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CLOSED SEMISTABLE OPERATORS AND SINGULAR DIFFERENTIAL EQUATIONS

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Abstract. We study a class of closed linear operators on a Banach space whose nonzero spectrum lies in the open left half plane, and for which 0 is at most a simple pole of the operator resolvent. Our spectral theory based methods enable us to give a simple proof of the characterization of C_0-semigroups of bounded linear operators with asynchronous exponential growth, and recover results of Thieme, Webb and van Neerven. The results are applied to the study of the asymptotic behavior of the solutions to a singularly perturbed differential equation in a Banach space.

Keywords: closed linear operator, C_0-semigroup, infinitesimal generator, semistable operator, singular differential equation

MSC 2000: 47A10, 47A60, 34G10

1. Introduction and preliminaries

Basic facts on closed linear operators needed in this paper can be found in the monographs by Kato [7] and Taylor and Lay [15]. By $\mathscr{C}(X)$ we denote the space of all closed linear operators A with domain and range in X; $\mathscr{D}(A)$, $\mathscr{N}(A)$ and $\mathscr{R}(A)$ denote the domain, nullspace and range of A, respectively. By $\mathscr{B}(X)$ we denote the space of all bounded linear operators defined on all of X. An operator $A \in \mathscr{C}(X)$ is *invertible* if there exists $B \in \mathscr{B}(X)$ such that $AB = I$ and $BAx = x$ for all $x \in \mathscr{D}(A)$; $B = A^{-1}$ is the *inverse* of A. If $A \in \mathscr{C}(X)$, then $\varrho(A)$ denotes the resolvent set of A, that is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is invertible. The complement $\sigma(A)$ of $\varrho(A)$ in $\mathbb{C}$ is the *spectrum* of A. The extended spectrum $\sigma_e(A)$ is the subset of the extended complex plane $\mathbb{C} \cup \{\infty\}$ equal to $\sigma(A)$ if $A \in \mathscr{B}(X)$, and to $\sigma(A) \cup \{\infty\}$ otherwise. If $A \in \mathscr{B}(X)$, we write $r(A)$ for the spectral radius of A. For $\lambda \in \varrho(A)$,

$R(\lambda; A)$ denotes the resolvent operator $(\lambda I - A)^{-1} \in \mathscr{B}(X)$ of A. For convenience, a pole of $R(\lambda; A)$ will be referred to as a pole of A.

We want to comment that 'essential spectrum' in this paper means the *Fredholm essential spectrum*, that is the set

$$\sigma^{\text{ess}}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not Fredholm}\}$$

(recall that $B \in \mathscr{C}(X)$ is Fredholm if $\mathscr{N}(A)$ is of finite dimension and $\mathscr{R}(A)$ of finite codimension). Other authors often use the *Browder essential spectrum* $\sigma_B^{\text{ess}}(A)$, which is the set of all complex numbers λ such that $\lambda I - A$ is not Browder (recall that $A \in \mathscr{C}(X)$ is Browder if A is Fredholm, and 0 is at most a pole of finite rank of A). In particular, Clément et al. [4], Martinez and Mazon [10] and Webb [18] all consider the Browder essential spectrum. (Nagel and Poland in [11] introduced—in the context of semigroups—a new kind of essential spectrum, the so-called 'critical spectrum'.) Both the Fredholm and Browder essential spectra are often referred to as 'essential spectra' without further qualification, and this may lead to confusion. We observe that

$$\sigma^{\text{ess}}(A) \subset \sigma_B^{\text{ess}}(A).$$

If $A \in \mathscr{B}(X)$, then $\sigma^{\text{ess}}(A) = \sigma(A + \mathscr{K}(X))$ in the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$; we write $r^{\text{ess}}(A)$ for the spectral radius of $A + \mathscr{K}(X)$ in $\mathscr{B}(X)/\mathscr{K}(X)$.

Let $A \in \mathscr{C}(X)$ with $\sigma(A) \neq \mathbb{C}$. A subset σ of $\sigma_e(A)$ is called a *spectral set* of A if it is both open and closed in the relative topology of $\sigma_e(A)$ as a subset of $\mathbb{C} \cup \{\infty\}$. A singleton $\{\mu\}$ is a spectral set of A if and only if μ is an isolated singularity of the resolvent $R(\lambda; A)$ of A.

If σ is a spectral set of A, then A admits a direct decomposition $A = A_1 \oplus A_2$ relative to a topological direct sum $X = X_1 \oplus X_2$ such that $\sigma(A_1) = \sigma$, $\sigma(A_2) = \sigma(A) \setminus \sigma$. The (bounded) projection P of X with $\mathscr{R}(P) = X_1$ and $\mathscr{N}(P) = X_2$ is the *spectral projection* of A corresponding to σ. If the spectral set σ is bounded, then $\mathscr{R}(P) \subset \mathscr{D}(A^n)$ for all n, and the restriction T_σ of T to $\mathscr{R}(P)$ is continuous [15, Theorem V.9.2].

Our terminology and notation for C_0-semigroups follow essentially Pazy [13] and van Neerven [12]. In particular, if $T(t)$ is a C_0-semigroup with the infinitesimal generator A, we write

$$\begin{aligned}
s(A) &= \sup\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(A)\},\\
s^{\text{ess}}(A) &= \sup\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma^{\text{ess}}(A)\},\\
\omega(A) &= \inf\{\omega \in \mathbb{R}\colon\ \mathrm{e}^{-\omega t}\|T(t)\| \text{ is bounded on } [0,\infty)\},\\
\omega^{\text{ess}}(A) &= \inf\{\omega \in \mathbb{R}\colon\ \mathrm{e}^{-\omega t}\|T(t) + \mathscr{K}(X)\| \text{ is bounded on } [0,\infty)\},
\end{aligned}$$

where $T(t)+\mathscr{K}(X)$ is the C_0-semigroup induced in the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ by $T(t)$. These quantities are the *spectral bound* of A, *essential spectral bound* of A, *growth bound* of $T(t)$ and *essential growth bound* of $T(t)$, respectively. Let us remark that, in addition, Nagel and Poland [11, Definition 2.3] defined the *critical growth bound* associated with their newly defined critical spectrum.

In the following theorem we list for future reference some well known relations between the above quantities important in theory of asymptotic behaviour of semigroups. Equation (1.1) can be found, for instance, in [12, Proposition 1.2.2], (1.2) in [14] or [12, Theorem 3.6.1]. Inequality (1.3) follows from the spectral inclusion

$$\exp(t\sigma^{\text{ess}}(A)) \subset \sigma^{\text{ess}}(T(t)), \quad t \geqslant 0,$$

for the essential spectra.

Theorem 1.1. *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A. Then*

$$e^{\omega(A)t} = r(T(t)) \text{ for all } t \geqslant 0, \tag{1.1}$$

$$\omega(A) = \max\{s(A), \omega^{\text{ess}}(A)\}, \tag{1.2}$$

$$s^{\text{ess}}(A) \leqslant \omega^{\text{ess}}(A). \tag{1.3}$$

In the sequel we need some auxiliary results whose proof we give for the sake of completeness.

Proposition 1.2. *Let $T(t)$ be C_0-semigroup with the infinitesimal generator A. If $B \in \mathscr{B}(X)$ is such that $ABx = BAx$ for all $x \in \mathscr{D}(A)$, then*

$$T(t)B = BT(t) \quad \text{for all } t \geqslant 0.$$

P r o o f. Let $\lambda > s(A)$. Since $(\lambda I - A)Bx = B(\lambda I - A)x$ for all $x \in \mathscr{D}(A)$, then also $(\lambda I - A)^{-1}B = B(\lambda I - A)^{-1}$, which implies that the operator B commutes with $A_\lambda = \lambda R(\lambda; A)$. Consequently, B commutes with $\exp(tA_\lambda)$ for all $t \geqslant 0$, and by the Yosida approximation

$$T(t)Bx = \lim_{\lambda\to\infty} \exp(tA_\lambda)Bx = \lim_{\lambda\to\infty} B\exp(tA_\lambda)x = BT(t)x$$

for all $x \in X$ and all $t \geqslant 0$ by [13, Corollary 1.3.5]. □

Proposition 1.3. *Let $U, V \in \mathscr{C}(X)$ be invertible operators with the same domain $\mathscr{D}$, let $P \in \mathscr{B}(X)$ be a projection such that $UPx = PUx$ and $VPx = PVx$ for all $x \in \mathscr{D}$, and let*

$$Wx = UPx + V(I-P)x, \quad x \in \mathscr{D}.$$

Then W is an invertible operator in $\mathscr{C}(X)$, and

$$W^{-1} = U^{-1}P + V^{-1}(I-P). \tag{1.4}$$

Proof. From the commutativity hypothesis we deduce that $U^{-1}P = PU^{-1}$ and $V^{-1}P = PV^{-1}$. The result follows from the definition of W^{-1} on verifying that

$$W(U^{-1}P + V^{-1}(I-P)) = I \quad \text{and} \quad (U^{-1}P + V^{-1}(I-P))Wx = x \ \text{ for all } x \in \mathscr{D}.$$

□

In the future we will need the following description of isolated spectral points. The theorem generalizes [8, Theorem 1.2] and gives a characterization of the spectral projection which does not require holomorphic calculus.

Theorem 1.4. *Let A be a closed linear operator with domain $\mathscr{D}(A)$. The point 0 is an isolated spectral point of A if and only if there exists a non-zero projection P such that*

(i) $\mathscr{R}(P) \subset \mathscr{D}(A)$,
(ii) $PAx = APx$ *for all* $x \in \mathscr{D}(A)$,
(iii) $\sigma(AP) = \{0\}$,
(iv) $A - P$ *is invertible.*

If (i)–(iv) *hold, P is the spectral projection of A corresponding to* 0.

Proof. Let 0 be an isolated spectral point of A and let P be the spectral projection of A corresponding to 0. By [15, Theorem V.9.2], $X = \mathscr{R}(P) \oplus \mathscr{N}(P)$, $A = A_1 \oplus A_2$ relative to this sum, $\sigma(A_1) = \{0\}$ and $0 \notin \sigma(A_2)$. Hence $AP = A_1 \oplus 0$ and $\sigma(AP) = \{0\}$. Further, $A - P = (A_1 - I) \oplus A_2$ si invertible since both $A_1 - I$ and A_2 are.

Conversely, if (i)–(iv) hold, then for any $\lambda \in \mathbb{C}$ and any $x \in \mathscr{D}(A)$ we have

$$(\lambda I - A)x = (\lambda I - AP)Px + (\lambda I - (A-P))(I-P)x, \tag{1.5}$$

where $\lambda I - AP$ is invertible for all $\lambda \neq 0$. Since the resolvent set of a closed operator is open, there exists $r > 0$ such that $\lambda \in \varrho(A-P)$ for all λ with $|\lambda| < r$. By (1.5) and Proposition 1.3, $\lambda I - A$ is invertible for all λ with $0 < |\lambda| < r$.

If A were invertible, A^{-1} would be bounded, and we would have $r(P) = r(A^{-1}AP) \leqslant r(A^{-1})r(AP) = 0$ (since A^{-1} and AP commute and AP is quasinilpotent). This contradicts $P \neq 0$. Hence 0 is an isolated spectral point of A.

Suppose that P has the properties specified in the theorem. Then $A = A_1 \oplus A_2$ and $P = I \oplus 0$ relative to $X = \mathscr{R}(P) \oplus \mathscr{N}(P)$. As above, $AP = A_1 \oplus 0$, and $\sigma(A_1) = \{0\}$. Further, $A - P = (A_1 - I) \oplus A_2$; since $A - P$ and $A_1 - I$ are invertible, so is A_2, and $\sigma(A_1) \cap \sigma(A_2) = \emptyset$. Therefore P is the spectral projection of A relative to 0. □

2. Stable operators

First we introduce some relevant notation and terminology. By H^- we denote the open left half of the complex plane, that is, the set of all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda < 0$. If $U\colon [0,\infty) \to \mathscr{B}(X)$ and there exist positive constants M, μ such that $\|U(t)\| \leqslant M\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$, we say that $U(t)$ *decays exponentially.* We write

$$T(t) \xrightarrow{s} P \quad (T(t) \xrightarrow{u} P) \quad \text{as } t \to \infty$$

for the convergence in the strong (uniform) operator topology, respectively.

Definition 2.1. An operator $A \in \mathscr{C}(X)$ is called *stable* if $\sigma(A) \subset H^-$.

We start with a C_0-semigroup version of a result which is well known for a uniformly continuous semigroup $S(t)$ with the infinitesimal generator C: $S(t) \xrightarrow{u} 0$ if and only if C is stable. It is known that for C_0-semigroups the spectral inclusion $\sigma(C) \subset H^-$, or even the stronger condition $s(A) < 0$, is no longer sufficient. This is seen from the following example.

Example 2.2 (Pazy [13, Example 4.4.2]**).** Let X be the Banach space of all measurable functions x on $[0,\infty)$ such that $|x|_1 = \int_0^\infty \mathrm{e}^s |x(s)|\,\mathrm{d}s < \infty$ and $x \in L^p(0,\infty)$, $1 < p < \infty$, equipped with the norm $\|x\| = |x|_1 + \|x\|_{L^p}$. The C_0-semigroup $S(t)$ acting on X is defined by $S(t)x(s) = x(t+s)$, $t \geqslant 0$, with the infinitesimal generator C the differential operator $Cx = x'$. It is shown that $s(C) < 0$, however $\|S(t)\| = 1$ for all $t \geqslant 0$, and so $\|S(t)\| \not\to 0$ as $t \to \infty$.

To obtain sufficient conditions for the uniform convergence of a C_0-semigroup $S(t)$ with generator C to 0 we need to augment the spectral inclusion $\sigma(C) \subset H^-$ by some additional conditions.

Theorem 2.3. *Let $S(t)$ be a C_0-semigroup with the infinitesimal generator C. Then the following conditions are equivalent.*

(i) $S(t) \xrightarrow{u} 0$ *as* $t \to \infty$.
(ii) $S(t)$ *decays exponentially.*
(iii) C *is stable and* $S(t) + \mathscr{K}(X) \xrightarrow{u} 0$ *in* $\mathscr{B}(X)/\mathscr{K}(X)$ *as* $t \to \infty$.
(iv) C *is stable and* $S(t) + \mathscr{K}(X)$ *decays exponentially.*
(v) C *is stable and* $S(t) = U(t) + V(t)$, *where* $U(t) \in \mathscr{B}(X)$ *decays exponentially and* $V(t) \in \mathscr{K}(X)$ *for all* $t \geqslant 0$.

Proof. (i)$\Longrightarrow$(ii) Let $\omega = \omega(C)$ be the growth bound of $S(t)$. By (1.1), $\mathrm{e}^{\omega t} = r(S(t)) \leqslant \|S(t)\| \to 0$ as $t \to \infty$; hence $\omega < 0$. Then there is μ satisfying $0 < \mu < -\omega$ for which $\|S(t)\|\mathrm{e}^{\mu t}$ has an upper bound $M > 0$ on $[0, \infty)$. Then (ii) follows.

(ii)$\Longrightarrow$(iii) Since $\omega(C) = \max\{s(C), \omega^{\mathrm{ess}}(C)\}$ by (1.2), (ii) implies $s(C) < 0$ and $\omega^{\mathrm{ess}}(C) < 0$.

(iii)$\Longrightarrow$(iv) is obtained by interpreting the proof of (i)$\Longrightarrow$(ii) for the Calkin algebra.

(iv)$\Longrightarrow$(v) Suppose that $\|S(t) + \mathscr{K}(X)\| N\mathrm{e}^{\nu t}$ for all $t \geqslant 0$ and some $N, \nu > 0$. Let $K > N$. Then for each $t \geqslant 0$ there exists $V(t) \in \mathscr{K}(X)$ such that $\|S(t) - V(t)\| < K\mathrm{e}^{-\nu t}$. The result follows on setting $U(t) = S(t) - V(t)$.

(v)$\Longrightarrow$(i) First we note that (v) implies (iii), and from (iii) it follows that $\omega^{\mathrm{ess}}(C) < 0$ and $s(C) \leqslant 0$. Suppose that $s(C) = 0$. Then there is an injective sequence (λ_n) in $\sigma(C)$ such that $\operatorname{Re}\lambda_n \to 0$. By the spectral inclusion theorem, $\mathrm{e}^{\lambda_n} \in \sigma(T(1))$ for all n. Since $|\mathrm{e}^{\lambda_n}| = \mathrm{e}^{\operatorname{Re}\lambda_n} \to 1$, the unit circle contains an accumulation point of the sequence (e^{λ_n}) contained in $\sigma(T(1))$; this implies [3, Theorem 3.2.11] that $r^{\mathrm{ess}}(T(1)) \geqslant 1$. Consequently, by the formula given in [12, p. 106], $\omega^{\mathrm{ess}}(C) = \log r^{\mathrm{ess}}(T(1)) \geqslant 0$; this contradiction proves that $s(C) < 0$. Then $\omega(C) = \max\{s(C), \omega^{\mathrm{ess}}(C)\} < 0$, and (i) follows. □

Thieme [16] recently derived a different characterization of C_0-semigroups $T(t)$ with the property that $\mathrm{e}^{\omega t}T(t)$ converges in the operator norm as $t \to \infty$ for some $\omega \in \mathbb{R}$. Our results on stable (and semistable) operators enable us to provide an alternative route to these results. First we need a definition.

Following [12, p. 37], we call a C_0-semigroup $S(t)$ with generator C *uniformly continuous at infinity* if

$$\lim_{t\to\infty} \limsup_{s\to 0+} \mathrm{e}^{-\omega(C)t}\|S(t+s) - S(t)\| = 0;$$

this class of C_0-semigroups was introduced by Martinez and Mazon [10] who called them norm-continuous at infinity. Such C_0-semigroups admit spectral mapping theorems for the peripheral spectrum. In particular, we have the following result.

Proposition 2.4 (Martinez and Mazon [10, Corollary 1.4 and Theorem 1.9]). *Let* $S(t)$ *be a* C_0*-semigroup with generator* C *uniformly continuous at infinity. Then*

(i) $s(C) = \omega(C)$;

(ii) *there exists* $\delta > 0$ *such that the set of all* $\lambda \in \sigma(C)$ *with* $\operatorname{Re}\lambda \geqslant s(C) - \delta$ *is bounded.*

We now derive our second characterization of $S(t) \xrightarrow{u} 0$.

Theorem 2.5. *Let* $S(t)$ *be a* C_0*-semigroup with generator* C. *Then the following conditions are equivalent.*

(i) $S(t) \xrightarrow{u} 0$ *as* $t \to \infty$.

(ii) C *is stable and* $S(t) = U(t) + V(t)$, *where* $U(t), V(t) \in \mathscr{B}(X)$, $\mathrm{e}^{-\omega(C)t}\|U(t)\| \to 0$ *as* $t \to \infty$ *and* $V(t)$ *is right continuous in the operator norm for each* $t \geqslant 0$.

(iii) C *is stable and* $S(t)$ *is uniformly continuous at infinity.*

P r o o f. (i)$\Longrightarrow$(ii) Follows from Theorem 2.3 when we set $U(t) = S(t)$ and $V(t) = 0$.

(ii)$\Longrightarrow$(iii) We write $\omega = \omega(C)$. Let $\varepsilon > 0$. Then there exists t_0 such that $\mathrm{e}^{-\omega t}\|U(t)\| \leqslant \varepsilon$ for all $t > t_0$. For any $t > t_0$ and any $s > 0$,

$$\mathrm{e}^{-\omega t}\|U(t+s) - U(t)\| \leqslant \mathrm{e}^{\omega s}\mathrm{e}^{-\omega(t+s)}\|U(t+s)\| + \mathrm{e}^{-\omega t}\|U(t)\| \leqslant (\mathrm{e}^{\omega s} + 1)\varepsilon,$$

and

$$\mathrm{e}^{-\omega t}\|S(t+s) - S(t)\| \leqslant (\mathrm{e}^{\omega s} + 1)\varepsilon + \mathrm{e}^{-\omega t}\|V(t+s) - V(t)\|.$$

Hence $\lim\limits_{t\to\infty} \limsup\limits_{s\to 0+} \mathrm{e}^{-\omega t}\|S(t+s) - S(t)\| \leqslant 2\varepsilon$ for any $\varepsilon > 0$, and $S(t)$ is uniformly continuous at infinity.

(iii)$\Longrightarrow$(i) From $\sigma(C) \subset H^-$ it follows that $s(C) \leqslant 0$. By Proposition 2.4 (ii) there exists $\delta > 0$ such that the set $\Sigma = \{\lambda \in \sigma(C)\colon\ \operatorname{Re}\lambda \geqslant s(C) - \delta\}$ is compact. Hence $s(C) = \max\{\operatorname{Re}\lambda\colon\ \lambda \in \Sigma\} < 0$. By Proposition 2.4 (i), $\omega(C) = s(C) < 0$. □

The equivalence (i) $\Longleftrightarrow$ (ii) of the foregoing theorem is a special case of a result due to Thieme (see Theorem 4.1). Thieme calls a C_0-semigroup $T(t)$ *essentially norm-continuous* if it can be expressed as the sum of operator families $U(t)$ and $V(t)$ with the properties specified in condition (ii) of the preceding theorem.

Further properties of C_0-semigroups norm-continuous at infinity and essentially norm-continuous have been recently investigated by Blake [1].

It will be shown that properties of semistable operators are inextricably linked to the asymptotic convergence of C_0-semigroups in the operator norm.

Definition 3.1. An operator $A \in \mathscr{C}(X)$ is called *semistable* if $\sigma(A) \subset H^- \cup \{0\}$ and 0 is at most a simple pole of A.

It is well-known (see, for instance, [8]) that if A is bounded, then it is semistable if and only if the limit $\lim\limits_{t\to\infty} \exp(tA) = P$ exists in $\mathscr{B}(X)$.

The next lemma and the theorem that follows are the main tools for reducing the general problem of the convergence $T(t) \to P$ as $t \to \infty$ to the case $P = 0$.

Lemma 3.2. *Suppose $A \in \mathscr{C}(X)$ and $P \in \mathscr{B}(X)$ is a projection such that $APx = PAx = 0$ for all $x \in \mathscr{D}(A)$. Then*

$$\sigma(A) \subset \sigma(A-P) \cup \{0\} \quad \text{and} \quad \sigma(A-P) \subset \sigma(A) \cup \{-1\}. \tag{3.1}$$

Proof. Write $Cx = Ax - Px$ for all $x \in \mathscr{D}(A)$. Then

$$(\lambda I - A)x = (\lambda I - C)(I-P)x + (\lambda I - AP)Px, \tag{3.2}$$

$$(\lambda I - C)x = (\lambda I - A)(I-P)x + ((\lambda+1)I - AP))Px. \tag{3.3}$$

We apply Proposition 1.3 to these two equations: If $\lambda \in \varrho(C)$ and $\lambda \neq 0$, then $\lambda \in \varrho(A)$ by (3.2); hence $\sigma(A) \subset \sigma(C) \cup \{0\}$. Similarly, from (3.3) we deduce $\sigma(C) \subset \sigma(A) \cup \{-1\}$. (If there is $x_0 \in \mathscr{D}(A)$ such that $Px_0 \neq 0$, then $\sigma(C) \cup \{0\} = \sigma(A) \cup \{-1\}$.) □

We obtain a generalization of a result known in the case of bounded linear operators or elements of Banach algebras [8, Example 2.3]:

Theorem 3.3. *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A. Then $T(t)$ converges in the strong (uniform) operator topology as $t \to \infty$ if and only if there exists $P \in \mathscr{B}(X)$ such that*

(i) *$P^2 = P$,*
(ii) *$AP = 0$ and $PAx = 0$ for all $x \in \mathscr{D}(A)$,*
(iii) *the C_0-semigroup $S(t)$ generated by $A - P$ satisfies $S(t) \xrightarrow{s} 0$ ($S(t) \xrightarrow{u} 0$) as $t \to \infty$.*

If (i)–(iii) *hold with the uniform convergence in* (iii), *then*

$$T(t)P = P = PT(t) \quad \text{for all } t \geqslant 0, \tag{3.4}$$

$$T(t) = S(t) + (1 - \mathrm{e}^{-t})P \quad \text{for all } t \geqslant 0, \tag{3.5}$$

$$A \text{ is semistable and } A - P \text{ is stable.} \tag{3.6}$$

P r o o f. (a) Suppose that there exists $P \in \mathscr{B}(X)$ with the specified properties. By Proposition 1.2, $T(t)$ and $\exp(-tP)$ commute. Then

$$S(t) = T(t)\exp(-tP) \tag{3.7}$$

is the C_0-semigroup with the infinitesimal generator $C = A - P$; the semigroup status is verified directly, and the infinitesimal generator is calculated from

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 S(t)x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 T(t)\exp(0P)x + T(0)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-tP)x = Ax - Px$$

for all $x \in \mathscr{D}(A)$.

Since $AR(\lambda; A)P = R(\lambda; A)AP = 0$, we have $(AR(\lambda; A))^n P = 0$ for all $n \geqslant 1$. Applying the Yosida approximation, we get

$$T(t)P = \lim_{\lambda\to\infty} \exp(t\lambda AR(\lambda; A))P = \lim_{\lambda\to\infty}\left(P + \sum_{n=1}^{\infty}\frac{(t\lambda)^n}{n!}(AR(\lambda; A))^n P\right) = P.$$

This proves (3.4).

Using the series expansion for the exponential we get $\exp(-tP) = I + (\mathrm{e}^{-t} - 1)P$; hence

$$S(t) = T(t)\exp(-tP) = T(t)(I + (\mathrm{e}^{-t} - 1)P) = T(t) + (\mathrm{e}^{-t} - 1)P, \quad t \geqslant 0,$$

and (3.5) holds. If $S(t) \xrightarrow{s} 0$ ($S(t) \xrightarrow{u} 0$), then $T(t) \xrightarrow{s} P$ ($T(t) \xrightarrow{u} P$).

(b) Conversely, suppose that $T(t) \xrightarrow{s} P$ ($T(t) \xrightarrow{u} P$) as $t \to \infty$. Then (3.4) is true, and (i) follows by differentiation. The semigroup generated by $A - P$ is given by (3.7), and (3.5) holds. Hence (ii) is true.

Suppose now that (i)–(iii) hold with $S(t) \xrightarrow{u} 0$. Then $\sigma(A - P) \subset H^-$ by Theorem 2.3. From Theorem 3.3 we see that A and P satisfy the condition of Theorem 1.4, and so 0 is an isolated spectral point of A with the spectral projection P; since $AP = 0$, 0 is a simple pole of A. By Lemma 3.2, $\sigma(A) \subset \sigma(A - P) \cup \{0\} \subset H^- \cup \{0\}$. □

When we combine Theorems 2.3 and 3.3, we obtain the following main result on asymptotic convergence of C_0-semigroups.

Theorem 3.4. *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A. Then the following conditions are equivalent.*

(i) *$T(t) \xrightarrow{u} P$ as $t \to \infty$.*

(ii) *$T(t) - P$ decays exponentially.*

(iii) *A is semistable with the spectral projection P, and $T(t) - P + \mathscr{K}(X) \xrightarrow{u} 0$ in $\mathscr{B}(X)/\mathscr{K}(X)$.*

(iv) *A is semistable with the spectral projection P, and $T(t) - P + \mathscr{K}(X)$ decays exponentially.*

(v) *A is semistable with the spectral projection P, and $T(t) - P = U(t) + V(t)$, where $U(t) \in \mathscr{B}(X)$ decays exponentially and $V(t) \in \mathscr{K}(X)$ for all $t \geqslant 0$.*

(vi) *There is a projection $P \in \mathscr{B}(X)$ such that $AP = 0$, $PAx = 0$ for all $x \in \mathscr{D}(A)$, $A - P$ is stable, and $T(t) - P = U(t) + V(t)$, where $U(t) \in \mathscr{B}(X)$ decays exponentially and $V(t) \in \mathscr{K}(X)$ for all $t \geqslant 0$.*

Proof. In this proof $S(t)$ is the C_0-semigroup generated by $A - P$.

(i)$\Longrightarrow$(ii) According to Theorems 3.3 and 2.3, $S(t)$ decays exponentially. Then $T(t) - P = S(t) - \mathrm{e}^{-t}P$ also decays exponentially.

(ii)$\Longrightarrow$(iii) A is semistable with the spectral projection P by Theorem 3.3. From $\|T(t) - P + \mathscr{K}(X)\| \leqslant \|T(t) - P\|$ it follows that $T(t) - P + \mathscr{K}(X) \xrightarrow{u} 0$ as $t \to \infty$.

(iii)$\Longrightarrow$(iv) By Lemma 3.2, $A - P$ is stable. By Theorem 3.3,

$$S(t) + \mathscr{K}(X) = (T(t) - P + \mathscr{K}(X)) + (\mathrm{e}^{-t}P + \mathscr{K}(X)) \xrightarrow{u} 0.$$

Then $S(t) + \mathscr{K}(X)$ decays exponentially by Theorem 2.3, and so does $T(t) - P + \mathscr{K}(X) = S(t) - \mathrm{e}^{-t}P + \mathscr{K}(X)$.

(iv)$\Longrightarrow$(v) By assumption, $\|T(t) - P + \mathscr{K}(X)\| \leqslant M\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$ and some $M, \mu > 0$. If $N > M$, then for each $t \geqslant 0$ there exists $V(t) \in \mathscr{K}(X)$ such that $\|T(t) - P - V(t)\| \leqslant N\mathrm{e}^{-\mu t}$. We set $U(t) = T(t) - P - V(t)$.

(v)$\Longrightarrow$(i) By Theorem 3.3,

$$S(t) = T(t) - P + \mathrm{e}^{-t}P = (U(t) + \mathrm{e}^{-t}P) + V(t),$$

and the result follows from Theorem 2.3.

(v) and (vi) are equivalent in view of Theorem 3.3. □

4. Asynchronous exponential growth

We say that a C_0-semigroup $T(t)$ has a *uniform asynchronous (or balanced) exponential growth* if there exists $\omega \in \mathbb{C}$ such that

$$\mathrm{e}^{\omega t}T(t) \xrightarrow{u} P \quad \text{as } t \to \infty.$$

This concept is due to Webb [18]. A recent paper [16] by Thieme gives a survey of the asynchronous exponential growth as well as new necessary and sufficient conditions for its validity.

The results of the previous section, in particular Theorem 3.4, can be interpreted as results on the asynchronous exponential growth in view of the following transformation: If $T(t)$ is a C_0-semigroup with generator A, then $U(t) = \mathrm{e}^{-\omega t}T(t)$ is a C_0-semigroup with generator $A-\omega I$. We can check that $\sigma(A-\omega I) = \sigma(A)-\omega$, and

$$\tau(A-\omega I) = \tau(A) - \operatorname{Re}\omega,$$

where $\tau \in \{\omega, \omega^{\mathrm{ess}}, s, s^{\mathrm{ess}}\}$. Hence it is enough to derive results on the asynchronous exponential growth assuming $\omega = 0$.

Using the reduction just described, we can deduce the following result from our Theorem 2.5. The equivalence of (i) and (ii) is [16, Theorem 2.7].

Theorem 4.1. *Let $T(t)$ be a C_0-semigroup with generator A, and let $\omega \in \mathbb{C}$. Then the following are equivalent:*

(i) $\mathrm{e}^{-\omega t}T(t) \xrightarrow{u} P$ *as* $t \to \infty$.

(ii) $A-\omega I$ *is semistable and* $T(t) = U(t)+V(t)$, *where* $U(t), V(t) \in \mathscr{B}(X)$ *are such that* $\lim\limits_{t\to\infty} \mathrm{e}^{-(\omega(A)+\operatorname{Re}\omega)t}\|U(t)\| = 0$ *and* $V(t)$ *is right continuous in the operator norm for each* $t \geqslant 0$.

(iii) $A-\omega I$ *is semistable and* $\mathrm{e}^{-\omega t}T(t)$ *is uniformly continuous at infinity.*

In the case that the spectral projection P is of finite rank, we get the following specialization of our main theorem. We may recall that a C_0-semigroup $T(t)$ with generator A is *essentially compact* if $\omega^{\mathrm{ess}}(A) < \omega(A)$.

Theorem 4.2. *If $T(t)$ is a C_0-semigroup with the infinitesimal generator A, then the following statements are equivalent.*

(i) *There exists a nonzero finite rank operator P such that* $T(t) \xrightarrow{u} P$.

(ii) *A is semistable with $0 \in \sigma(A)$ and $T(t)$ is essentially compact.*

(iii) *A is semistable with $0 \in \sigma(A)$ and $T(t) = U(t) + V(t)$, where $U(t) \in \mathscr{B}(X)$ decays exponentially and $V(t) \in \mathscr{K}(X)$ for all $t \geqslant 0$.*

P r o o f. (i)$\Longrightarrow$(ii) From Theorem 3.4 we obtain the semistability of A and the existence of $N, \nu > 0$ such that $\|T(t) - P + \mathscr{K}(X)\| \leqslant N\mathrm{e}^{-\nu t}$ for all $t \geqslant 0$. Since P has finite rank, $\|T(t)+\mathscr{K}(X)\| \leqslant N\mathrm{e}^{-\nu t}$ for all $t \geqslant 0$, which shows that $\omega^{\mathrm{ess}}(A) < 0$. If $\omega(A)$ were equal to $\omega^{\mathrm{ess}}(A)$, we would have $s(A) \leqslant \omega(A) = \omega^{\mathrm{ess}}(A) < 0$. This contradicts $0 \in \sigma(A)$ $(P \neq 0)$.

(ii)$\Longrightarrow$(iii) According to (1.2), $\omega(A) = \max\{s(A), \omega^{\mathrm{ess}}(A)\}$; $\omega^{\mathrm{ess}}(A) < \omega(A)$ implies $\omega(A) = s(A) \leqslant 0$ and $\omega^{\mathrm{ess}}(A) < 0$. By (1.3), $s^{\mathrm{ess}}(A) \leqslant \omega^{\mathrm{ess}}(A) < 0$, which means that 0 is an isolated spectral point of A contained in $\sigma(A)\backslash\sigma^{\mathrm{ess}}(A)$. Hence A is Fredholm, and the spectral projection P of A at 0 is of finite rank. From $\omega^{\mathrm{ess}}(A) < 0$

it follows that there are $N, \nu > 0$ such that $\|T(t) + \mathscr{K}(X)\| \leqslant N\mathrm{e}^{-\nu t}$ for all $t \geqslant 0$. Since P is of finite rank,

$$\|T(t) - P + \mathscr{K}(X)\| = \|T(t) + \mathscr{K}(X)\| \leqslant N\mathrm{e}^{-\nu t}$$

for all $t \geqslant 0$. For each $t \geqslant 0$ there exists $W(t) \in \mathscr{K}(X)$ such that $\|T(t) - P - W(t)\| \leqslant 2N\mathrm{e}^{-\nu t}$. The result follows on setting $V(t) = P + W(t)$ and $U(t) = T(t) - V(t)$.

(iii)$\Longrightarrow$(i) Let P be the spectral projection of A, and $S(t)$ the C_0-semigroup generated by $A - P$. Then

$$T(t) - P = T(t)(I - P) = U(t)(I - P) + V(t)(I - P) = \widetilde{U}(t) + \widetilde{V}(t)$$

where $\widetilde{U}(t)$ decays exponentially and $\widetilde{V}(t) \in \mathscr{K}(X)$ for all $t \geqslant 0$. Then $T(t) \xrightarrow{u} P$ by Theorem 3.4. The projection P is of finite rank since $P = T(t)P = U(t)P + V(t)P$ and $\operatorname{dist}(P, \mathscr{K}(X)) \leqslant \|P - V(t)P\| = \|U(t)P\| \to 0$. □

From the preceding theorem we can recover Webb's result [18, Proposition 2.3] on the asynchronous exponential growth which has significant applications to population dynamics. First we recall [4] that

$$\omega_B^{\mathrm{ess}}(A) = \lim_{t\to\infty} \frac{\log \alpha[T(t)]}{t},$$

where $\alpha(L)$ is the measure of noncompactness of $L \in \mathscr{B}(X)$.

Theorem 4.3 (Webb [18, Proposition 2.3]). *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A, and let $\omega \in \mathbb{C}$. Then the following statements are equivalent.*

(i) *There exists a nonzero finite rank operator P such that $\mathrm{e}^{-\omega t}T(t) \xrightarrow{u} P$.*

(ii) *$A - \omega I$ is semistable with $\omega \in \sigma(A)$ and $\omega_B^{\mathrm{ess}}(A) < \omega(A)$.*

Proof. It is enough to prove the theorem with $\omega = 0$.

(i)$\Longrightarrow$(ii) By Theorem 3.4, A is semistable and there are constants $M \geqslant 1$, $\mu > 0$ such that $\|T(t) - P\| \leqslant M\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$. Note that $T(t)P = P$ is compact. By [17, Proposition 4.9], $\alpha[T(t)] \leqslant \alpha[T(t)P] + \alpha[T(t)(I - P)] \leqslant M\mathrm{e}^{-\mu t}$ for all sufficiently large t. Hence $\log \alpha[T(t)]/t \leqslant (\log M/t) - \mu$, which proves $\omega_B^{\mathrm{ess}}(A) < 0 = s(A)$. Therefore $\omega_B^{\mathrm{ess}}(A) < \omega(A)$ by (1.2).

(ii)$\Longrightarrow$(i) From $\omega_B^{\mathrm{ess}}(A) < \omega(A)$ we obtain that $\omega^{\mathrm{ess}}(A) < \omega(A)$. The result then follows from Theorem 4.2. □

Theorem 4.2 has another useful corollary in the case that the spectral projection is of rank one. This result is essentially a theorem due to van Neerven [12, Theorem 3.6.2].

Theorem 4.4. *If $T(t)$ is a C_0-semigroup with the infinitesimal generator A, then the following statements are equivalent.*

(i) *There is a rank one operator P such that $T(t) \xrightarrow{u} P$.*

(ii) *$T(t)$ is essentially compact, $\sigma(A) \subset H^- \cup \{0\}$, and 0 is an isolated spectral point of A with the spectral projection of rank one.*

Proof. (i)$\Longrightarrow$(ii) As in the proof of Theorem 4.2.

(ii)$\Longrightarrow$(i) If the spectral projection P is of finite rank, then AP is nilpotent by Theorem 1.4. If P is of rank one, it can be easily verified that $AP = 0$, which means that 0 is a simple pole of A. The result follows in view of Theorem 4.2. □

5. Singularly perturbed differential equations

In this section we are concerned with the singularly perturbed differential problem

$$\varepsilon \frac{\mathrm{d}x_\varepsilon(t)}{\mathrm{d}t} = (A + \varepsilon B)x_\varepsilon(t), \quad \varepsilon \in (0, \varepsilon_0), \qquad x_\varepsilon(0) = x, \tag{5.1}$$

where $x_\varepsilon \colon [0, \infty) \to X$, and where x is an element of X; the operator A is an infinitesimal generator of a C_0-semigroup $T(t)$, and B is a bounded linear operator on X. We investigate the convergence of the solutions $x_\varepsilon(t)$ of (5.1) as $\varepsilon \to 0+$. This problem was studied, for instance, by Campbell [2] in the case that A, B are matrices, and by the present authors [9] in the case that A, B are bounded linear operators. A related problem, under more elaborate conditions, is treated by Greiner, Heesterbeek and Metz [5] in the setting of C_0-semigroups.

Theorem 5.1. *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A such that $T(t) \xrightarrow{u} P$ as $t \to \infty$, and let $B \in \mathscr{B}(X)$. Then the solutions $x_\varepsilon(t)$ to (5.1) converge to the limit $x_0(t)$ as $\varepsilon \to 0+$ uniformly on compact subsets of $(0, +\infty)$, where*

$$x_0(t) = \exp(tPB)Px \tag{5.2}$$

is the solution to the reduced problem

$$\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = PBx_0(t), \quad x_0(0) = Px. \tag{5.3}$$

P r o o f. First we rewrite the differential equation (5.1) as

$$\frac{\mathrm{d}x_\varepsilon(t)}{\mathrm{d}t} = (\varepsilon^{-1}A + B)x_\varepsilon(t), \quad x_\varepsilon(0) = x. \tag{5.4}$$

Let $T_\varepsilon(t)$ and $S_\varepsilon(t)$ be the C_0-semigroups generated by A/ε and $A/\varepsilon + B$, respectively. The solution to (5.4) is given by $x_\varepsilon(t) = S_\varepsilon(t)x$. The perturbation formula (2.1) in [13, p. 77] gives

$$S_\varepsilon(t)x = T_\varepsilon(t)x + \int_0^t T_\varepsilon(t-s)BS_\varepsilon(s)x\,\mathrm{d}s.$$

Observing that

$$\int_0^t PB\exp(sPB)Px\,\mathrm{d}s = \exp(tPB)Px - Px,$$

we can write

$$\begin{aligned} S_\varepsilon(t)x - \exp(tPB)Px &= (T_\varepsilon(t)x - Px) + \int_0^t T_\varepsilon(t-s)BS_\varepsilon(s)x\,\mathrm{d}s \\ &\quad - \int_0^t PB\exp(sPB)Px\,\mathrm{d}s \\ &= (T_\varepsilon(t)x - Px) + \int_0^t (T_\varepsilon(t-s) - P)BS_\varepsilon(s)x\,\mathrm{d}s \\ &\quad + \int_0^t PB(S_\varepsilon(s) - \exp(sPB))Px\,\mathrm{d}s. \end{aligned} \tag{5.5}$$

By hypothesis on $T(t)$, there is a constant $M > 0$ such that $\|T(t)\| \leqslant M$ for all $t \geqslant 0$; then also $\|T_\varepsilon(t)\| \leqslant M$ for all $\varepsilon > 0$ and all $t \geqslant 0$. By [13, Corollary 1.3],

$$\|S_\varepsilon(t) - T_\varepsilon(t)\| \leqslant M(\mathrm{e}^{M\|B\|t} - 1), \quad t \geqslant 0,$$

and

$$\|S_\varepsilon(s)\| \leqslant \|S_\varepsilon(s) - T_\varepsilon(s)\| + \|T_\varepsilon(s)\| \leqslant M(\mathrm{e}^{M\|B\|s} - 1) + M = M\mathrm{e}^{M\|B\|s}.$$

From (5.5) we obtain the following estimate:

$$\begin{aligned} \|S_\varepsilon(t)x - \exp(tPB)Px\| &\leqslant \|T_\varepsilon(t) - P\|\,\|x\| \\ &\quad + \int_0^t \|T_\varepsilon(t-s) - P\|\,\|B\|\mathrm{e}^{M\|B\|s}\|x\|\,\mathrm{d}s \\ &\quad + \|PB\| \int_0^t \|S_\varepsilon(s)x - \exp(sPB)Px\|\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

Observe that $T_\varepsilon(t) = T(t/\varepsilon)$. Since $\|T(t) - P\| \leqslant N\mathrm{e}^{-\mu t}$ for some $N, \mu > 0$ and all $t \geqslant 0$, the sum of the first two terms on the right in (5.6) has an upper bound

$$h_\varepsilon(t) = c_1\mathrm{e}^{-\mu t/\varepsilon} + c_2 \int_0^t \mathrm{e}^{-\mu(t-s)/\varepsilon}\mathrm{e}^{\nu s}\,\mathrm{d}s = c_1\mathrm{e}^{-\mu t/\varepsilon} + \frac{c_2\varepsilon}{\mu + \nu\varepsilon}(\mathrm{e}^{\nu t} - \mathrm{e}^{-\mu t/\varepsilon})$$

with c_1, c_2, ν positive constants. From (5.6) we obtain

$$\varphi_\varepsilon(t) \leqslant h_\varepsilon(t) + c\int_0^t \varphi_\varepsilon(s)\,\mathrm{d}s, \tag{5.7}$$

where $\varphi_\varepsilon(t) = \|S_\varepsilon(t)x - \exp(tPB)Px\|$ and c is a positive constant, which can be chosen to satisfy $c > \nu$. Applying the Gronwall lemma [6], after a calculation we obtain

$$\begin{aligned}\varphi_\varepsilon(t) &\leqslant h_\varepsilon(t) + c\mathrm{e}^{ct}\int_0^t h_\varepsilon(s)\mathrm{e}^{-cs}\,\mathrm{d}s\\ &\leqslant c_1\mathrm{e}^{-\mu t/\varepsilon} + \frac{\varepsilon c_2\mathrm{e}^{\nu t}}{\mu + \nu\varepsilon}\\ &\quad + \frac{\varepsilon c\mathrm{e}^{ct}}{(c-\nu)(\mu+\nu\varepsilon)(\mu + c\varepsilon)}\\ &\quad \times \big((c_1 c + c_2)(\mu + \nu\varepsilon) + (c_1\nu(\mu+\nu\varepsilon) + c_2 c\varepsilon)\mathrm{e}^{-(c+\mu/\varepsilon)t}\big)\end{aligned}$$

for all $t > 0$ and all $\varepsilon > 0$. Since the last expression converges to 0 as $\varepsilon \to 0+$ uniformly for $t \in [t_1, t_2]$, where $0 < t_1 < t_2$, the same is true for $\varphi_\varepsilon(t)$. □

This theorem generalizes [9, Theorem 2.2] proved by the present authors for bounded linear operators on X, and can be used to derive results on age-structured population models similar to those of Greiner, Heesterbeek and Metz [5]. In the theorem, the condition $T(t) \xrightarrow{u} P$ can be replaced by any of the equivalent conditions of Theorem 3.4. In particular, we have the following result.

Theorem 5.2. *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A, and let $P \in \mathscr{B}(X)$ be a nonzero projection such that*
(i) *$AP = 0$ and $PAx = 0$ for all $x \in \mathscr{D}(A)$,*
(ii) *$A - P$ is stable,*
(iii) *$T(t) = P + U(t) + V(t)$, where $U(t) \in \mathscr{B}(X)$ decays exponentially and $V(t)$ is compact for all $t \geqslant 0$.*
Then, for any $B \in \mathscr{B}(X)$, the solutions $x_\varepsilon(t)$ of (5.1) converge as $\varepsilon \to 0+$ to $x_0(t)$ uniformly on compact subsets of $(0, +\infty)$, where $x_0(t)$ is given by (5.2).

In view of Theorem 4.1, condition (iii) of the preceding theorem can be replaced by (iii)′ $T(t) = U(t) + V(t)$, where $U(t), V(t) \in \mathscr{B}(X)$ are such that $e^{\omega(A)t}\|U(t)\| \to 0$ as $t \to \infty$ and $V(t)$ is right continuous for each $t \geqslant 0$.

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