For $\varrho \in \operatorname{con} G$ and $x, y \in G\uparrow$ we put $x\varrho^1 y$ iff $x\varrho y$. Next, for $\tau \in \operatorname{con} G$ and $u, v \in G$ we set $u\tau' v$ iff $(u^{-1}v)\tau e$.

3.3. Lemma. *Let $\varrho \in \operatorname{Con} G$ and $\tau \in \operatorname{con} G$. Then*

(i) *$\varrho^1 \in \operatorname{con} G\uparrow$; moreover, if $\varrho \in \operatorname{Con} G$, then $\varrho^1 \in ConG\uparrow$;*

(ii) *$\tau' \in \operatorname{con} G$; if $\tau \in \operatorname{Con} G\uparrow$, then $\tau' \in \operatorname{Con} G$.*

Proof. The assertion (i) is obvious. By applying 3.2 and using the same method as in Section 3 of [4] we conclude that $\tau' \in \operatorname{con} G$. If, moreover, $\tau \in \operatorname{Con} G\uparrow$, then the results of [4] yield that τ' belongs to $\operatorname{Con} G$. $\square$

For $\varrho \in \operatorname{Con} G$ the symbol G/ϱ has the obvious meaning. If we assume only that $\varrho \in \operatorname{con} G$, then (G/ϱ) denotes the corresponding factor group. For $\tau \in \operatorname{Con} G\uparrow$ or $\tau \in \operatorname{con} G\uparrow$ the symbols $G\uparrow/\tau$ or $(G\uparrow/\tau)$ have analogous meanings.

Suppose that

$$\varphi\colon\ G\uparrow \longrightarrow \Gamma_{i\in I} A_i \tag{1}$$

is a lexicographic product decomposition of the linearly ordered group $G\uparrow$ and that $G\downarrow \neq \emptyset$. For $i \in I$ and $x, y \in G\uparrow$ we put $x\tau^i y$ if

$$\varphi(x)_i = \varphi(y)_i.$$

3.4. Lemma. *For each $i \in I$, τ^i belongs to $\operatorname{con} G\uparrow$.*

Proof. This is an immediate consequence of (1). $\square$

Let us remark that, in general, τ^i need not belong to $\operatorname{Con} G\uparrow$.

In what follows we suppose that $A_i \neq \{e\}$ for each $i \in I$. For $i \in I$ and $a_i \in A_i$ we denote by a_i^0 the element of $G\uparrow$ such that

$$(\varphi(a_i^0))_i = a_i, \quad (\varphi(a_i^0))_{i(1)} = e \quad \text{for each} \quad i(1) \in I \setminus \{i\};$$

in view of the definition of the lexicographic product, such an element a_i^0 does exist in $G\uparrow$. Next we put

$$A_i^0 = \{a_i^0\colon\ a_i \in A_i\}.$$

For $x \in G\uparrow$ we set

$$\chi_i(\overline{x}(\tau^i)) = \varphi(x)_i^0.$$

Then χ_i is correctly defined (i.e., the result of applying χ_i does not depend on the choice of the element $x \in \overline{x}(\tau^i)$).

For $x, y \in G\!\uparrow$ we define $\overline{x}(\tau^i) \leqslant \overline{y}(\tau^i)$ to be valid in $(G\!\uparrow/\tau^i)$ if and only if

$$\varphi(x)^0_i \leqslant \varphi(y)^0_i.$$

Also, the relation $\leqslant$ on $(G\!\uparrow/\tau^i)$ is correctly defined. We obviously have

3.5. Lemma. *Under the relation $\leqslant$, $(G\!\uparrow/\tau^i)$ is a linearly ordered group and χ_i is an isomorphism of this linearly ordered group onto A_i.*

Put $H_i^{(1)} = (G\!\uparrow/\tau^i)$ under the linear order defined above. For $x \in G\!\uparrow$ let

$$\varphi^{(1)}(x) = (\overline{x}(\tau^i))_{i \in I}.$$

Then 3.5 and (1) yields that we have a lexicographic product decomposition

$$\varphi^{(1)}\colon\ G\!\uparrow \longrightarrow \Gamma_{i\in I} H_i^{(1)}. \tag{2}$$

Let us have a fixed element i of the set I. We construct $\tau^i \in \operatorname{con} G\!\uparrow$ and $(\tau^i)' \in \operatorname{con} G$ as above. Put

$$H_i^{(2)} = \{\overline{y}((\tau^i)')\colon\ y \in G\!\downarrow\},$$
$$G_i = H_i^{(1)} \cup H_i^{(2)}.$$

Then G_i is a group, namely, $G_i = (G/(\tau^i)')$.

Choose a fixed element $g^{(1)}$ of $G\!\downarrow$. By means of $g^{(1)}$ we define a relation $\leqslant$ on $H_i^{(2)}$ as follows.

Let $h^{(1)}, h^{(2)} \in H_i^{(2)}$. There are $y_1, y_2 \in G\!\downarrow$ such that

$$h^{(j)} = \overline{y}_j((\tau^i)') \quad (j = 1, 2). \tag{$*$}$$

Then $y_1 y_2^{-1} \in G\!\uparrow$. We put

$$h^{(1)} \leqslant h^{(2)}$$

if

$$\overline{y_1 g^{(1)}}(\tau^i) \leqslant \overline{y_2 g^{(2)}}(\tau_i).$$

The relation $\leqslant$ is correctly defined on $H_i^{(2)}$ (i.e., it does not depend of the choice of y_1, y_2 satisfying $(*)$). It is a routine to verify that this relation is reflexive, transitive and antisymmetric. Finally, we have either $h^{(1)} \leqslant h^{(2)}$ or $h^{(2)} \leqslant h^{(1)}$. Hence $\leqslant$ is a linear order on $H_i^{(2)}$. Also, $H_i^{(2)}$ is isomorphic to $H_i^{(1)}$.

If $h^{(1)} \in H_i^{(1)}$ and $h^{(2)} \in H_i^{(2)}$, then we consider $h^{(1)}$ and $h^{(2)}$ to be incomparable. Thus $\leqslant$ turns out to be a partial order on G_i.

Now let us verify that G_i is a half linearly ordered group; we have to consider the conditions 1)–4) from 1.1.

Since $H_i^{(1)}$ is isomorphic to $A_i \neq \{0\}$ and A_i is linearly ordered we conclude that the partial order on G_i is non-trivial, thus 1) holds. The condition 2) is obviously valid. Clearly $G_i{\uparrow} = H_i^{(1)}$ and $G_i{\downarrow} = H_i^{(2)}$. Thus 3) and 4) are also satisfied.

For $g_1, g_2 \in G$ and $i \in I$ we put $g_1 \varrho^i g_2$ if either

$$g_1, g_2 \in G{\uparrow} \quad \text{and} \quad g_1 \tau_i g_2,$$

or

$$g_1, g_2 \in G{\downarrow} \quad \text{and} \quad g_1 \tau_i' g_2.$$

Then $\varrho^i \in \operatorname{con} G$. For each $g \in G$ we put

$$\varphi_1(g) = (\overline{g}(\varrho^i))_{i \in I}.$$

Hence φ_1 is a mapping of G into the cartesian product of the half linearly ordered groups G_i $(i \in I)$.

3.6. Lemma. *φ_1 is an isomorphism of the group G into the cartesian product of the groups G_i $(i \in I)$.*

P r o o f. For each $i \in I$, the mapping

$$g \longrightarrow \overline{g}(\varrho^i) \quad (g \in G)$$

is a homomorphism of the group G into the group G_i. Hence φ_1 is a homomorphism of the group G into the cartesian product of the groups G_i $(i \in I)$.

Let $g, g' \in G$ and suppose that $\varphi_1(g) = \varphi_1(g')$. If $g \in G{\uparrow}$ and $g' \in G{\downarrow}$, then $\overline{g}(\varrho^i) \neq \overline{g'}(\varrho^i)$ for each $i \in I$, whence $\varphi_1(g) \neq \varphi_1(g')$. Thus either (i) $g, g' \in G{\uparrow}$, or (ii) $g, g' \in G{\downarrow}$.

Let (i) be valid. Then in view of (2) we obtain that $g = g'$. Next suppose that (ii) holds and let $g^{(1)}$ be as above. Then $gg^{(1)}, g'g^{(1)} \in G{\uparrow}$ and $\varphi_1(gg^{(1)}) = \varphi_1(g'g^{(1)})$. Thus $gg^{(1)} = g'g^{(1)}$ yielding that $g = g'$. Therefore φ_1 is a monomorphism. □

3.7. Lemma. *The set $\varphi_1(G)$ coincides with the underlying set of the lexicographic product $\Gamma_{i \in I} G_i$ (constructed with respect to the element $\varphi_1(g^{(1)})$).*

P r o o f. In view of 2.4 we have to verify that the conditions (i_1) and (ii_1) from Section 2 are satisfied. The relation (2) yields that (i_1) is valid. Let $g \in G$ and suppose that $(\varphi_1(g))_i \in G_i{\downarrow} = H_i^{(2)}$ for each $i \in I$. Then $g^{(1)}g^{-1} \in G{\uparrow}$ and hence the condition (i_1) holds for $\varphi_1(g^{(1)}g^{-1})$; thus (ii_1) is satisfied. □

3.8. Lemma. *Let $g, g' \in G$. Then $g \leqslant g'$ if and only if $\varphi_1(g) \leqslant \varphi_1(g')$.*

Proof. Let $g \leqslant g'$. Then either (i) $g, g' \in G\uparrow$, or (ii) $g', g' \in G\downarrow$. Suppose that (i) holds. Then in view of (2), $\varphi_1(g) \leqslant \varphi_1(g')$. Next, let (ii) be valid. Thus $gg^{(1)}, g'g^{(1)} \in G\uparrow$ and $gg^{(1)} \leqslant g'g^{(1)}$. Hence $\varphi_1(gg^{(1)}) \leqslant \varphi_1(g'g^{(1)})$ and then $\varphi_1(g)\varphi_1(g^{(1)}) \leqslant \varphi_1(g')\varphi_1(g^{(1)})$ yielding that $\varphi(g) \leqslant \varphi(g')$.

Conversely, suppose that $\varphi_1(g) \leqslant \varphi_1(g')$. From this we infer that we have either (i) or (ii). This shows that g, g' are comparable and that $g > g'$ cannot hold. $\square$

3.9. Theorem. *Let $G \in \mathcal{HL}_1$ and suppose that for $G\uparrow$ the relation (1) is valid. Then φ_1 is a lexicographic product decomposition of G.*

Proof. This is a consequence of 3.6, 3.7 and 3.8. $\square$

Consider a lexicographic product decomposition

$$\psi\colon\ G \longrightarrow \Gamma_{i\in I} T_i. \tag{3}$$

We denote by φ the mapping ψ reduced to the subset $G\uparrow$ of G; next we put $T_i\uparrow = A_i$ for each $i \in I$. Then we obtain that (1) holds.

For $g, g' \in G$ and $i \in I$ we put $g\varrho_i^* g'$ if $\psi(g)_i = \psi(g')_i$. Thus if $g\varrho_i^* g'$ then either (i) $g, g' \in G\uparrow$, or (ii) $g, g' \in G\downarrow$.

We apply the symbols τ_i and ϱ_i as above. If (i) is valid, then

$$g\varrho_i^* g' \iff g\tau_i g' \iff g\varrho_i g'.$$

If (ii) holds, then we obtain

$$g\varrho_i^* g' \iff (gg^{(1)})\varrho_i^*(g'g^{(1)}) \iff (gg^{(1)})\varrho_i(g'g^{(1)}) \iff g\varrho_i g^{(1)}.$$

Thus $\varrho_i = \varrho_i^*$ for each $i \in I$.

In view of (3), the group (G/ϱ_i^*) is isomorphic to T_i. Hence we have

3.10. Lemma. *For each $i \in I$, the groups T_i and G_i are isomorphic.*

In more detail, the isomorphism under consideration is constructed as follows. Let $i \in I$ and $t^i \in T_i$. We denote by X the class of all $g \in G$ such that $\psi(g)_i = t^i$. Then $X \in G/\varrho_i^* = G/\varrho_i = G_i$ and we assign the element X of G_i to the element t^i.

Let $(t^i)'$ be another element of T and let $X' \in G_i$ be assigned to $(t^i)'$. Then according to the above defined partial order on G/ϱ_i we have $X < X'$ if and only if $t < t'$. Hence the mapping of T_i onto G_i under consideration turns out to be also an isomorphism with respect to the partial order. By summarizing, we get

3.11. Proposition. *Let us apply the notation as above and let $i \in I$. Then the half linearly ordered groups T_i and G_i are isomorphic.*

4. Isomorphic refinements

Again, let $G \in \mathcal{HL}$ and let us have two lexicographic product decompositions

$$\alpha\colon\ G \longrightarrow \Gamma_{i\in I} G_i,$$
$$\beta\colon\ G \longrightarrow \Gamma_{k\in K} T_k.$$

These lexicographic product decompositions are said to be isomorphic if there exists a monotone bijection $b\colon\ I \longrightarrow K$ such that for each $i \in I$, G_i is isomorphic to $T_{b(i)}$.

4.1. Definition. The lexicographic product decomposition β is said to be a refinement of α if for each $i \in I$ there exists a subset $K(i)$ of K and a lexicographic product decomposition

$$\alpha_i\colon\ G_i \longrightarrow \Gamma_{k\in K(i)} T_k$$

such that, whenever $g \in G$, $i \in I$ and $k \in K(i)$, then

$$\beta(g)_k = \alpha_i(\alpha(g)_i)_k.$$

It is easy to verify that this definition is equivalent to the notion of refinement as applied in [1], [5] (though we use a different notation).

We obviously have

4.2. Lemma. *Let α and β be isomorphic lexicographic product decompositions of G and let α' be a refinement of α. Then there exists a refinement β' of β such that α' and β' are isomorphic.*

Suppose that the relation (1) from Section 3 is valid. Next suppose that we have a lexicographic product decomposition

$$\chi\colon\ G\!\uparrow \longrightarrow \Gamma_{j\in J} B_j \tag{1'}$$

such that (1′) is a refinement of (1).

By applying the lexicographic product decomposition φ we construct $\varphi^{(1)}$ as in Section 3; there we have proved that (2) holds.

Analogously, by applying χ we construct a lexicographic product decomposition

$$\chi^{(1)}\colon\ G\!\uparrow \longrightarrow \Gamma_{j\in J} K_j^{(1)}. \tag{2'}$$

Since χ is a refinement of φ, from the construction of $\varphi^{(1)}$ and $\chi^{(1)}$ we obtain

4.3. Lemma. *$\chi^{(1)}$ is a refinement of $\varphi^{(1)}$.*

Again, let φ_1 be as in Section 3. In view of 3.9 we have a lexicographic product decompositions

$$\varphi_1\colon\ G \longrightarrow \Gamma_{i\in I} G_i.$$

By using $\chi^{(1)}$ we obtain analogously

$$\chi_1\colon\ G \longrightarrow \Gamma_{j\in J} K_j,$$

where, under a similar notation as in Section 3, $K_j = K_j^{(1)} \cup K_j^{(2)}$. By a routine verification we get

4.4. Lemma. *χ_1 is a refinement of φ_1.*

4.5. Theorem. *Any two lexicographic product decompositions of a half linearly ordered group G have isomorphic refinements.*

P r o o f. If G is a linearly ordered group, then the assertion is valid in view of [5].

Suppose that $G \in \mathcal{HL}_1$ and that α, β are lexicographic product decompositions of G. Let us denote by α_0 and β_0 the mappings α and β, respectively, reduced to the subset $G\!\uparrow$ of G. Hence α_0 and β_0 are lexicographic product decompositions of $G\!\uparrow$. (Cf. Fig. 1.)

$$\begin{tikzcd}
 & \alpha \arrow[r] \arrow[d, dash, dashed] & \alpha_0 \arrow[d] & \beta_0 \arrow[d] & \beta \arrow[l] \arrow[d, dash, dashed] & \\
 & \alpha_{01} \arrow[d] & \alpha_0^1 \arrow[l] \arrow[d] & \beta_0^1 \arrow[r] \arrow[d] & \beta_{01} \arrow[d] & \\
\mu_2 \arrow[uur] \arrow[r, dash, dashed] & \mu_1 \arrow[r, dash, dashed] & \mu \arrow[r, dash, dashed] & \nu \arrow[r, dash, dashed] & \nu_1 \arrow[r, dash, dashed] & \nu_2 \arrow[uul]
\end{tikzcd}$$

Fig. 1

If we construct $\alpha_0^{(1)}$ and α_{01} (similarly as we have constructed $\varphi^{(1)}$ and φ_1 above) then in view of 3.11 the lexicographic product decompositions α and α_{01} are isomorphic.

Under analogous notation, the lexicographic product decompositions β and β_{01} are isomorphic.

Since $G\!\uparrow$ is a linearly ordered group, according to [5] there exist lexicographic product decompositions μ and ν of $G\!\uparrow$ such that

μ is a refinement of $\alpha_0^{(1)}$,

ν is a refinement of $\beta_0^{(1)}$,

μ and ν are isomorphic.

Now we construct the lexicographic product decompositions μ_1 and ν_1 of G in the same way as we did for φ_1. In view of 4.4, μ_1 is a refinement of α_{01}, and ν_1 is a refinement of β_{01}.

Hence according to 4.2 there exist lexicographic product decompositions μ_2 and ν_2 such that

μ_2 is a refinement of α,

ν_2 is a refinement of β,

μ_2 and ν_2 are isomorphic.

This completes the proof. $\square$

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