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THE HENSTOCK-KURZWEIL-PETTIS INTEGRALS AND EXISTENCE THEOREMS FOR THE CAUCHY PROBLEM

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Abstract. In this paper we prove an existence theorem for the Cauchy problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \in I_\alpha = [0, \alpha]$$

using the Henstock-Kurzweil-Pettis integral and its properties. The requirements on the function f are not too restrictive: scalar measurability and weak sequential continuity with respect to the second variable. Moreover, we suppose that the function f satisfies some conditions expressed in terms of measures of weak noncompactness.

Keywords: pseudo-solution, Pettis integral, Henstock-Kurzweil integral, Cauchy problem

MSC 2000: 34G20, 28B05

1. Introduction

The Henstock-Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals [16], [17], [24], [27]. A particular feature of this integral is that integrals of highly oscillating function such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0, 1]$ and $F(0) = 0$ can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957–1958 and has since proved useful in the study of ordinary differential equations [1], [8], [9], [13]. In the paper [6], Cao defined the Henstock integral in Banach space, which is a generalization of the Bochner integral.

The Pettis integral is also a generalization of the Bochner integral [14], [29]. This notion is strictly relative to weak topologies in Banach spaces.

We generalize both concepts of integrals introducing the Henstock-Kurzweil-Pettis integral.

Let E be a Banach space and let E^* be a dual space. Moreover, let $(C(I_\alpha, E), \omega)$ denote the space of all continuous functions from I_α to E endowed with the topology

$\sigma(C(I_\alpha, E),\ C(I_\alpha, E)^*)$. Recall that a function $f\colon\ I_\alpha \longrightarrow E$ is said to be weakly continuous if it is continuous from I_α to E endowed with its weak topology.

In this paper we will deal with the Cauchy problem:

$$\tag{1} \begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \qquad t \in [0, \alpha] = I_\alpha,$$

where f is a Henstock-Kurzweil-Pettis integrable function. In fact, our existence theorem is based on an idea of Kurzweil from [24].

We will consider the problem:

$$\tag{2} x(t) = x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s, \quad t \in I_\alpha,$$

where the integral is taken in the sense of Henstock-Kurzweil-Pettis.

In the sequel we will denote by $\int f(t)\,\mathrm{d}t$ the Henstock-Kurzweil-Pettis integral. When it is necessary to distinguish between different classes of integrals we will use the prefixes (L) for Lebesgue integrals, (HK) for Henstock-Kurzweil integrals and (P) for Pettis integrals.

In this paper we use the measure of weak noncompactness developed by De Blasi [5].

Let A be a bounded nonempty subset of E.

The *measure of weak noncompactness* $\mu(A)$ is defined by

$$\mu(A) = \inf\{t > 0\colon \text{ there exists } C \in \mathscr{K}^w \text{ such that } A \subset C + tB_0\},$$

where $\mathscr{K}^w$ is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

For the properties of measure of weak noncompactness $\mu(A)$ see [5].

We can construct many other measures of weak noncompactness with suitable sets of properties, by using the scheme from [4] or [10]. The following lemma is important in our proof:

Lemma 1 ([26]). *Let $H \subset C(I_\alpha, E)$ be a family of strongly equicontinuous functions. Then $\mu(H(I_\alpha)) = \sup_{t \in I_\alpha} \mu(H(t))$ and the function $t \mapsto \mu(H(t))$ is continuous.*

Fix $x^* \in E^*$ and consider the problem

$$\tag{1'} (x^*x)'(t) = x^*f(t, x(t)), \quad x(0) = x_0, \quad t \in I_\alpha.$$

Let us introduce the following definition:

Definition 1 ([29]). Let $F\colon [a,b] \longrightarrow E$ and let $A \subset [a,b]$. The function $f\colon A \longrightarrow E$ is a *pseudoderivative* of F on A if for each x^* in E^* the real-valued function x^*F is differentiable almost everywhere on A and $(x^*F)' = x^*f$ almost everywhere on A.

Regarding the above definition it is clear that the left-hand side of $(1')$ can be rewritten in the form $x^*(x'(t))$ where x' denotes the pseudoderivative.

Definition 2 ([7]). A family $\mathscr{F}$ of functions F is said to be *uniformly absolutely continuous* in the restricted sense on X or, in short, uniformly $AC_*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every F in $\mathscr{F}$ and for every finite or infinite sequence of non-overlapping intervals $\{[a_i,b_i]\}$ with $a_i, b_i \in X$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F,[a_i,b_i]) < \varepsilon$, where ω denotes the oscillation of F over $[a_i,b_i]$ (i.e. $\omega(F,[a_i,b_i]) = \sup\{|F(r)-F(s)|\colon r,s \in [a_i,b_i]\}$).

A family $\mathscr{F}$ of functions F is said to be *uniformly generalized absolutely continuous* in the restricted sense on $[a,b]$ or uniformly ACG_* on $[a,b]$ if $[a,b]$ is the union of a sequence of closed sets E_i such that on each E_i the family $\mathscr{F}$ is uniformly $AC_*(E_i)$.

Now, let us present an important lemma (the uniform integrability of a family of functions means that the function $\delta(\cdot)$ in Definition 5 is common to all functions from this family) (cf. [7], [17]).

Lemma 2. *Let $f_n, f\colon I_\alpha \longrightarrow \mathbb{R}$ and assume that $f_n\colon I_\alpha \longrightarrow \mathbb{R}$ are* (HK) *integrable on I_α. Let F_n be a primitive of f_n. If we assume that:*

(i) *$f_n(t) \to f(t)$ a.e. on I_α,*

(ii) *the family $G = \{F_n\colon n \in \mathbb{N}\}$ is uniformly ACG_* on I_α,*

(iii) *G is equicontinuous on I_α,*

then (f_n) is uniformly (HK) *integrable on I_α.*

P r o o f. This is a simple consequence of Theorems 13.26 and 13.29 in the book of Gordon [17]. Similar result for ACG functions is well known (cf. [7, Lemma 2, p. 48]). $\square$

Now we are able to introduce the definition of pseudo-solution which we will use in the sequel.

Definition 3 (cf. [10], [11], [21], [24]). A function $x\colon I_\alpha \longrightarrow E$ is said to be a pseudo-solution of the Cauchy problem (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is ACG_*,

(ii) $x(0) = x_0$,

(iii) for each $x^* \in E^*$ there exists a set $A(x^*)$, with a Lebesgue measure zero, such that for each $t \notin A(x^*)$

$$x^*(x'(t)) = x^*(f(t, x(t))).$$

Here "$'$" denotes the pseudoderivative (see [29]).

A function $g\colon E \longrightarrow E_1$, where E and E_1 are Banach spaces, is said to be weakly-weakly sequentially continuous if for each weakly convergent sequence $(x_n) \subset E$, the sequence $(g(x_n))$ is weakly convergent in E_1.

A very interesting discussion (including examples) about different types of continuity can be found in [2] and [3]. The notion of weak sequential continuity seems to be the most convenient in use. It is not always possible to show that a given operator between Banach spaces is weakly continuous, quite often its weak sequential continuity presents no problem. This follows from the fact that the Lebesgue dominated convergence theorem is valid for sequences but not for nets.

The fact that a sequence x_n tends weakly to x_0 in E will be denoted by $x_n \xrightarrow{\omega} x_0$.

2. Henstock-Kurzweil-Pettis integral in Banach spaces

In this part we define the Henstock-Kurzweil-Pettis integral and we give properties of this integral. For basic definitions we refer the reader to [17] or [20].

Definition 5 ([6])**.** A function $f\colon [a,b] \longrightarrow E$ is Henstock-Kurzweil integrable on $[a,b]$ if there exists $A \in E$ with the following property: for every $\varepsilon > 0$ there exists a positive function $\delta(\cdot)$ on $[a,b]$ such that for every division $\mathscr{D}$ of $[a,b]$ given by $a = x_0 < x_1 < \ldots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ satisfying $\xi_i \in [cx_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$, we have

$$\left\| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right\| < \varepsilon.$$

We write $(\mathrm{HK})\int_a^b f(t)\,\mathrm{d}t = A$. We say that $\mathscr{D}$ is δ-fine and we can write $\mathscr{D} = \{[u,v];\xi\}$ with $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$.

We will write $f \in \mathrm{HK}([a,b], E)$ if f is Henstock-Kurzweil integrable on $[a,b]$.

This definition includes the generalized Riemann integral defined by Gordon ([18]).

Definition 6 ([6])**.** A function $f\colon [a,b] \longrightarrow E$ is (HL) integrable on $[a,b]$ ($f \in \mathrm{HL}([a,b], E)$) if there exists a function $F\colon [a,b] \longrightarrow E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$ there exists a positive function $\delta(\cdot)$ on $[a,b]$ such that if $\mathscr{D}=\{[u,v],\xi\}$ is a δ-fine division of $[a,b]$, we have

$$\sum_{\mathscr{D}}\big\|f(\xi)(v-u)-(F(v)-F(u))]\big\|<\varepsilon.$$

Remark 1. We note that by the triangle inequality: $f\in \mathrm{HL}([a,b],E)$ implies $f\in \mathrm{HK}([a,b],E)$. In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

Definition 7 ([29])**.** The function $f\colon I_\alpha\longrightarrow E$ is Pettis integrable ((P) integrable for short) if

(i) $\forall\, x^*\in E^*\quad x^*f$ is Lebesgue integrable on I_α,

(ii) $\forall A\subset I_\alpha,\ A$ measurable $\exists\, g\in E\ \forall\, x^*\in E^*\ \ x^*g=(L)\int_A x^*f(s)\,\mathrm{d}s$.

Now we present a definition of the integral which is a generalization of both Pettis and Henstock-Kurzweil integrals.

Definition 8. A function $f\colon I_\alpha\longrightarrow E$ is Henstock-Kurzweil-Pettis integrable ((HKP) integrable for short) if there exists a function $g\colon I_\alpha\longrightarrow E$ with the following properties:

(i) $\forall\, x^*\in E^*x^*\ \ f$ is Henstock-Kurzweil integrable on I_α,

(ii) $\forall\, t\in I_\alpha\ \forall\, x^*\in E^*\ \ x^*g(t)=(\mathrm{HK})\int_0^t x^*f(s)\,\mathrm{d}s$.

This function g will be called a primitive of f and by $g(\alpha)=\int_0^\alpha f(t)\,\mathrm{d}t$ we will denote the Henstock-Kurzweil-Pettis integral of f on the interval I_α.

Remark 2. Each function which is (HL) integrable is integrable in the sense of the Henstock-Kurzweil-Pettis. Our notion of integral is essentially more general than the previous ones (in Banach spaces):

(1°) Pettis integral: by the definition of the Pettis integral and since each Lebesgue integrable function is (HK) integrable we can put the Lebesgue integral in condition (i) and as a consequence we obtain that any (P) integrable function is (HKP) integrable.

(2°) Bochner, Riemann and Riemann-Pettis integrals (cf. [18]).

(3°) McShane integral (cf. [19] or [27]).

(4°) Henstock-Kurzweil (HL) integral: we present below an example.

Example. We present an example of function which is (HKP) integrable and neither (HL) integrable nor (P) integrable.

Let $f\colon\ [0,1]\longrightarrow (L^\infty[0,1],\|\cdot\|_\infty)$ and let $f(t)=\chi_{[0,t]}+A(t)\cdot F'(t)$, where

$$F(t)=t^2\sin t^{-2},\quad \chi_{[0,t]}(\tau)=\begin{cases}1, & \tau\in[0,t],\\ 0, & \tau\notin[0,t],\end{cases}\qquad t,\tau\in[0,1],$$

$F(0)=0$ and $A(t)(\tau)=1$ for $\tau,t\in[0,1]$.

Put $f_1(t) = \chi_{[0,t]}$, $f_2(t) = A(t) \cdot F'(t)$.

We will show that a function $f(t) = f_1(t) + f_2(t)$ is integrable in the sense of Henstock-Kurzweil-Pettis.

Observe that

$$x^* f(t) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).$$

Moreover, the function $x^*(f_1(t))$ is Lebesgue integrable (in fact f_1 is Pettis integrable [15]), so is Henstock-Kurzweil integrable, and the function $x^*(f_2(t))$ is Henstock-Kurzweil integrable by Definition 8.

The function f is not Lebesgue integrable because $x^* f_2$ is not Lebesgue integrable so f is not Pettis integrable. Moreover, the function f_1 is not strongly measurable (cf. [15]), so by Theorem 9 from [6] this function is not (HL) integrable. Hence, the function f is not integrable in the sense of Pettis and moreover this function is not (HL) integrable.

In the sequel we will investigate some properties of the (HKP) integral which are important in the next part of our paper.

Theorem 1. *Let $f\colon [a,b] \longrightarrow E$ be* (HKP) *integrable on $[a,b]$ and let $F(x) = \int_a^x f(s)\,\mathrm{d}s$.*

(a) *For each x^* in E^* the function $x^* f$ is* (HK) *integrable on $[a,b]$ and* (HK) $\int_a^x x^* f(s)\,\mathrm{d}s = x^* F(x)$.

(b) *The function F is weakly continuous on $[a,b]$ and f is a pseudoderivative of F on $[a,b]$.*

P r o o f. (a) See Definition 8.

(b) The function $x^* f$ is a real valued and (HK) integrable, and $x^* F(x) =$ (HK) $\int_a^x x^* f(s)\,\mathrm{d}s$ (by part (a)), thus $G(x) = \int_a^x x^* f(s)\,\mathrm{d}s$ is continuous (by Theorem 1(a)) i.e., F is a weakly continuous function.

By Theorem 1(a) there exists a set $A(x^*)$, $\operatorname{mes} A(x^*) = 0$, $A(x^*) \subset [a,b]$, such that $G'(x) = x^* f(x)$, but $G'(x) = (x^* F)'(x)$. $\square$

Theorem 2 ([12]). *Let $f_n, f\colon I_\alpha \longrightarrow E$ and assume that $f_n\colon I_\alpha \longrightarrow E$ are* (HKP) *integrable on I_α. Let F_n be a primitive of f_n. If we assume that:*

(i) $\forall\, x^* \in E^*$ $x^* f_n(t) \longrightarrow x^* f(t)$ *a.e. on I_α,*

(ii) *for each $x^* \in E^*$ the family $G = \{x^* F_n\colon n = 1, 2, 3, \ldots\}$ is uniformly ACG_* on I_α (i.e. weakly uniformly ACG_* on I_α),*

(iii) *for each $x^* \in E^*$ the set G is equicontinuous on I_α,*

then f is (HKP) *integrable on I_α and $\int_0^t f_n(s)\,\mathrm{d}s$ tends weakly in E to $\int_0^t f(s)\,\mathrm{d}s$ for each $t \in I_\alpha$.*

Theorem 3 (Mean value theorem for the (HKP) integral). *If the function $f\colon I_\alpha \longrightarrow E$ is* (HKP) *integrable, then:*

$$\int_I f(t)\,\mathrm{d}t \in |I| \cdot \overline{\mathrm{conv}} f(I),$$

where I is an arbitrary subinterval of I_α and $|I|$ is the lenght of I.

Proof. Taking an arbitrary $x^* \in E^*$ by the mean value theorem for (HK) integral we have:

$$\int_I x^*(f(t))\,\mathrm{d}t \in |I| \cdot \overline{\mathrm{conv}} x^* f(I) = x^*(|I| \cdot \overline{\mathrm{conv}} f(I)).$$

But, by the definition of Henstock-Kurzweil-Pettis integral, there exists $\int_I f(t)\,\mathrm{d}t$ such that $\int_I x^* f(t)\,\mathrm{d}t = x^* \int_I f(t)\,\mathrm{d}t$.

So $x^*\big(\int_I f(t)\,\mathrm{d}t\big) \in x^*(|I| \cdot \overline{\mathrm{conv}} f(I))$ for each $x^* \in E^*$. Because the set $|I| \cdot \overline{\mathrm{conv}} f(I)$ is a closed convex set, this implies $\int_I f(t)\,\mathrm{d}t \in |I| \cdot \overline{\mathrm{conv}} f(I)$. □

3. Main result

Now we prove an existence theorem for the problem (1) under the weakest assumptions on f, as it is known. We will use the following results.

Theorem 4 ([22]). *Let E be a metrizable locally convex topological vector space. Let $\mathscr{D}$ be a closed convex subset of E, and let F be a weakly sequentially continuous map of $\mathscr{D}$ into itself. If for some $x \in \mathscr{D}$ the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{x\} \cup F(V)) \Longrightarrow V \text{ is relatively weakly compact} \tag{3}$$

holds for every subset V of $\mathscr{D}$, then F has a fixed point.

Let

$$C(x_0, \alpha) = \{x \in C(I_\alpha, E)\colon\ x(0) = x_0,\ \ \|x\| \leqslant \|x_0\| + b\}$$

(α, b are some positive numbers). This set is closed and convex.

Let F_x be defined by $(F_x)(t) = x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s$, for $t \in I_\alpha$ and $x \in C(x_0, \alpha)$, $G = \{F_x\colon\ x \in C(x_0, \alpha)\}$.

Theorem 5. *Assume that for each ACG_* function $x\colon I_\alpha \longrightarrow E$, $f(\cdot, x(\cdot))$ is* (HKP) *integrable, $f(t,\cdot)$ is weakly-weakly sequentially continuous and*

$$\mu(f(I\times X)) \leqslant c\cdot\mu(X), \quad 0\leqslant c\alpha<1, \tag{4}$$

for each bounded subset $X\subset E$ and for each subinterval I of I_α.

Suppose that the set G is strongly equicontinuous and weakly uniformly ACG_* on I_α. Then there exists at least one pseudo-solution of the problem (1) *on I_β, for some number $0<\beta\leqslant\alpha$.*

Proof. We will prove, in fact, the existence of a solution for the problem (2).

By Theorem 1(a) each solution of the problem (2) is a solution of the problem (1). Fix an arbitrary $b\geqslant 0$. By the equicontinuity of G, there exists a number β, $0<\beta\leqslant\alpha$, such that

$$\left\|\int_0^t f(s,x(s))\,\mathrm{d}s\right\|\leqslant b,$$

for $t\in I_\beta$ and $x\in C(x_0,\alpha)$.

By our assumptions the operator F is well defined and maps $C(x_0,\beta)$ into $C(x_0,\beta)$.

We will show that the operator F is sequentially continuous. By Lemma 9 of [26] a sequence $x_n(\cdot)$ is weakly convergent in $C(I_\beta,E)$ to $x(\cdot)$ iff $x_n(t)$ tends weakly to $x(t)$ for each $t\in I_\beta$, so if $x_n \xrightarrow{\omega} x$ in $C(I_\beta,E)$ then $f(t,x_n(t)) \xrightarrow{\omega} f(t,x(t))$ in E for $t\in I_\beta$, and by Theorem 2 we have

$$\lim_{n\to\infty}\int_0^t f(s,x_n(s))\,\mathrm{d}s=\int_0^t f(s,x(s))\,\mathrm{d}s$$

weakly in E, for each $t\in I_\beta$.

We see that $F_{x_n}(t)\longrightarrow F_x(t)$ weakly in E for each $t\in I_\beta$ so $F_{x_n}\longrightarrow F_x$ in $(C(I_\beta,E),\omega)$.

Suppose that $V\subset C(x_0,\beta)$ satisfies the condition $\overline{V}=\overline{\mathrm{conv}}(F(V)\cup\{x\})$ for some $x\in C(x_0,\beta)$. We will prove that V is relatively weakly compact in $C(x_0,\beta)$, thus (3) is satisfied. Theorem 4 will ensure, that F has a fixed point (cf. [11]).

Let

$$F(V(t))=\{F_x(t)\colon\ x\in V\}=\left\{x_0+\int_0^t f(s,x(s))\,\mathrm{d}s\colon\ x\in V\right\}.$$

By properties of the measure of weak noncompactness and the assumption (4) we have

$$
\begin{aligned}
\mu(F(V(t))) &= \mu\left\{x_0 + \int_0^t f(s,x(s))\,\mathrm{d}s\colon\ x \in V\right\} \\
&\leqslant \mu\left\{\left(\int_0^t f(s,x(s))\,\mathrm{d}s\colon\ x \in V\right)\right\} \\
&\leqslant \mu(t \cdot \overline{\mathrm{conv}} f([0,t] \times V([0,t]))) \\
&\leqslant t \cdot \mu(f([0,t] \times V([0,t]))) \\
&\leqslant \beta \cdot \mu(f(I_\beta \times V(I_\beta))) \leqslant \beta \cdot c \cdot \mu(V(I_\beta)).
\end{aligned}
$$

Hence $\mu(F(V(t))) \leqslant \beta \cdot c \cdot \mu(V(I_\beta))$ for each $t \in I_\beta$.

Because $V = \overline{\mathrm{conv}}(F(V) \cup \{x\})$ then

$$
\mu(V(t)) = \mu(\overline{\mathrm{conv}}(F(V(t)) \cup \{x\})) \leqslant \mu(F(V(t))) \leqslant \beta \cdot c \cdot \mu(V(I_\beta)).
$$

By Lemma 1 we have

$$
\mu(V(I_\beta)) \leqslant \beta \cdot c \cdot \mu(V(I_\beta)) \leqslant \alpha \cdot c \cdot \mu(V(I_\beta)).
$$

So $\mu(V(I_\beta)) = 0$ and $\mu(V(t)) = 0$ for each $t \in I_\beta$. By the Arzelà-Ascoli theorem V is relatively weakly compact in $C(I_\beta, E)$. Using Theorem 4 there exists a fixed point of the operator F which is a pseudo-solution of (1). $\square$

Remark 4. The condition (4) in our Theorem 6 can be also generalized to the Sadovskii condition: $\mu(F(I \times X)) < \mu(X)$, whenever $\mu(X) > 0$, where μ can be replaced by some axiomatic measure of weak noncompactness (cf. [10]).

As we generalize both Pettis and Henstock-Kurzweil integrals our existence theorem is an extension of previous results; for example Chew and Flordelija [8], Cichoń [10], Cichoń, Kubiaczyk [11], Congxin, Baolin and Lee [13], Knight [21], Kubiaczyk [23], Kurzweil [24], Mitchell, Smith [26], O'Regan [28].

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