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DUAL CONVERGENCES OF ITERATION PROCESSES FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper we establish a dual weak convergence theorem for the Ishikawa iteration process for nonexpansive mappings in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and then apply this result to study the problem of the weak convergence of the iteration process.

Keywords: Banach limit, dual convergence theorem, duality mapping, Ishikawa iteration process, nonexpansive mapping

MSC 2000: 47H09, 47H10

1. Introduction

Let D be a nonempty subset of a Banach space E. A mapping $T\colon D \to E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for all x, y in D. In [4], Ishikawa introduced a new iteration process as

$$(1)\qquad \begin{cases} x_{n+1} = (1 - t_n)x_n + t_n T y_n \\ y_n = (1 - s_n)x_n + s_n T x_n, \quad n = 1, 2, \ldots, \end{cases}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,1]$ satisfying certain restrictions. If $s_n = 0$, the Ishikawa iteration process reduces to the Mann iteration process [7]

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, \quad n = 1, 2, \ldots$$

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The convergence of the sequence $\{x_n\}$ defined by (1) for a nonexpansive self-mapping T in a uniformly convex Banach space with a Fréchet differentiable norm has been studied by Deng [2], Reich [8], and Tan and Xu [10].

In this paper we first establish a dual weak convergence theorem for the Ishikawa iteration process (1) for nonexpansive mappings in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Then we apply this result to study the weak convergence of the iteration process (1) in the same Banach space. Further, we obtain the dual weak convergence for the iteration process (1) under different restrictions on $\{t_n\}$ and $\{s_n\}$ in the same Banach space. We should point out that the Banach space E in the main results is not assumed to be uniformly convex and that our results also apply to all L^p spaces $(1 < p < \infty)$.

2. Preliminaries and Lemmas

Let E be a real Banach space and let I denote the identity operator. Recall that a Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in $U = \{x \in E\colon \|x\| = 1\}$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. The norm is said to be Fréchet differentiable if for each $x \in U$, this limit is attained uniformly for $y \in U$. Finally, the norm is said to be uniformly Fréchet differentiable if the limit is attained uniformly for $(x, y) \in U \times U$. In this case, E is said to be uniformly smooth. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The reverse is false (cf. [9]).

The duality mapping from E into the family of nonempty subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^*\colon (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$

It is single valued if and only if E is smooth. If E is smooth, the duality mapping J is said to be weakly sequentially continuous at 0 if $\{J(x_n)\}$ converges to 0 in the sense of the weak-star topology of E^*, whenever $\{x_n\}$ converges weakly to 0 in E.

Let μ be a mean on positive integers $\mathbb{N}$, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n\colon n \in \mathbb{N}\} \leqslant \mu(a) \leqslant \sup\{a_n\colon n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leqslant \mu_n(a_n) \leqslant \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in E. Then we can define a real valued continuous convex function φ on E by

$$\varphi(z) = \mu_n \|x_n - z\|^2$$

for each $z \in E$. The following lemma was proved in [5] (cf. [9]).

Lemma 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let μ be a Banach limit and $u \in E$. Then*

$$\mu_n \|x_n - u\|^2 = \inf_{z\in E} \mu_n \|x_n - z\|^2$$

if and only if

$$\mu_n(z, J(x_n - u)) = 0$$

for all $z \in E$, where J is the duality mapping of E into E^*.

In the sequel, we need the following lemmas for the proof of our main results.

Lemma 2 ([2]). *Let D be a subset of a normed space E and let $T\colon D \to E$ be a nonexpansive mapping. Let a sequence $\{x_n\}$ in E be defined by*

$$\begin{cases} x_{n+1} = (1 - t_n)x_n + t_n T y_n, \\ y_n = (1 - s_n)x_n + s_n T x_n, \quad n = 1,\ 2, \ldots, \end{cases}$$

where two real sequences $\{t_n\}$ and $\{s_n\}$ satisfy the following conditions

(i) $0 \leqslant t_n \leqslant t < 1$ *and* $\sum\limits_{n=1}^{\infty} t_n = \infty$,

(ii) $0 \leqslant s_n \leqslant 1$ *and* $\sum\limits_{n=1}^{\infty} s_n < \infty$.

If $\{x_n\}$ is bounded, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Lemma 3 ([10]). *Let D be a subset of a normed space E, and $T\colon D \to X$ a nonexpansive mapping. Let a sequence $\{x_n\}$ in D be defined by*

$$\begin{cases} x_{n+1} = (1-t_n)x_n + t_n T y_n, \\ y_n = (1-s_n)x_n + s_n T x_n, \quad n = 1, 2, \ldots, \end{cases}$$

where $0 \leqslant t_n$, $s_n \leqslant 1$ for all $n \geqslant 1$. Then

$$\|x_{n+1} - p\| \leqslant \|x_n - p\|$$

for all $n \geqslant 1$ and all $p \in F(T)$, where $F(T)$ denotes the set of fixed points of T.

Finally, we recall the following definition: a sequence $\{a_n\} \in \ell^\infty$ is said to be almost convergent if all Banach limits agree (Lorentz's characterization is that $\lim\limits_{n\to\infty} \Big(\sum\limits_{i=1}^{n} a_{i+k}\Big)/n$ exists uniformly in $k \geqslant 0$ [6]). We also say that a sequence $\{x_n\}$ in a Banach space E is weakly almost convergent to $z \in E$ if the weak $\lim\limits_{n\to\infty} \Big(\sum\limits_{i=1}^{n} x_{i+k}\Big)/n = z$ uniformly in $k \geqslant 0$.

3. Main results

Now, we establish a dual weak convergence theorem for the Ishikawa iteration process in a reflexive and strictly convex Banach space.

Theorem 1. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and let $T\colon E \to E$ be a nonexpansive mapping. Let a sequence $\{x_n\}$ in E be defined by*

$$\begin{cases} x_{n+1} = (1-t_n)x_n + t_n T y_n, \\ y_n = (1-s_n)x_n + s_n T x_n, \quad n = 1, 2, \ldots, \end{cases}$$

where two real sequences $\{t_n\}$ and $\{s_n\}$ satisfy the following conditions:

(i) $0 \leqslant t_n \leqslant t < 1$ *and* $\sum\limits_{n=1}^{\infty} t_n = \infty$,

(ii) $0 \leqslant s_n \leqslant 1$ *and* $\sum\limits_{n=1}^{\infty} s_n < \infty$.

If the fixed point set $F(T)$ of T is nonempty, then there exists a point v in $F(T)$ such that $\{J(x_n - v)\}$ weakly almost converges to 0.

Proof. Let $p \in F(T)$ and let x_0 be an initial point of $\{x_n\}$. Then $\|x_n - p\| \leqslant \|x_0 - p\|$ for all n and hence $\{x_n\}$ is bounded by Lemma 3. Let μ be a Banach limit and define a real valued function φ on E by

$$\varphi(z) = \mu_n \|x_n - z\|^2$$

for each $z \in E$. Then φ is a continuous convex function and $\varphi(z) \to \infty$ as $\|z\| \to \infty$. Since X is reflexive, φ attains its infimum over E (cf. [3, p. 12]). Let

$$K = \{u \in E\colon\ \varphi(u) = \inf_{z \in E} \varphi(z)\}.$$

Then it is easy to verify that K is nonempty closed convex and bounded. Furthermore, K is invariant under T. In fact, by Lemma 2, $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, and so we have, for each $u \in K$,

$$\varphi(Tu) = \mu_n \|x_n - Tu\|^2 = \mu_n \|Tx_n - Tu\|^2 \leqslant \mu_n \|x_n - u\| = \varphi(u).$$

We also observe that K contains a fixed point v of T. To see this, let $w \in F(T)$ and define

$$K' = \{u \in K\colon\ \|u - w\| = d(w, K)\},$$

where $d(w, K)$ denotes the distance of K from a point w. Then, since E is strictly convex, K' is a singleton. Let $K' = \{v\}$. Then we have

$$\|Tv - w\| = \|Tv - Tw\| \leqslant \|v - w\|,$$

and so $Tv = v$.

On the other hand, since $\{\|x_n - p\|\}$ is nonincreasing for any $p \in F(T)$ by Lemma 3, it converges in real numbers $\mathbb{R}$ and so $\varphi(p)$ is independent of Banach limits. Thus we may assume that v minimizes φ for any Banach limit μ. It follows from Lemma 1 that

$$\mu_n (z, J(x_n - v)) = 0$$

for all $z \in E$ and any Banach limit μ. Thus $\{(z, J(x_n - v))\}$ is almost convergent to 0. In other words, $\{J(x_n - v)\}$ is weakly almost convergent to 0. □

Applying Theorem 1, we obtain the following result.

Theorem 2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and let T and $\{x_n\}$, $\{t_n\}$ and $\{s_n\}$ be as in Theorem 1. If $F(T)$ is nonempty and $J^{-1}\colon\ E^* \to E$ is weakly sequentially continuous at 0, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since

$$\begin{aligned}\|x_{n+1} - x_n\| = t_n \|x_n - Ty_n\| &\leqslant t_n (\|x_n - Tx_n\| + \|Tx_n - Ty_n\|)\\ &\leqslant (1 + s_n) t_n \|x_n - Tx_n\|\end{aligned}$$

for all $n = 1, 2, \ldots$, where $y_n = (1 - s_n)x_n + s_n T x_n$, by Lemma 2, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. By Theorem 1, there exists a point v in $F(T)$ such that $\{J(x_n - v)\}$ is weakly almost convergent to 0. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Thus, since $\{x_n\}$ is bounded and $x_n - x_{n+1} \to 0$, $\{J(x_n - v) - J(x_{n+1} - v)\}$ converges weakly to 0. However, this is a Tauberian condition for almost convergence, so $\{J(x_n - v)\}$ converges weakly to 0. Since J^{-1} is weakly sequentially continuous at 0, $\{x_n\}$ converges weakly to v. □

Remark 1. For a closed bounded convex subset D of E and a nonexpansive self-mapping $T\colon D \to D$, the weak convergence of the sequence $\{x_n\}$ in Theorem 2 has been known in a uniformly convex Banach space with a Fréchet differentiable norm or with a duality mapping that is weakly sequentially continuous at 0 under different choices of t_n and s_n (see [2, 8, 10]).

As a consequence of Theorem 2, we obtain the following result.

Corollary 1. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and let $T\colon E \to E$ be a nonexpansive mapping. Let a sequence $\{x_n\}$ in E be defined by*

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, \quad n = 1, 2, \ldots,$$

where $\{t_n\}$ is a real sequence such that $0 \leqslant t_n \leqslant t < 1$ and $\sum\limits_{n=1}^{\infty} t_n = \infty$. If $F(T)$ is nonempty and $J^{-1}\colon E \to E^*$ is weakly sequentially continuous at 0, then $\{x_n\}$ converges weakly to a fixed point of T.

Corollary 2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $\{x_n\}$, $\{t_n\}$, $\{s_n\}$, J^{-1} be as in Theorem 2. If $F(T)$ is nonempty, then $\{x_n\}$ converges weakly to a fixed point of T.*

Finally, using the property of the Banach limit that $\mu_n(a_n) = \mu_n(a_{n+1})$, we give a dual weak convergence theorem under different restrictions on $\{t_n\}$ and $\{s_n\}$.

Theorem 3. *Let E be a reflexive and strict convex Banach space with a uniformly Gâteaux differentiable norm. Let $T\colon E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let a sequence $\{x_n\}$ in E be defined by*

$$\begin{cases} x_{n+1} = (1 - t_n)x_n + t_n T y_n, \\ y_n = (1 - s_n)x_n + s_n T x_n, \quad n = 1, 2, \ldots, \end{cases}$$

where two real sequences $\{t_n\}$ and $\{s_n\}$ satisfy the following conditions:

(i) $0 \leqslant t_n \leqslant 1$ *and* $\lim_{n\to\infty} t_n = 1$,

(ii) $0 \leqslant s_n \leqslant 1$ *and* $\lim_{n\to\infty} s_n = 0$.

Then there exists a point v in $F(T)$ such that $\{J(x_n - v)\}$ is weakly almost convergent to 0.

P r o o f. As in the proof of Theorem 1, for a $p \in F(T)$ and an initial point x_0 of $\{x_n\}$, we have $\|x_n - y\| \leqslant \|x_0 - y\|$ for all n and hence $\{x_n\}$ is bounded. Also it follows from Lemma 3 that $\{\|x_n - w\|\}$ is decreasing and so $\{\|x_n - w\|\}$ converges in real numbers $\mathbb{R}$ for all $w \in F(T)$.

Now, since

$$\begin{aligned}\|x_{n+1} - Tx_n\| &= \|(1 - t_n)x_n + t_n Ty_n - Tx_n\| \\ &\leqslant (1 - t_n)\|x_n - Tx_n\| + t_n\|y_n - x_n\| \\ &\leqslant [(1 - t_n) + t_n s_n]\|x_n - Tx_n\|\end{aligned}$$

for all $n = 1, 2, \ldots$ by conditions (1) and (ii), we have

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0. \tag{2}$$

As in the proof of Theorem 1, for a Banach limit μ we define

$$\varphi(z) = \mu_n\|x_n - z\|^2$$

for each $z \in E$. Let K be the set of minimizers of φ over E. Then K is invariant under T. In fact, by (2), we have for each $u \in K$,

$$\varphi(Tu) = \mu_n\|x_n - Tu\|^2 = \mu_n\|x_{n+1} - Tu\|^2 = \mu_n\|Tx_n - Tu\|^2 \leqslant \mu_n\|x_n - u\|^2 = \varphi(u).$$

(Here we have used the fact that $\mu_n(a_n) = \mu_n(a_{n+1})$.) So, by the argument used in the proof of Theorem 1, φ attains its minimizer v over E for any μ, which is a fixed point of T. Thus the remainder also follows from that in the proof of Theorem 1. □

Corollary 3. *Let E be a reflexive and strict convex Banach space with a uniformly Gâteaux differentiable norm, and let $T\colon E \to E$ be a nonexpansive mapping with a fixed point. Let $\{x_n\}$ be defined by*

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n = 1, 2, \ldots,$$

where $\{t_n\}$ is a real sequence such that $0 \leqslant t_n \leqslant 1$ and $t_n \to 1$. Then there exists a fixed point v of T such that $\{J(x_n - v)\}$ is weakly almost convergent to 0.

Corollary 4. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, and let $T\colon E \to E$ be a nonexpansive mapping with a fixed point. Let $\{x_n\}$ be defined by*

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, \quad n = 1, 2, \ldots,$$

where $\{t_n\}$ is a real sequence such that $0 \leqslant t_n \leqslant 1$ and $t_n \to 1$. Then there exists a fixed point v of T such that $\{J(x_n - v)\}$ is weakly almost convergent to 0.

Remark 2. Using the fixed point property for nonexpansive self-mappings, Bruck and Reich [1] established Corollary 3 in uniformly smooth Banach spaces.

Remark 3. Theorems 1 and 3 apply to all L^p spaces $(1 < p < \infty)$.

Remark 4. Since the duality mapping J in a Hilbert space is an identity mapping, we obtain the weak almost convergences of the Ishikawa and Mann iteration processes by virtue of our results.

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