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RECOVERY OF BAND-LIMITED FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS FROM IRREGULAR SAMPLES

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Abstract. Using the techniques of approximation and factorization of convolution operators we study the problem of irregular sampling of band-limited functions on a locally compact Abelian group G. The results of this paper relate to earlier work by Feichtinger and Gröchenig in a similar way as Kluvánek's work published in 1969 relates to the classical Shannon Sampling Theorem. Generally speaking we claim that reconstruction is possible as long as there is sufficient high sampling density. Moreover, the iterative reconstruction algorithms apply simultaneously to families of Banach spaces.

Keywords: irregular sampling, band-limited functions, locally compact Abelian group, solid Banach spaces

MSC 2000: 43A15, 47B38, 42C15, 22B99

1. Introduction and notations

In this paper we propose three theorems on the representation of band-limited functions in a solid Banach space of functions on a locally compact Abelian (lca.) group G. Although the method of proof is closely related to the techniques described in an earlier paper by Feichtinger and Gröchenig [8] (for $\mathrm{G} = \mathbb{R}^d$) the results are more general in the following aspects:

- they are valid for general lca. groups, thus providing the irregular analogue of Kluvánek's sampling theorem over general lca. groups, which in turn has found attention in the applied community recently (cf. [1]);

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- for Euclidean spaces the restriction to polynomial weights, hence to Banach spaces of tempered distributions is replaced by the much weaker *Beurling-Domar non-quasianalyticity condition*;
- whereas it has been assumed in [7] that the sampling set is relatively separated the results presented here apply also to sampling sets of arbitrary high density, including those containing clusters, a situation which occurs naturally in many applications.

Throughout this paper we shall use the following standard notations from harmonic analysis. The group operation on the lca. group G will be written additively, we use $\mathrm{d}x$ as a symbol for its translation invariant Haar measure, and $\hat{G}$ for the dual group, consisting of characters χ, the continuous homomorphism from G into the unit circle, satisfying thus $\chi(x+y)=\chi(x)\chi(y)$. Translation of a function by $x \in G$ is written as $L_x f(z) = f(z-x)$. Furthermore the symbol w will be reserved to some (fixed, non-quasianalytic) *Beurling-Domar weight*, i.e., to a continuous (or at least locally integrable) function w satisfying the following conditions:

1. $w(x) \geqslant \delta > 0$ for all $x \in G$;
2. $w(x+y) \leqslant w(x)w(y)$, for all $x, y \in G$;
3. the Beurling-Domar condition (BD)†

$$\sum_n \log(w(nx))/n^2 < \infty \text{ for all } x \in G.$$

We write $L^p_w(G)$ for the space of all (equivalence classes of) complex-valued measurable functions f on G such that $fw \in L^p$, with norm

$$\|f\|_{p,w} = \left(\int_G |f(x)|^p w^p(x)\,\mathrm{d}x\right)^{1/p}. \tag{1}$$

$L^p_w(G)$ is a Banach space with respect to this norm. For $p=1$ we obtain a so-called *Beurling algebra* $L^1_w(G)$ [13], which is a commutative Banach algebra with respect to convolution, defined as usual within $L^1(G)$. The space $C_c(G)$ of all complex-valued continuous functions on G with compact support satisfies $C_c(G) \subset L^p_w(G) \subseteq L^p(G)$ for $1 \leqslant p \leqslant \infty$, and is dense in $L^p_w(G)$, for $p < \infty$. The dual of a weighted L^p-spaces L^p_w is identified (using the standard identifcation of the unweighted case) $L^{p'}_{1/w}(G)$, with $1/p + 1/p' = 1$ for $p \in [1,\infty)$. $C^0(G)$ denotes the closed subspace of $L^\infty(G)$ consisting of all continuous functions vanishing at infinity, which coincides with the closure of $C_c(G)$ in $L^\infty(G)$.

If T is a bounded linear operator on a normed space $(B, \|\cdot\|_B)$, we write $|||T|||$ or $|||T|||_B$ for its operator norm. We use the symbol "$\hookrightarrow$" to describe a continuous embedding between topological vector spaces.

† It guarantees that there are many non-zero band-limited functions in $L^1_w(G)$ (see [13]).

For the definition of *Wiener amalgam spaces* over lca. groups, such as $W(L^1, l^\infty_{1/w})$ or $W(C^0, B)$, we refer to [3], [11] or [6]. For $f \in L^1(G)$ and $\chi \in \hat{G}$ the (ordinary) Fourier transform is given as usual, by $\hat{f}(\chi) = \int_G f(x)\chi(-x)\,\mathrm{d}x$. The indicator function of a set Q is denoted by $\mathbf{1}_Q$.

2. Spaces of band-limited functions

The aim of this paper is to provide reconstruction theorems for spaces of band-limited functions. In order to emphasize that the estimates described below do not just apply to individual Banach spaces but hold simultaneously for large classes of Banach spaces, we have to describe those first as follows: for a given (BD)-weight function w we denote by $\mathscr{B}_w$ the collection of all Banach spaces $(B, \|\cdot\|_B)$ which satisfy the following five conditions:

(B1) $(B, \|\cdot\|_B)$ is a Banach space, continuously embedded into $L^1_{\mathrm{loc}}(G)$, i.e., for every compact set $K \subseteq G$, there exists a constant $C_K > 0$ such that $\int_K |f(x)|\,\mathrm{d}x \leqslant C_K \|f\|_B$ for all $f \in B$.

(B2) $(B, \|\cdot\|_B)$ is *solid*, i.e., if $f \in L^1_{\mathrm{loc}}(G)$ satisfies $|f(x)| \leqslant |g(x)|$ almost everywhere on G for some $g \in B$, then $f \in B$ and $\|f\|_B \leqslant \|g\|_B$.

(B3) B is invariant under translations, i.e., $L_x f \in B$, for all $f \in B$, $x \in G$.

(B4) The weight function w controls the operator norm of L_x on B, i.e., $|||L_x|||_B \leqslant w(x)$, or more explicitly by:

$$\|L_x f\|_B \leqslant w(x)\|f\|_B \;\; \forall x \in G, \;\; f \in B. \tag{2}$$

(B5) $(B, \|\cdot\|_B)$ is a Banach convolution module over $L^1_w(G)$, i.e.,

$$L^1_w * B \subseteq B, \;\; \text{and} \;\; \|g * f\|_B \leqslant \|g\|_{1,w}\,\|f\|_B \;\; \forall f \in B, \;\; g \in L^1_w. \tag{3}$$

As it will not restrict the generality of our statements we assume for convenience throughout this paper that the weight function is symmetric, i.e., satisfies the condition $w(-x) = w(x)$, for $x \in G$. Moreover, we mention that (B5) is a consequence of (B4) whenever $C_c(G)$ is dense in $(B, \|\cdot\|_B)$ (the statement following via vector-valued integration).

Examples. For a given Beurling-Domar weight w, the family $\mathscr{B}_w$ contains the spaces $L^p_{w^\alpha}$, for $-1 \leqslant \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$, but also a large variety of spaces, such as spaces based on dyadic decompositions and many others.

Lemma 2.1. *For B satisfying the conditions* (B1)–(B4), *one has:*

$$B \hookrightarrow W(L^1, l^\infty_{1/w}).$$

Proof. For any $\varphi \in C_c(G)$, the set $K := \operatorname{supp}(\varphi)$ is a compact. Hence

$$\|f \cdot L_y\varphi\|_1 = \|L_y(L_{-y}f \cdot \varphi)\|_1 = \|L_{-y}f \cdot \varphi\|_1 \leqslant C_K \|L_{-y}f \cdot \varphi\|_B,$$

because $\operatorname{supp}(L_y f \cdot \varphi) \subseteq K$ and $B \hookrightarrow L^1_{\mathrm{loc}}(G)$. Using the solidity of $(B, \|\cdot\|_B)$ and the boundedness of φ it follows from (2) that

$$\|L_{-y}f \cdot \varphi\|_B \leqslant \|\varphi\|_\infty \|L_{-y}f\|_B \leqslant Cw(-y)\|f\|_B.$$

Altogether we may conclude that the function $\|f \cdot L_y\varphi\|_1$ belongs to $L^\infty_{1/w}$, which by the definition of Wiener amalgam spaces implies the inclusion stated in the lemma. The continuous embedding is expressed by the inequality

$$\|f\| \cdot L_y\varphi\|_1 w^{-1}(y) \leqslant C_K C \|f\|_B.$$

$\square$

Remark. If $G = \mathbb{R}^n$ and $w(x) = (1+|x|)^\alpha$ for some $\alpha \geqslant 0$ then Lemma 2.1 implies $B \hookrightarrow \mathscr{S}'$, since $\mathscr{S}(\mathbb{R}^n) \hookrightarrow W(C^0, l^1_\alpha)$ for any $\alpha \geqslant 0$. In the present paper we can use more general weight functions, such as $w(x) = \mathrm{e}^{\alpha|x|^\gamma}$, for $\alpha \geqslant 0$, $\gamma \in [0,1)$. For $\gamma > 0$ the Fourier transform has to be understood in the sense of ultra-distribution on $\mathbb{R}^n$. For details see [5].

From Lemma 2.1 it follows that $B \in \mathscr{B}_w$ is a well-defined Banach space of distributions on G, for which a generalized Fourier transform is already defined (for details see Theorem 3, [5]). Indeed, one has $\mathscr{F}W(L^1, l^\infty_{1/w}) \subset W(\mathscr{F}L^\infty_{1/w}, l^\infty)$. In particular, the following definition makes sense:

Given a compact set $\Omega \subseteq \hat{G}$ with non-void interior the space of all band-limited functions in B with spectrum in Ω is well defined via

$$B^\Omega = \{f \in B \colon \operatorname{spec}(f) = \operatorname{supp} \hat{f} \subseteq \Omega\}. \tag{4}$$

For a fixed neighborhood U_0 of the identity we define the *local maximal function* $f^\#$ by

$$f^\#(x) = \sup_{z \in x+U_0} |f(z)|. \tag{5}$$

For a neighborhood U of the identity we define the local U oscillation by

$$\mathrm{Osc}_U f(x) = \sup_{z \in U} |f(x+z) - f(x)|. \tag{6}$$

Throughout we will only use neighborhoods $U \subseteq U_0$ because then

$$\mathrm{Osc}_U f(x) \leqslant 2f^{\#}(x) \tag{7}$$

while generally one has the following *pointwise* estimates

$$(f * h)^{\#}(x) \leqslant |f| * h^{\#}(x), \tag{8}$$

$$\mathrm{Osc}_U (f * h)(x) \leqslant |f| * \mathrm{Osc}_U h(x). \tag{9}$$

The *density of discrete families* in G will we described as follows: A set $X = (x_i)_{i \in I}$ is U*-dense* provided $\bigcup_{i \in I} (x_i + U) = G$.

If U is open and X is a U-dense family in G it is always (by the local compactness of G) possible to construct a uniform partition of unity $\Psi = (\psi_i)_{i \in I}$ of size U, i.e., Ψ satisfies the following conditions (cf. [4]):

1. ψ_i is measurable and $0 \leqslant \psi_i(x) \leqslant 1$, $\forall i \in I$.
2. $\mathrm{supp}(\psi_i) \subseteq x_i + U$, $\forall i \in I$.
3. $\sum_{i \in I} \psi_i(x) = 1$, $\forall x \in G$.

In this situation we will refer to the family Ψ shortly as partition of unity of size U (for short UPU), associated with the family $X = (x_i)$. Families of B-splines on R are typical UPUs associated with regular families of the form αZ (satisfying an additional finite overlapping condition).

Following Feichtinger and Gröchenig [7], we define for such a UPU the following irregular spline approximation operator

$$\mathrm{Sp}_\psi f = \sum_{i \in I} f(x_i) \psi_i,$$

which may be considered as an irregular spline approximation of f.

The assumption $U \subseteq U_0$ implies for Ψ as above the pointwise estimate

$$|\mathrm{Sp}_\psi f(x)| \leqslant f^{\#}(x), \tag{10}$$

which yields—if $f^{\#} \in B$—by the solidity of $(B, \|\cdot\|_B)$

$$\|\mathrm{Sp}_\psi f\|_B \leqslant \|f^{\#}\|_B. \tag{11}$$

In the present paper we prove the following three theorems on the representation of band-limited functions:

Theorem 2.2. *Assume that $h \in L^1_w$ satisfies $\hat{h}(t) = 1$ on an open neighborhood of Ω and* $\operatorname{spec}(h) \subseteq \Omega_0$, *and that $U \subseteq U_0$ is small enough such that for some $g \in L^1_w$ with $\hat{g} = 1$ on Ω_0, $\|\operatorname{Osc}_U g\|_{1,w} < \|h\|^{-1}_{1,w}$. Then, for any $B \in \mathscr{B}_w$ the following is true: Every $f \in B^\Omega$ can be reconstructed from its sampled values on any U-dense discrete subset $X = (x_i)_{i\in I}$ of G, in the form of a series expansion*

$$f = \sum_{i\in I} f(x_i) e_i, \tag{12}$$

where $e_i \in L^1_w(G)$, and $\operatorname{spec}(e_i) \subseteq \Omega_0$. *Furthermore, the series* (12) *converges in B and uniformly on compact sets.*

Theorem 2.3. *Assume that $g, h \in L^1_w(G)$ are band-limited with $\hat{g}(x) \neq 0$ on Ω,* $\operatorname{spec}(h) \subseteq \Omega_0$ *and $\hat{h}(x) = 1$ on* $\operatorname{spec}(g)$. *Then for any sufficiently small neighborhood U of the identity (say $U \subseteq U_1$, with U_1 only depending on g, h, w) there exists $C = C(U) > 0$ such that for any $B \in \mathscr{B}_w$* and *any U-dense family $Y = (y_j)_{j\in J}$, $f \in B^\Omega$ has a representation of the form*

$$f = \sum_{j\in J} c_j(f) L_{y_j} g \tag{13}$$

with coefficients satisfying

$$\Big\| \sum_{j\in J} c_j(f)\varphi_j \Big\|_B \leqslant C\|f\|_B, \tag{14}$$

where $\Phi = (\varphi_j)_{j\in J}$ is any partition of unity of associated with the discrete sampling set Y.

Theorem 2.4. *If $\Omega \subseteq \hat{G}$ is compact, and $g \in L^1_w(G)$ is band-limited with $\hat{g}(\omega) \neq 0$ on Ω, then there exist U_1 and U_2 such that for any two U_1-dense resp. U_2-dense discrete families of points $X = (x_i)_{i\in I}$ and $Y = (y_j)_{j\in J}$, and for $B \in \mathscr{B}_w$, every $f \in B^\Omega$ has the representation of the form*

$$f = \sum_{j\in J} c_j(f(x_i)_{i\in I}) L_{y_j} g, \tag{15}$$

where the series converges in B and uniformly on compact sets. The notation is meant to indicate that the coefficients c_j depend only on the sampling values $f(x_i)$

and the linear coefficient map $C\colon\ (f(x_i))_{i\in I} \longrightarrow (c_j)_{j\in J}$ *is continuous in the sense that for some* $C_0 > 0$ *(depending only on* U_1 *and* U_2*) one has*

$$\Big\|\sum_{j\in J} c_j(f)\varphi_j\Big\|_B \leqslant C_0\Big\|\sum_{i\in I} f(x_i)\psi_i\Big\|_B, \tag{16}$$

where $\Psi = (\psi_i)_{i\in I}$ *and* $\Phi = (\varphi_j)_{j\in J}$ *are arbitrary* UPUs *associated with* X *and* Y *respectively.*

3. Auxiliary results

The proofs of the theorems are based on a series of lemmas, which will be given in the subsequent sections. We begin with some general observations.

Lemma 3.1. *Any* $g \in L^1_w(G)$ *which is band-limited also belongs to* $W(C^0, l^1_w)$*, in particular* $g^{\#} \in L^1_w$*, and* g *is uniformly continuous and bounded.*

Proof. Since $g \in L^1_w$, we have $\hat g \in \mathscr{F}L^1_w$. By the assumption $\hat g$ has compact support and thus $\hat g \in W(\mathscr{F}L^1_w, l^1)$. By [5], Theorem 2

$$g = \mathscr{F}^{-1}\hat g \in W(\mathscr{F}L^1, l^1_w) \subseteq W(C^0, l^1_w) \subseteq C^0(G),$$

hence uniformly continuous (and bounded). Since a continuous function g is in $W(C^0, l^1_w)$ if and only if $g^{\#} \in L^1_w(G)$ the proof of lemma is complete. $\square$

Lemma 3.2. *For every* $g \in L^1_w(G)$ *with compact* $\operatorname{spec}(g) = \Omega_1$*, and any* $\varepsilon > 0$*, there exists a neighborhood* U *of the identity such that*

$$\|\mathrm{Osc}_U(g)\|_{1,w} < \varepsilon.$$

Proof. Let g be a band-limited function in L^1_w. By Lemma 3.1 $g^{\#} \in L^1_w(G)$ and thus one can find for $\varepsilon > 0$ a compact set $K \subseteq G$ such that

$$\int_{G\setminus K} g^{\#}(x)w(x)\,\mathrm{d}x < \varepsilon/4,$$

and hence using (7)

$$\int_{G\setminus K} \mathrm{Osc}_U(g)(x)w(x)\,\mathrm{d}x < \varepsilon/2. \tag{17}$$

Over K we have the following estimate:

$$\int_K \mathrm{Osc}_U(g)(x)w(x)\,\mathrm{d}x \leqslant \max_{x\in K} w(x)\|\mathrm{Osc}_U\, g\|_\infty |K| = C_{K,w}\|\mathrm{Osc}_U\, g\|_\infty,$$

where $|K|$ is the Haar measure of K. Since g is a uniformly continuous function, one may choose U small enough to obtain

$$C_{K,w}\|\mathrm{Osc}_U(g)\|_\infty < \varepsilon/2. \tag{18}$$

From (17) and (18) we infer that

$$\|\mathrm{Osc}_U(g)\|_{1,w} < \varepsilon.$$

□

Lemma 3.3. *Given a* (BD)*-weight w and a compact set $\Omega \subseteq \hat{G}$, one can find for every $\eta > 0$ some neighborhood $U \subseteq U_0$ such that for any $B \in \mathscr{B}_w$*

$$\|\mathrm{Osc}_U(f)\|_B \leqslant \eta\|f\|_B, \quad \forall f \in B^\Omega.$$

In particular, $\{f \in B^\Omega,\ \|f\|_B \leqslant 1\}$ is an equicontinuous family in $(B, \|\cdot\|_B)$, for any space $B \in \mathscr{B}_w$.

Proof. For any compact set Ω one can find some band-limited $g \in L^1_w(G)$ with $\hat{g}(\omega) = 1$ for all $\omega \in \Omega$, since the Beurling-Domar condition implies that $\mathscr{F}L^1_w$ is a Wiener algebra in Reiter's sense [13]. Now $f = f * g$ for $f \in B^\Omega$ implies via (9)

$$\|\mathrm{Osc}_U\, f\|_B \leqslant \|\mathrm{Osc}_U\, g\|_{1,w}\|f\|_B$$

for any U. Choosing now U according to Lemma 3.2 the proof is finished. □

Remark. The above argument also shows that a similar estimate is valid for $\mathrm{Osc}^\#_U\, f$ (just use (8) and replace $\|\mathrm{Osc}_U\, g\|_{1,w}$ by $\|\mathrm{Osc}^\#_U\, g\|_{1,w}$), as it is also not difficult to find that Lemma 3.2 is also valid for $\mathrm{Osc}^\#_U\, g$.

Lemma 3.4. *For given w and any compact set $\Omega \subseteq \hat{G}$ there exists $C_\Omega = C(w, \Omega)$ such that for all $B \in \mathscr{B}_w$ one has for all $f \in B^\Omega$*

$$\|f^\#\|_B \leqslant C_\Omega\|f\|_B.$$

Proof. We choose $g \in (L^1_w)^{\Omega_1}$ such that $\hat{g}(\omega) = 1$ on Ω. Then we have $f = f*g$, and therefore $f^\# \leqslant |f|*g^\#$ by (8). From this pointwise estimate the norm inequality follows by applying the solid B-norm and assumption (B5), with $C_\Omega = \|g^\#\|_{1,w}$. □

4. Approximation of convolution operators

Let $X = (x_i)_{i\in I}$ and $Y = (y_j)_{j\in J}$ be any two sets of sampling points and $(\psi_i)_{i\in I}$ and $(\Phi_j)_{j\in J}$ be partitions of unity associated with X and Y respectively. We suppose h is a continuous function with $h^{\#} \in L^1_w(G)$. As in [4], we consider the following approximations of the convolution operator $C_h\colon\ f \to h * f$.

$$A_1 f = (\mathrm{Sp}_\Psi f) * h = \Big(\sum_{i\in I} f(x_i)\psi_i\Big) * h, \tag{19}$$

$$A_2 f = (D_\Psi f) * h = \sum_{i\in I} \langle \psi_i, f\rangle L_{x_i} h, \tag{20}$$

$$A_3 f = (D^+_\Psi f) * h = \sum_{i\in I}\ f(x_i)\Big(\int \psi_i(x)\,\mathrm{d}x\Big) L_{x_i} h, \tag{21}$$

$$A_4 f = [D_\Phi(\mathrm{Sp}_\Psi f)] * h = \sum_{j\in j}\Big[\sum_{i\in I} f(x_i)\Big(\int \psi_i(x)\varphi_j(x)\,\mathrm{d}x\Big)\Big] L_{y_j} h. \tag{22}$$

If $C_c(G)$ is dense in B, the solidity of B implies that the partial sums of the series $\sum\limits_{i\in I} f(x_i)\psi_i$ are norm convergent in B (otherwise one has at least pointwise convergence). Hence we may write $A_1 f = \sum\limits_{i\in I} f(x_i)(\psi_i * h)$ if this is more suitable for our purpose, and likewise for other operators. Essentially as a consequence of the convolution theorems for Wiener amalgam spaces the operators A_1 and A_3 are well defined and bounded on $W(C,B)$, while A_2, A_4 are bounded from B to $W(C,B)$. In particular, all of them are well defined on B^{Ω_0}, due to Lemma 3.4 (recall that $\mathrm{spec}(h) \subseteq \Omega_0$).

The following two lemmas will be the key to the main results:

Lemma 4.1. *Assume that $h \in L^1_w$ satisfies $\hat h(\omega) = 1$ on $\Omega \subseteq \hat G$. Then for every $\gamma > 0$ there is neighborhood of the identity $U_1 = U_1(\eta)$ such that*

$$\|C_h f - A_1 f\|_B < \gamma \|f\|_B, \quad \forall f \in B^\Omega, \quad B \in \mathscr{B}_w,$$

as long as the sampling set $X = (x_i)_{i\in I}$ is U_1-dense. If in addition h is band-limited the same is true for suitable neighborhoods U_2 and U_3 for the approximation operators A_2 and A_3 respectively.

P r o o f. Case $i = 1$. Since Ψ is of size $U \subseteq U_0$ the pointwise estimate

$$|f - \mathrm{Sp}_\psi f|(x) \leqslant \mathrm{Osc}_U f(x) \tag{23}$$

is valid, and we obtain by Lemma 3.3 and the assumption (B5)

$$\begin{aligned}\|C_h f - A_1 f\|_B &\leqslant \|(f - \operatorname{Sp}_\psi f) * |h|\,\|_B \\ &\leqslant \|\operatorname{Osc}_U f\|_B\, \|h\|_{1,w} \leqslant \eta \|f\|_B\, \|h\|_{1,w}.\end{aligned}$$

Case $i = 2$. Rewriting the definition of A_2 we obtain

$$C_h f - A_2 f = f * h - (D_\Psi f) * h = \Big(\sum_{i\in I} \psi_i f\Big) * h - \Big\langle \sum_{i\in I} \psi_i, f\Big\rangle L_{x_i} h.$$

For any fixed $i \in I$ the expression can be written as

$$\begin{aligned}\int_G \psi_i(y) f(y) h(x-y)\,\mathrm{d}y - \int_G \psi_i(y) f(y) h(x-x_i)\,\mathrm{d}y \\ = \int_G \psi_i(y) f(y) [h(x-y) - h(x-x_i)]\,\mathrm{d}y.\end{aligned}$$

Taking absolute values and summing over i in I one obtains

$$|C_h f - A_2 f|(x) \leqslant |f| * \operatorname{Osc}_U h(x),$$

which implies by the solidity of B and the choice of U according to Lemma 3.3 that for $f \in B$

$$\|C_h f - A_2 f\|_B \leqslant \|f\|_B\, \|\operatorname{Osc}_U h\|_{1,w} \leqslant \eta \|f\|_B\, \|h\|_{1,w}.$$

Case $i = 3$. We have $C_h - A_3 = (C_h - A_1) + (A_1 - A_3)$. Since

$$\begin{aligned}|A_1 f(x) - A_3 f(x)| &= \Big|\sum_{i\in I} f(x_i)\psi_i * h - \sum_{i\in I} f(x_i)\Big(\int_G \psi_i(y)\,\mathrm{d}y\Big) L_{x_i} h(x)\Big| \\ &= \Big|\sum_{i\in I} f(x_i) \int_G \psi_i(y) h(x-y)\,\mathrm{d}y \\ &\quad - \sum_{i\in I} f(x_i) \int_G \psi_i(y) h(x-x_i)\,\mathrm{d}y\Big| \\ &\leqslant \sum_{i\in I} |f(x_i)| \int_G \psi_i(y) |h(x-y) - h(x-x_i)|\,\mathrm{d}y \\ &\leqslant \sum_{i\in I} |f(x_i)| \int_G \psi_i(y) \operatorname{Osc}_U h(x-y)\,\mathrm{d}y \\ &\leqslant \operatorname{Sp}_\psi(|f|) * \operatorname{Osc}_U h(x),\end{aligned}$$

it follows, using again Lemma 3.3, that

$$\|A_1 f - A_3 f\|_B \leqslant \|\operatorname{Sp}_\psi(|f|)\|_B \|\operatorname{Osc}_U h\|_{1,w},$$

and thus using (11) and Lemma 3.4 for U small enough (by Lemma 3.3)

$$\|A_1 f - A_3 f\| \leqslant \eta \|f\|_B \, \|h\|_{1,w}.$$

This completes the proof of the lemma. $\square$

Lemma 4.2. *If $h, h_1, h_2 \in L^1_w(G)$ are band-limited functions with common spectrum Ω_0, then for any $\eta > 0$ there exist neighborhoods U_4, U_5 such that for all $B \in \mathscr{B}_w$ and all* UPUs *Ψ of size U_4 and all* UPUs *Φ of size U_5:*

(i) $\|C_h f - A_4 f\|_B \leqslant \eta \|f\|_B \, \|h_1\|_{1,w}$ *for all* $f \in B^{\Omega_0}$,

(ii) $\|C_{h_1 * h_2} f - A_5 f\|_B \leqslant \eta \|f\|_B \, \|h_1\|_{1,w} \, \|h_2\|_{1,w}$,

where

$$A_5 f = [D_\Phi(\mathrm{Sp}_\Psi f * h_1)] * h_2 = \sum_{i \in I,\, j \in J} f(x_i) \Big(\int \psi_i * h(z) \varphi_j(z) \,\mathrm{d}z \Big) L_{y_j} h_2. \tag{24}$$

P r o o f. For the proof of part (i) see [7]. Concerning (ii) observe that for given $\eta > 0$ and $\Omega_0, h_1, h_2 \in L^1_w$ with the $\mathrm{spec}(h_1) \cup \mathrm{spec}(h_2) \subseteq \Omega_0$, we can find neighborhoods U_4 and U_5 (using $h = h_1$ in the definition of A_4) such that $\|C_h f - A_4 f\|_B \leqslant \eta \|f\|_B \, \|h_1\|_{1,w}$ by (i). Since $A_5 = C_{h_2} A_4$, this implies

$$\begin{aligned} \|C_{h_1 * h_2} f - A_5 f\|_B &\leqslant \|h_2\|_{1,w} \, \|(C_{h_1} - A_4) f\|_B \\ &\leqslant \eta \|f\|_B \, \|h_1\|_{1,w} \, \|h_2\|_{1,w}. \end{aligned}$$

$\square$

5. Factorization of convolutions

Let $g \in L^1_w(G)$ be any band-limited function. We choose another band-limited function $h \in L^1_w(G)$ with $\hat{h}(x) = 1$ on $\mathrm{spec}(g)$ and write Ω_0 for $\mathrm{spec}(h)$. Consequently $g * h = g$, or $C_g C_h = C_g$.

In the sequel some of the operators to be discussed may have nice mapping properties (e.g. C_h will map all of B into B^{Ω_0}), but on the other hand it will be also of interest to consider them only as operators on B^{Ω_0} itself, because they may have better properties there. In order to express this difference more clearly we will use the notation $|||T|||_{B^{\Omega_0}}$ in order to describe the operator norm of some operator on B^{Ω_0}. Actually, in this way we can find that certain operators are contractions and thus allow to build a Neumann series expansion from them.

In order to get into this situation let us assume that we deal with one of the operators $A(h, X, \Psi)$ from the previous section, which is a good approximation to C_h on B^{Ω_0}, such that the remainder $R := C_h - A$ is a contraction on B^{Ω_0}. In other words we assume

$$|||R|||_{B^{\Omega_0}} = |||C_h - A|||_{B^{\Omega_0}} < 1.$$

This implies that

$$D = \sum_{k=0}^{\infty} R^k$$

is well defined on B^{Ω_0}. For these operators we prove the following factorization results.

Lemma 5.1. *Under the above assumptions C_g factorizes as given below:*
(i) $C_g = C_g AD$ *on* B^{Ω_0}*;*
(ii) $C_g = DAC_g$ *on all of* B*;*
(iii) $C_g = C_g DA$ *on* B^{Ω_0}.

Proof. (i) Since $C_g = C_g C_h$, we have

$$C_g = C_g(A + R) = C_g A + C_g R.$$

Repeating this process n times, we get $C_g = C_g A \sum_{k=0}^{n} R^k + C_g R^{n+1}$ for all $n \geqslant 0$, hence $C_g = C_g AD$, by going to the limit $n \to \infty$ (R is a contraction on B^{Ω_0}).

(ii) Again $C_g = C_h C_g$ implies

$$C_g = C_h C_g = AC_g + RC_g = AC_g + RAC_g + R^2 C_g.$$

Iterating this process and passing to the limit leads to

$$C_g = \left(\sum_{k=0}^{\infty} R^k\right) AC_g = DAC_g.$$

Since C_g maps B into B^{Ω_0} and A, D are well defined on all of B, the assertion (ii) holds for the whole space B.

(iii) Since for $f \in B^{\Omega_0}$ we have $C_g f = C_g C_g f$ and $C_g = DAC_g$ it suffices to verify that $C_g R^k A f = C_g A R^k f$ for all $k \geqslant 0$. This relation is obvious for $k = 0$. Assuming that it is true for all $k \leqslant n$ we derive therefrom

$$\begin{aligned}
C_g R^{n+1} A f &= C_g (C_h - A) R^n A f \\
&= C_g C_h R^n A f - C_g A R^n A f \\
&= C_g R^n A f - C_g A R^n A f \\
&= C_g A R^n f - C_g A R^n A f \\
&= C_g A R^n C_h f - C_g A R^n A f \\
&= C_g A R^n (C_h - A) f = C_g A R^{n+1} f.
\end{aligned}$$

□

6. Proof of the main theorems

In this last section the observations made so far will be put together in order to derive the main results of this paper.

P r o o f of Theorem 2.2. Since $\hat{h}(t) = 1$ on an open neighborhood of Ω, we assume that $\hat{h}(t) = 1$ on Ω', where Ω' is an open neighborhood of Ω. Hence there exists $g \in L^1_w(G)$ such that $\hat{g}(x) = 1$ on Ω and $\operatorname{spec}(g) \subseteq \Omega'$. This entails that $g * h = g$, i.e., $C_g C_h = C_g$. Since we can find for $\eta < 1$ some U according to Lemma 4.1, we have $|||C_h - A_1|||_B < 1$ on B^{Ω_0} for any U-dense set $X = (x_i)$ and any UPU Ψ associated to X. Now, using Lemma 5.1, we obtain $C_g = D_1 A_1 C_g$ on B, which implies that $C_g = C_g D_1 A_1$ on B^{Ω}. Since $f * g = f$ for all $f \in B^{\Omega}$ we have

$$f = C_g f = D_1 A_1 C_g f = D_1 \left(\sum_{i \in I} f(x_i) \psi_i * h \right) = \sum_{i \in I} f(x_i) D_1(\psi_i * h).$$

As D_1 is continuous on B^{Ω_0}, the above series converges in B.

As the series $\sum\limits_{i \in I} f(x_i)(\psi_i * h)$ is unconditionally convergent in B and D_1 is continuous on B^{Ω_0}, the series on the right-hand side is convergent in B (actually in $W(C, B)$). On the other hand we have $\psi_i * h \in L^1_w$, and $L^1_w \in \mathscr{B}_w$. Therefore, the continuity of D_1 on L^1_w implies that

$$e_i = D_1(\psi_i * h) \in L^1_w(G)$$

and $\operatorname{spec}(e_i) \subseteq \operatorname{spec}(h) = \Omega_0$, for all $i \in I$. Thus Theorem 2.2 holds. □

In the same way we can use the operators A_3 and A_4 to find the corresponding representations for $f \in B^{\Omega}$. For applications, A_3 is the most useful operator.

P r o o f of Theorem 2.3. On account of Lemma 5.1 we have $C_g = C_g A_2 D_2$. Since $\hat{h}(t) = 1$ on $\operatorname{spec}(g)$, which contains an open neighborhood of Ω, by the Wiener-Levy theorem ([13], Chapter 6) there exists a local inverse of g, that is, a band-limited function $g_1 \in L^1_w(G)$ such that $\operatorname{spec}(g_1) \subseteq \operatorname{spec}(g)$ and $\hat{g}_1 \hat{g} \equiv 1$ on Ω, hence $f = f * g_1 * g$. It follows that

$$\begin{aligned} f = C_g C_{g_1} f = C_g A_2 D_2 C_{g_1} f &= C_g \sum_{j \in J} \langle \varphi_j, D_2 C_{g_1} f \rangle C_g L_{y_i} h \\ &= \sum_{j \in J} \langle \varphi_j, D_2 C_{g_1} f \rangle L_{y_i} h = \sum_{j \in J} c_j(f) L_{y_i} g, \end{aligned}$$

where

$$c_j(f) = \langle \varphi_j, D_2(f * g_1) \rangle.$$

To obtain an estimate for the sequence $(c_j(f))$ we write $F = D_2(f * g_1) = D_2 C_{g_1} f$. Then we have

$$\|F\|_B \leqslant |||D_2||| \cdot \|g_1\|_{1,w} \, \|f\|_B.$$

To obtain the required estimate (14) we observe that for any $x \in G$ the sum $\sum_{j \in J} \langle \varphi_j, F \rangle \varphi_j(x)$ is absolutely convergent and contains only those non-zero terms where $x \in U + y_j$. Denoting the characteristic function of $U - U$ by k we have

$$|\langle \varphi_j, F \rangle \varphi_j(x)| \leqslant \langle L_x k, |F| \rangle \varphi_j(x) = (|F| * k(x)) \varphi_j(x).$$

Taking the B norm on both sides, we get

$$\Big\| \sum_{j \in J} \langle \varphi_j, F \rangle \varphi_j \Big\|_B \leqslant \| \, |F| * k \|_B \leqslant \|F\|_B \, \|k\|_{1,w}.$$

This completes the proof of Theorem 2.3. □

P r o o f of Theorem 2.4. The proof of Theorem 2.4 can be obtained either by combining Theorem 2.2 and Theorem 2.3 or following the arguments of [4]. This somewhat lengthy and technical method requires the use of Lemma 4.2 (ii). It is left to the interested reader to fill in the details.

Final remarks. In this paper we have emphasized the fact that the reconstruction method can be described in a unified way, for a variety of function spaces. This is of relevance for applications, since it would not be so useful if together with the sampling values also assumptions on the function (e.g. to belong to some weighted L^p-space) would have to be made, or relevant information would have to be provided by the user. The results as presented are a good basis for error analysis, such as jitter, aliasing or truncation errors. This will be a topic of a subsequent paper (see [10]).

From the practical side one might prefer to stress the fact that all the algorithms also have a description in the form of an iterative algorithm, with a geometric decay rate. Thus they could be redescribed by saying that for all the function spaces under consideration a certain sampling density U would allow to guarantee a relative approximation error (in any of the function space norms $\|\cdot\|_B$, for $B \in \mathscr{B}_w$) by running a *fixed number of iterations only.*

Last but not least we want to mention that even for the case of the ordinary Hilbert space $L^2(\mathbb{R})$ these iterative algorithms (requiring admittedly a bit of oversampling) deliver a reconstruction of a band-limited function from its sampling values with building blocks which are *better concentrated near the sampling points* than the standard frame algorithm. This aspect has been investigated in the master thesis by Tobias Werther [14], cf. also [9].

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