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WEAK MULTIPLICATION MODULES

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Abstract. In this paper we characterize weak multiplication modules.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. A proper submodule N of a module M over a ring R is said to be prime (P-prime) if $ra \in N$ for $r \in R$ and $a \in M$ implies that either $a \in N$ or $r \in (N : M) = P$ (see, for example, [4], [6]). The set of all prime submodules in an R-module M is denoted $\operatorname{Spec}_R M$ or $\operatorname{Spec} M$.

Recall that if R is an integral domain with the quotient field K, the rank of an R-module M ($\operatorname{rank} M$ or $\operatorname{rank}_R M$) is defined to be the maximal number of elements of M linearly independent over R. We have $\operatorname{rank} M =$ the dimension of the vector space KM over K, that is $\operatorname{rank} M = \operatorname{rank}_K KM$ ([7]).

An R-module M is called a multiplication module if for every submodule N of M we have $N = IM$, where I is an ideal of R ([3]).

2. Weak multiplication modules

Definition. An R-module M is called a *weak multiplication module* if $\operatorname{Spec} M = \emptyset$ or for every prime submodule N of M we have $N = IM$, where I is an ideal of R.

One can easily show that if M is a weak multiplication module, then $N = (N : M)M$ for every prime submodule N of M ([1]).

As is seen in [1], Q is a weak multiplication Z-module which is not a multiplication module.

If R is a ring (not necessarily an integral domain) and M is an R-module, the subset $T(M)$ of M is defined by

$$T(M) = \{m \in M \mid \exists\, 0 \neq r \in R \text{ such that } rm = 0\}.$$

Obviously, if R is an interal domain, then $T(M)$ is a submodule of M.

It is well known that if R is a ring in which every proper ideal is prime, then R is a field. Compare it with the following result.

Proposition 2.1. *Let R be a ring and $O \neq M$ an R-module, then R is a field if and only if every proper submodule of M is a prime submodule of M and $T(M) \neq M$.*

Proof. $\Rightarrow$ Is obvious.

$\Leftarrow$ Let $a \in M - T(M)$, so $\mathrm{Ann}(a) = O$. In view of the assumption, it is easy to see that every proper submodule of the R-module $M^* = Ra$ is a prime submodule of M^* and $M^* = Ra \cong R$ as R-modules, therefore every proper ideal of R is a prime ideal, hence R is a field. $\square$

Note. The condition $T(M) \neq M$ in the previous result is necessary. For example, let R be a ring which is not a field and let m be a maximal ideal of R, then for the R-module $M = \frac{R}{m}$ every proper submodule is prime, indeed the only proper submodule of M is $\frac{m}{m}$ which is prime as well.

Lemma 2.2. *Let P be a prime ideal of R, let S be a multiplicatively closed set such that $P \cap S = \emptyset$ and let M be an R-module. Then there exists a one-to-one correspondence between the P-prime submodules of M and the $S^{-1}P$-prime submodules of $S^{-1}M$.*

Proof. See [5, Proposition 1]. $\square$

Lemma 2.3. *An R-module M is a weak multiplication module if and only if the R_P-module M_P is a weak multiplication module for every prime (or maximal) ideal P of R.*

Proof. Let M be a weak multiplication R-module and N a prime submodule of M_P where P is a prime ideal of R. According to Lemma 2.2, we know that $N \cap M$ is a prime submodule of M. So $N \cap M = IM$, therefore $N = (N \cap M)_P = I_P M_P$.

Conversely, let N be a prime submodule of M. We show that $\left(\frac{N}{(N:M)M}\right)_P = O$ for every maximal ideal P.

If $(N : M) \subseteq P$, then by Lemma 2.2, N_P is a prime submodule, so $N_P = (N_P : M_P)M_P$, and by Corollary 1 of [5], $(N_P : M_P) = (N : M)_P$. Hence $\left(\frac{N}{(N:M)M}\right)_P = \frac{N_P}{(N:M)_P M_P} = \frac{N_P}{(N_P:M_P)M_P} = O$. If $(N : M) \nsubseteq P$, then clearly $N_P = M_P$ and $(N : M)_P = R_P$, so obviously

$$\left(\frac{N}{(N:M)M}\right)_P = \frac{N_P}{(N:M)_P M_P} = \frac{M_P}{M_P} = O.$$

$\square$

Proposition 2.4. *If M is a weak multiplication module over an integral domain, then*

(i) *If M is a non-zero torsion-free module, then* $\operatorname{rank} M = 1$.

(ii) *If M is a torsion module, then* $\operatorname{rank} M = 0$.

(iii) *M is either torsion or torsion-free.*

P r o o f. (i) First let $O \neq M$ be a vector space which is a weak multiplication module. If $\operatorname{rank} M > 1$, then let $O \neq W \subset M$. According to Proposition 2.1, W is a prime submodule of M, and since M is a weak multiplication module, $W = IM$ where I is an ideal of the field R. So $I = O$ or $I = R$, which is a contradiction. Hence $\operatorname{rank} M \leqslant 1$, and since $0 \neq M$, then $\operatorname{rank} M = 1$.

Now in the general case, if M is a non-zero torsion-free R-module, then $KM \neq O$, where K is the quotient field of R. By Lemma 2.3, KM is a weak multiplication K-module (vector space), and as we have proved above, $\operatorname{rank}_K KM = 1$. Hence $\operatorname{rank} M = \operatorname{rank}_K KM = 1$.

(ii) Suppose that M is a torsion module, then $KM = O$ and therefore $\operatorname{rank} M = \operatorname{rank}_K kM = 0$.

(iii) If $T(M) \neq M$, we show that $T(M) = O$. If $T(M) \neq O$, then $KM \neq 0$ and by Lemma 2.3, KM is a non-zero weak multiplication K-module, so by part (i), $\operatorname{rank}_K KM = 1$, that is $\operatorname{rank} M = \operatorname{rank}_K KM = 1$. It is easy to see that $T(M)$ is a prime submodule of M, so $T(M) = (T(M) : M)T(M)$ and since $T(M) \neq O$, $(T(M) : M) \neq O$. Let $0 \neq r \in (T(M) : M)$. Since $\operatorname{rank} M = 1$, let $\{x\}$ be a linearly independent set in M. Now, $rx \in rM \subseteq T(M)$, so there exists $0 \neq r_1 \in R$ such that $r_1 rx = 0$, and this is a contradiction, because $\{x\}$ is linearly independent. $\square$

Proposition 2.5. *A finitely generated module is a multiplication module if and only if it is locally cyclic.*

P r o o f. See [3, Proposition 5]. $\square$

Theorem 2.6. *Let R be a local ring with a maximal ideal m and let M be a finitely generated R-module. If $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots, \bar{u}_n\}$ is a basis of the vector space $\overline{M} = \frac{M}{mM}$ over the field $\frac{R}{m}$, then $\{u_1, u_2, u_3, \ldots, u_n\}$ is a minimal basis of M.*

Proof. See [7, Theorem 2.3]. □

Theorem 2.7. *Every finitely generated weak multiplication module is a multiplication module.*

Proof. Suppose that M is a finitely generated weak multiplication R-module. We show that M is locally cyclic, and by Proposition 2.5, M is a multiplication module. By localization and Lemma 2.3, we can assume that M is a finitely generated weak multiplication R-module where R is a local ring. Let m be the only maximal ideal of R. Obviously $\frac{M}{mM}$ is a finitely generated weak multiplication $\frac{R}{m}$-module. If $mM = M$, then by Nakayama's Lemma $M = O$, so it is cyclic.

If $mM \neq M$, then $\operatorname{rank}_{R/m} \frac{M}{mM} = 1$, by Proposition 2.4 (i) and by Theorem 2.6, M is a cyclic R-module. □

Theorem 2.8. *If R is a ring , then the following are equivalent.*

(i) $\dim R = 0$.

(ii) *For every weak multiplication R-module M, if $T(M) = 0$, then M is cyclic.*

(iii) *For every weak multiplication R-module M, if $T(M) = 0$, then M is a multiplication module.*

Proof. (i) $\Rightarrow$ (ii). First let R be a field. Let M be a torsion-free weak multiplication R-module. If $M = 0$, then M is cyclic. So let $0 \neq M$. M is a non-zero weak multiplication vector space over the field R. According to Proposition 2.4 (i), we have $\operatorname{rank} M = 1$. That is $M \cong R$, and evidently M is cyclic.

Now we prove the general case. Let $0 \neq M$. It is easy to see that $T(M) = 0$ is a prime submodule of M. Hence $(T(M) : M)$ is a prime ideal of R and since $\dim R = 0$, $\frac{R}{(T(M):M)}$ is a field. Since $T(M) = 0$, one can easily show that $M \cong \frac{M}{0} = \frac{M}{T(M)}$ is a torsion-free weak multiplication $\frac{R}{(T(M):M)}$-module. So M is a torsion-free weak multiplication module over the field $\frac{R}{(T(M):M)}$. And as we have proved above M is a cyclic $\frac{R}{(T(M):M)}$-module and clearly M is a cyclic R-module.

(ii) $\Rightarrow$ (iii). Is obvious.

(iii) $\Rightarrow$ (i). Let P be a prime ideal of R. It is enough to prove that $\frac{R}{P}$ is a field.

If K is the quotient field of the integral domain $\frac{R}{P}$, then by Theorem 1 in [5], $\operatorname{Spec}_{\frac{R}{P}}(K) = \{O\}$. So K is a torsion-free weak multiplication $\frac{R}{P}$-module. Therefore by assumption it is a multiplication module. And since $\frac{R}{P} \leqslant K$, we have $\frac{R}{P} = IK$, where I is a non-zero ideal of $\frac{R}{P}$ and obviously $IK = K$. Hence $\frac{R}{P} = K$, and this completes the proof. □

Corollary 2.9. *If R is an integral domain, then the following are equivalent.*
(i) *R is a field.*
(ii) *Every weak multiplication R-module is cyclic.*
(iii) *Every weak multiplication R-module is a multiplication module.*

Proof. If R is a field, then since every weak multiplication R-module is a vector space, it is a torsion-free weak multiplication R-module, so the proof follows by Theorem 2.8. □

Lemma 2.10. *Let R be a ring and M an R-module whose annihilator is contained in only finitely many maximal ideals $m_1, m_2, \ldots, m_n$ of R. If M_{m_i} is a cyclic R_{m_i}-module for $1, 2, \ldots, n$, then M is a cyclic R-module.*

Proof. See Lemma 3 of [3]. □

In [3, Proposition 8], Barnard proved:

Every finitely generated Artinian multiplication R-module M is cyclic. In this case we know that $\frac{R}{\operatorname{Ann} M}$ is an Artinian ring and obviously M is a multiplication $\frac{R}{\operatorname{Ann} M}$-module. So the following result is a generalization of this result.

Proposition 2.11. *Every weak multiplication module over an Artinian ring is cyclic.*

Proof. Let M' be a weak multiplication module over an Artinian ring R'. We prove that M' is locally cyclic and by Lemma 2.10, M' is cyclic. Let P be a prime ideal. Put $M'_P = M$ and $R'_P = R$. So R is a local Artinian ring and by Lemma 2.3, M is a weak multiplication R-module. Suppose that P is the only prime ideal of R, then $P^n = O$ for some natural number n. If $PM = M$, obviously $O = P^n M = M$, so let $PM \neq M$. $\frac{M}{PM}$ is a weak multiplication $\frac{R}{P}$-module. Therefore, by Proposition 2.4 (i), we have $\operatorname{rank}_{\frac{R}{P}} \frac{M}{PM} = 1$. That means PM is a maximal submodule of M. If $x \in M - PM$, then $PM \subset PM + Rx \subseteq M$, and therefore $PM + Rx = M$. Thus $O = P^n \frac{M}{Rx} = P \frac{M}{Rx} = \frac{M}{Rx}$, so $M = Rx$. □

Proposition 2.12. *If m is a maximal ideal of the ring R which is a minimal prime ideal and $m \neq m^2$, then the following are equivalent.*
(i) *m is a weak multiplication R-module.*
(ii) *There is no ideal between m^2 and m.*
(iii) $\operatorname{Spec}_R m = \{m^2\}$.

Proof. By localization and Lemma 2.3 we can assume that R is a local ring with the only prime ideal m.

(i)$\Rightarrow$(ii). Let m be a weak multiplication R-module. If $m^2 \subseteq I \subset m$ where I is an ideal of R, we show that I is a prime submodule of m. Let $r_1 r_2 \in I$, where $r_1 \in R$ and $r_2 \in m$. Suppose that $r_2 \notin I$, then r_1 is not a unit, hence $r_1 \in m$, hence $r_1 m \subseteq m^2 \subset I$, that is I is a prime submodule of m.

Since m is a weak multiplication module, and I is a prime submodule, then $I = mm_1$ for some ideal m_1 of R. If $m_1 = R$, then $I = mm_1 = m$, which is impossible. So $m_1 \subseteq m$, that is $m^2 \subseteq I = mm_1 \subseteq m^2$, thus there is no ideal between m^2 and m.

(ii)$\Rightarrow$(iii). Suppose that there is no ideal between m^2 and m. If I is a prime submodule of the R-module m, then $(I : m)$ is a prime ideal. Further, since m is the only prime ideal of R, we have $(I : m) = m$. Therefore $m^2 \subseteq I \subset m$, and by assumption $I = m^2$, hence $\operatorname{Spec}_R m = \{m^2\}$.

(iii)$\Rightarrow$(i) Is clear.

The following theorem is a known result, but we will also prove it by the above result. □

Corollary 2.13. *If R is a local Artinian ring and m is a maximal ideal of R, then m is cyclic if and only if* $\operatorname{rank}_{\frac{R}{m}} \frac{m}{m^2} \leqslant 1$.

P r o o f. $\Rightarrow$ Is obvious.

$\Leftarrow$ If $\operatorname{rank}_{\frac{R}{m}} \frac{m}{m^2} = 0$, then $m^2 = m$, and by Nakayama's lemma we have $m = 0$. If $\operatorname{rank}_{\frac{R}{m}} \frac{m}{m^2} = 1$, then there is no ideal between m^2 and m, so by Proposition 2.12, m is a weak multiplication R-module and the proof follows by Proposition 2.11. □

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