Samir H. Saker; Sui-Sun Cheng

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KAMENEV TYPE OSCILLATION CRITERIA FOR NONLINEAR DIFFERENCE EQUATIONS

Samir H. Saker, Mansoura, and Sui Sun Cheng, Hsinchu

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Abstract. By means of Riccati transformation techniques, we establish some new oscillation criteria for second-order nonlinear difference equation which are sharp.

Keywords: oscillation, nonlinear difference equation, Riccati transformation

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1. Introduction

In recent years, the asymptotic behavior of second order nonlinear difference equations has been the subject of investigation by many authors, see e.g. [1]–[19]. In particular, oscillatory behavior of second order nonlinear difference equations of the form

$$\Delta(p_n(\Delta x_n)^\gamma) + q_{n+1} f(x_{n+1}) = 0, \quad n = 0, 1, 2, \ldots \tag{1}$$

where γ is a ratio of positive odd integers, $\{p_n\}_{n=0}^{\infty}$ is a positive sequence and $\{q_n\}_{n=1}^{\infty}$ is a nonnegative sequence which has a positive subsequence, are obtained under various additional conditions such as

(H1) $\sum\limits_{n=0}^{\infty} (1/p_n)^{1/\gamma} = \infty$,

(H2) $\sum\limits_{n=0}^{\infty} (1/p_n)^{1/\gamma} < \infty$,

The paper was finished when the first author was in Faculty of Mathematics and Computer Science, Adam Mickiewicz Univeristy, Matejki 48/49, 60-769 Poznań, Poland.

(H3) $f\colon \mathbb{R} \to \mathbb{R}$ is such that $xf(x) > 0$ and $f(x) \geqslant \kappa x$ for $x \neq 0$ and some $\kappa > 0$,

(H4) $f\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing,

(H5) $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f'(x) \geqslant 0$ for $x \neq 0$,

(H6) $f\colon \mathbb{R} \to \mathbb{R}$ is such that $f(u) - f(v) = g(u,v)(u-v)^\delta$ and g is nonnegative.

Some of these results will be briefly stated below. Before doing so, let us first recall that a solution of (1) is a nontrivial real sequence $\{x_n\}$ satisfying equation (1) for $n \geqslant 0$. A solution $\{x_n\}$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory. In [16], the authors considered the second order nonlinear difference equation

$$\Delta^2 x_n + q_{n+1} f(x_{n+1}) = 0, \quad n = 0, 1, 2, \dots \tag{2}$$

and proved that if (H4) holds and

$$\sum_{n=0}^{\infty} q_{n+1} = \infty, \tag{3}$$

then every solution oscillates. In [13], Thandapani et al. considered also (2) when (H6) holds with $\delta = 1$ and obtained some different oscillation criteria. In [4], the authors considered the equation (1) when $\gamma = 1$ and proved that if (H1), (H4) and (3) hold then every solution of (1) oscillates. In [9], [14], the authors employed the techniques similar to those in [13] to obtain oscillation criteria for (1), and proved that every solution of (1) oscillates if (H1) and (3) hold. Unfortunately, the oscillation criteria impose assumptions on the unknown solutions, which diminishes their applicability. Furthermore, the restrictive condition (H6) is required in [14], where $\delta = 1$, and the condition (H5) is required in [9]. In [10], the authors studied the existence of positive and negative solutions of (1) under the condition (H6) where $\delta = 1$. In [6] the authors considered equation (1) and gave some oscillation criteria when (H1), (H2) or (H6) hold, where $g(u,v)$ satisfies $g(u,v) \geqslant \tau > 0$

In this paper we intend to use the Riccati transformation technique for obtaining several new oscillation criteria for (1) when (H1), (H2), (H3) or (H5) hold. However, some of our results do not require the nonlinear function f to be nondecreasing, nor condition (H6). Furthermore, when (H3) holds, we present some sufficient conditions which guarantee that every solution of (1) oscillates or tends to zero. Our results improve the results in [4], [6], [9], [13], [14], [15], [16] when (H1) is satisfied.

For oscillation and nonoscillation of different classes of second order difference equations we refer the reader to the monographs of Agarwal [1] and Agarwal and Wong [2].

2. The case when (H1) holds

In this section, we assume that (H1) holds.

Theorem 2.1. *Assume that* (H1) *and* (H3) *hold. Furthermore, assume that there exists a positive sequence* $\{\varrho_n\}_{n=0}^{\infty}$ *such that for every positive constant* M*,*

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\Big[\kappa\varrho_l q_{l+1}-\frac{p_{l+1}^{1/\gamma}(\Delta\varrho_l)^2}{4M^{(\gamma-1)/\gamma}\varrho_l}\Big]=\infty. \tag{4}$$

Then every solution of equation (1) *oscillates.*

Proof. Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x_n>0$ for all large n. We shall consider only this case, since the substitution $y_n=-x_n$ transforms equation (1) into an equation of the same form. In view of (1) we have

$$\Delta(p_n(\Delta x_n)^\gamma)=-q_{n+1}f(x_{n+1})\leqslant 0,$$

for all large n, and so $\{p_n(\Delta x_n)^\gamma\}$ is an eventually nonincreasing sequence. We first show that $\{p_n(\Delta x_n)^\gamma\}$ is eventually positive. Indeed, since $\{q_n\}$ has a positive subsequence, the nondecreasing sequence $\{p_n(\Delta x_n)^\gamma\}$ is either eventually positive or eventually negative. Suppose there exists an integer $n_1\geqslant 0$ such that $p_n(\Delta x_n)^\gamma\leqslant p_{n_1}(\Delta x_{n_1})^\gamma=c<0$ for $n\geqslant n_1$, then

$$\Delta x_n\leqslant c^{1/\gamma}\Big(\frac{1}{p_n}\Big)^{1/\gamma},$$

which implies that

$$x_n\leqslant x_{n_1}+c^{1/\gamma}\sum_{i=n_1}^{n-1}\Big(\frac{1}{p_i}\Big)^{1/\gamma}\to-\infty\quad\text{as } n\to\infty,$$

which contradicts the fact that $x_n>0$ for all large n. Hence $\{p_n(\Delta x_n)^\gamma\}$ is eventually positive. Therefore, we see that there is n_0 such that

$$x_n>0,\quad \Delta x_n>0,\quad \Delta(p_n(\Delta x_n)^\gamma)\leqslant 0,\quad n\geqslant n_0. \tag{5}$$

Define a sequence $\{w_n\}$ by

$$w_n=\varrho_n\frac{p_n(\Delta x_n)^\gamma}{x_{n+1}},\quad n\geqslant n_0. \tag{6}$$

Then $w_n > 0$ and

$$\Delta w_n = p_{n+1}(\Delta x_{n+1})^\gamma \Delta\Big[\frac{\varrho_n}{x_{n+1}}\Big] + \frac{\varrho_n \Delta(p_n(\Delta x_n)^\gamma)}{x_{n+1}}. \tag{7}$$

In view of (1), (H3) and (7),

$$\Delta w_n \leqslant -\kappa \varrho_n q_{n+1} + \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} - \frac{\varrho_n p_{n+1}(\Delta x_{n+1})^{\gamma+1}}{x_{n+1}x_{n+2}}. \tag{8}$$

But from (5) we have

$$p_n(\Delta x_n)^\gamma \geqslant p_{n+1}(\Delta x_{n+1})^\gamma, \quad x_{n+2} \geqslant x_{n+1}, \tag{9}$$

and thus from (8) and (9) we have

$$\Delta w_n \leqslant -\kappa \varrho_n q_{n+1} + \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} - \frac{\varrho_n p_{n+1}(\Delta x_{n+1})^{\gamma+1}}{(x_{n+2})^2}, \tag{10}$$

so that

$$\Delta w_n \leqslant -\kappa \varrho_n q_{n+1} + \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} - \frac{\varrho_n}{(\varrho_{n+1})^2 p_{n+1}} w_{n+1}^2 \frac{1}{(\Delta x_{n+1})^{\gamma-1}}. \tag{11}$$

Now, by virtue of the fact that $\{p_n(\Delta x_n)^\gamma\}$ is a positive and nonincreasing sequence, there exists $n_2 \geqslant n_1$ sufficiently large such that $p_n(\Delta x_n)^\gamma \leqslant 1/M$ for some positive constant M and $n \geqslant n_2$, and hence by (9) we have $p_{n+1}(\Delta x_{n+1})^\gamma \leqslant 1/M$, so that

$$\frac{1}{(\Delta x_{n+1})^{\gamma-1}} \geqslant (M p_{n+1})^{(\gamma-1)/\gamma}. \tag{12}$$

Substituting from (12) into (11) we obtain

$$\Delta w_n \leqslant -\kappa \varrho_n q_{n+1} + \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} - M^{(\gamma-1)/\gamma} \frac{\varrho_n}{p_{n+1}^{1/\gamma}(\varrho_{n+1})^2} w_{n+1}^2, \tag{13}$$

so that

$$\begin{aligned}\Delta w_n \leqslant & -\kappa \varrho_n q_{n+1} + \frac{p_{n+1}^{1/\gamma}(\Delta \varrho_n)^2}{4M^{(\gamma-1)/\gamma}\varrho_n} - \left[\frac{\sqrt{M^{(\gamma-1)/\gamma}\varrho_n}}{\varrho_{n+1}\sqrt{p_{n+1}^{1/\gamma}}} w_{n+1} - \frac{\sqrt{p_{n+1}^{1/\gamma}}\Delta \varrho_n}{2\sqrt{M^{(\gamma-1)/\gamma}\varrho_n}}\right]^2 \\ < & -\left[\kappa \varrho_n q_{n+1} - \frac{p_{n+1}^{1/\gamma}(\Delta \varrho_n)^2}{4M^{(\gamma-1)/\gamma}\varrho_n}\right].\end{aligned}$$

Hence, we have

$$\Delta w_n < -\left[\kappa \varrho_n q_{n+1} - \frac{p_{n+1}^{1/\gamma}(\Delta \varrho_n)^2}{4M^{(\gamma-1)/\gamma}\varrho_n}\right]. \tag{14}$$

Summing (14) from n_2 to n, we obtain

$$-w_{n_2} < w_{n+1} - w_{n_2} < -\sum_{l=n_2}^{n}\left[\kappa \varrho_l q_{l+1} - \frac{p_{n+1}^{1/\gamma}(\Delta \varrho_l)^2}{4M^{(\gamma-1)/\gamma}\varrho_l}\right], \tag{15}$$

which yields

$$\sum_{l=n_2}^{n}\left[\kappa \varrho_l q_{l+1} - \frac{p_{n+1}^{1/\gamma}(\Delta \varrho_l)^2}{4M^{(\gamma-1)/\gamma}\varrho_l}\right] < w_{n_2},$$

for all large n. This is contrary to (4). The proof is complete. □

Note that if $\varrho_n = n+1$, $\gamma = 1$ and $p_n = 1$, then (4) reduces to

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[\kappa(l+1)q_{l+1} - \frac{1}{4(l+1)}\right] = \infty.$$

This is an improvement of condition (3).

Note that when $\gamma = 1$, equation (1) reduces to the nonlinear difference equation

$$\Delta(p_n \Delta x_n) + q_{n+1} f(x_{n+1}) = 0, \quad n = 0, 1, 2 \ldots \tag{16}$$

and then Theorem 2.1 reduces to the following corollary which improves Theorem 13 in [4].

Corollary 2.1. *Assume that* (H1) *and* (H3) *hold and there exists a positive sequence* $\{\varrho_n\}_{n=0}^{\infty}$ *such that*

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[\kappa \varrho_l q_{l+1} - \frac{p_{l+1}(\Delta \varrho_l)^2}{4\varrho_l}\right] = \infty. \tag{17}$$

Then every solution of (16) *oscillates.*

From Theorem 2.1 we can obtain different conditions for oscillation of all solutions of (1) by different choices of $\{\varrho_n\}$. For instance, let $\varrho_n = (n+1)^{\lambda}$, $n \geqslant n_0$ and $\lambda \geqslant 1$. By Theorem 2.1 we have the following result.

Corollary 2.2. *Assume that* (H1) *and* (H3) *hold. Furthermore, assume that there is $\lambda \geqslant 1$ such that for every positive constant M,*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[\kappa(s+1)^{\lambda} q_{s+1} - \frac{p_{s+1}^{1/\gamma}((s+2)^{\lambda}-(s+1)^{\lambda})^2}{4M^{(\gamma-1)/\gamma}(s+1)^{\lambda}} \right] = \infty. \tag{18}$$

Then every solution of (1) *oscillates.*

As an example, consider the discrete Euler equation

$$\Delta^2 x_n + \frac{\mu}{(n+1)^2} x_{n+1} = 0, \quad n \geqslant 1, \tag{19}$$

where $\mu > 1/4$. Then $p_n = 1$ and $\gamma = 1$. If we take $\varrho_n = n+1$, then we have

$$\sum_{s=n_0}^{n} \left[(s+1) q_{s+1} - \frac{((s+2)-(s+1))^2}{4(s+1)} \right] = \sum_{s=1}^{n} \left[\frac{\mu(s+1)}{(s+1)^2} - \frac{1}{4(s+1)} \right]$$
$$= \sum_{s=1}^{n} \frac{(4\mu-1)}{4(s+1)} \to \infty,$$

as $n \to \infty$. By Corollary 2.1, every solution of (19) oscillates. It is known from [18], [19] that when $\mu \leqslant 1/4$, (19) has a nonoscillatory solution. Hence, Theorem 2.1 and Corollary 2.1 are sharp. Note that the results in [4], [9], [14], [16] cannot be applied to (19).

As another example, consider the nonlinear difference equation

$$\Delta\Big(\frac{n}{n+1}\Delta x_n\Big) + \frac{\mu}{(n+1)^2} x_{n+1}(1+x_{n+1}^2) = 0, \quad n \geqslant 1 \tag{20}$$

where $\mu > 1/4$, $f(u)/u \geqslant 1$, and $\gamma = 1$. If we take $\varrho_n = n+1$, then we have

$$\sum_{s=n_0}^{n} \left[(s+1) q_{s+1} - \frac{p_{s+1}((s+2)-(s+1))^2}{4(s+1)} \right]$$
$$= \sum_{s=1}^{n} \left[\frac{\mu(s+1)}{(s+1)^2} - \frac{(s+1)/(s+2)}{4(s+1)} \right]$$
$$= \sum_{s=1}^{n} \left[\frac{\mu}{s+1} - \frac{1}{4(s+2)} \right] \geqslant \sum_{s=1}^{n} \frac{4\mu-1}{s+1} \to \infty,$$

as $n \to \infty$. Thus Corollary 2.2 asserts that every solution of (20) is oscillatory when $\mu > 1/4$. The results in [4], [9], [14], [15], however, are not applicable.

As a variant of the Riccati transformation technique used above, we will derive a Kamenev type oscillation criteria which can be considered a discrete analogy of Philos' condition for oscillation of second order differential equations [11].

Theorem 2.2. *Assume that* (H1) *and* (H3) *hold. Let* $\{\varrho_n\}_{n=0}^{\infty}$ *be a positive sequence. Furthermore, assume that there exists a double sequence* $\{H_{m,n}\colon m \geqslant n \geqslant 0\}$ *such that*

(i) $H_{m,m} = 0$ *for* $m \geqslant 0$,

(ii) $H_{m,n} > 0$ *for* $m > n \geqslant 0$,

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leqslant 0$ *for* $m \geqslant n \geqslant 0$.

If

$$\limsup_{m\to\infty} \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \biggl[\kappa H_{m,n}\varrho_n q_{n+1} - \frac{p_{n+1}^{1/\gamma}\varrho_{n+1}^2}{4M^{(\gamma-1)/\gamma}\varrho_n}\Bigl(h_{m,n} - \frac{\Delta\varrho_n}{\varrho_{n+1}}\sqrt{H_{m,n}}\Bigr)^2\biggr] = \infty, \tag{21}$$

where

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \qquad m > n \geqslant 0,$$

for every positive number M, *then every solution of* (1) *oscillates.*

Proof. We proceed as in the proof of Theorem 2.1. We may assume that (1) has a nonoscillatory solution $\{x_n\}_{n=0}^{\infty}$ such that $x_n > 0$, $\Delta x_n > 0$, $\Delta(p_n(\Delta x_n)^{\gamma}) \leqslant 0$ for $n \geqslant n_1 \geqslant 0$. Define $\{w_n\}$ by (6) as before, then we have $w_n > 0$ and there is $M > 0$ such that (13) holds. For the sake of convenience, let us set

$$\bar{\varrho}_n = \frac{M^{(\gamma-1)/\gamma}\varrho_n}{p_{n+1}^{1/\gamma}}.$$

Then

$$\Delta w_n \leqslant -\kappa\varrho_n q_{n+1} + \frac{\Delta\varrho_n}{\varrho_{n+1}} w_{n+1} - \frac{\bar{\varrho}_n}{\varrho_{n+1}^2} w_{n+1}^2, \tag{22}$$

or

$$\kappa\varrho_n q_{n+1} \leqslant -\Delta w_n + \frac{\Delta\varrho_n}{\varrho_{n+1}} w_{n+1} - \frac{\bar{\varrho}_n}{\varrho_{n+1}^2} w_{n+1}^2. \tag{23}$$

Therefore,

$$\sum_{n=n_2}^{m-1} \kappa H_{m,n}\varrho_n q_{n+1} \leqslant -\sum_{n=n_2}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\varrho_n}{\varrho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\varrho}_n}{\varrho_{n+1}^2} w_{n+1}^2,$$

which yields, after summing by parts,

$$\begin{aligned}
&\sum_{n=n_2}^{m-1} \kappa H_{m,n} \varrho_n q_{n+1} \\
&\quad \leqslant H_{m,n_2} w_{n_2} + \sum_{n=n_2}^{m-1} w_{n+1} \Delta_2 H_{m,n} + \sum_{n=n_2}^{m-1} H_{m,n} \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} \\
&\qquad - \sum_{n=n_2}^{m-1} H_{m,n} \frac{\overline{\varrho}_n}{\varrho_{n+1}^2} w_{n+1}^2 \\
&\quad = H_{m,n_2} w_{n_2} - \sum_{n=n_2}^{m-1} h_{m,n} \sqrt{H_{m,n}} w_{n+1} + \sum_{n=n_2}^{m-1} H_{m,n} \frac{\Delta \varrho_n}{\varrho_{n+1}} w_{n+1} \\
&\qquad - \sum_{n=n_2}^{m-1} H_{m,n} \frac{\overline{\varrho}_n}{\varrho_{n+1}^2} w_{n+1}^2 \\
&\quad = H_{m,n_2} w_{n_2} \\
&\qquad - \sum_{n=n_2}^{m-1} \left[\frac{\sqrt{H_{m,n} \overline{\varrho}_n}}{\varrho_{n+1}} w_{n+1} + \frac{\varrho_{n+1}}{2\sqrt{H_{m,n} \overline{\varrho}_n}} \left(h_{m,n} \sqrt{H_{m,n}} - \frac{\Delta \varrho_n}{\varrho_{n+1}} H_{m,n} \right) \right]^2 \\
&\qquad + \frac{1}{4} \sum_{n=n_2}^{m-1} \frac{(\varrho_{n+1})^2}{\overline{\varrho}_n} \left(h_{m,n} - \frac{\Delta \varrho_n}{\varrho_{n+1}} \sqrt{H_{m,n}} \right)^2
\end{aligned}$$

Then

$$\sum_{n=n_2}^{m-1} \left[\kappa H_{m,n} \varrho_n q_{n+1} - \frac{\varrho_{n+1}^2}{4\overline{\varrho}_n} \left(h_{m,n} - \frac{\Delta \varrho_n}{\varrho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < H_{m,n_2} w_{n_2} \leqslant H_{m,0} w_{n_2}$$

which implies that

$$\sum_{n=0}^{m-1} \left[\kappa H_{m,n} \varrho_n q_{n+1} - \frac{\varrho_{n+1}^2}{4\overline{\varrho}_n} \left(h_{m,n} - \frac{\Delta \varrho_n}{\varrho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < \kappa H_{m,0} \sum_{n=0}^{n_2-1} \varrho_n q_{n+1} + H_{m,0} w_{n_2}.$$

Hence

$$\limsup_{m\to\infty} \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left[\kappa H_{m,n} \varrho_n q_{n+1} - \frac{\varrho_{n+1}^2}{4\overline{\varrho}_n} \left(h_{m,n} - \frac{\Delta \varrho_n}{\varrho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < \infty,$$

which is contrary to (21). The proof is complete. □

By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1). First, let us consider a double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n} = (m-n)^{\lambda}, \quad \lambda \geqslant 1, \ \ m \geqslant n \geqslant 0. \tag{24}$$

Then $H_{m,m} = 0$ for $m \geqslant 0$, $H_{m,n} > 0$, $\Delta_2 H_{m,n} \leqslant 0$ for $m > n$ and

$$h_{m,n} = -\frac{(m-n-1)^\lambda - (m-n)^\lambda}{(m-n)^{\lambda/2}} = \frac{(m-n)^\lambda - (m-n-1)^\lambda}{(m-n)^{\lambda/2}}.$$

Using the inequality

$$x^\gamma - y^\gamma \leqslant \gamma x^{\gamma-1}(x-y) \quad \text{for all } x \neq y \text{ and } \gamma \geqslant 1,$$

we have

$$h_{m,n} \leqslant \lambda(m-n)^{\lambda/2-1}.$$

Hence from Theorem 2.2 we have the following result.

Corollary 2.3. *Assume that all the assumptions of Theorem* 2.2 *hold, except that the condition* (21) *is replaced by*

$$(25)\quad \limsup_{m\to\infty} \frac{1}{m^\lambda} \sum_{n=0}^{m-1} \Bigg[\kappa(m-n)^\lambda \varrho_n q_{n+1} - \frac{p_{n+1}^{1/\gamma} \varrho_{n+1}^2}{4M^{(\gamma-1)/\gamma} \varrho_n} \Big(\lambda(m-n)^{\lambda/2-1} - \frac{\Delta \varrho_n}{\varrho_{n+1}} \sqrt{(m-n)^\lambda} \Big)^2 \Bigg] = \infty,$$

for every positive number M and some $\lambda \geqslant 1$. Then every solution of (1) *oscillates.*

Next, consider the double sequence $\{H_{m,n}\}$ defined by

$$(26)\qquad H_{m,n} = \Big(\ln \frac{m+1}{n+1} \Big)^\lambda, \quad \lambda \geqslant 1, \ m \geqslant n \geqslant 0.$$

Then $H_{m,m} = 0$ for $m \geqslant 0$, $H_{m,n} > 0$, $\Delta_2 H_{m,n} \leqslant 0$ for $m > n$ and

$$h_{m,n} = -\frac{\big(\ln \frac{m+1}{n+2}\big)^\lambda - \big(\ln \frac{m+1}{n+1}\big)^\lambda}{(m-n)^{\lambda/2}} = \frac{\big(\ln \frac{m+1}{n+1}\big)^\lambda - \big(\ln \frac{m+1}{n+2}\big)^\lambda}{\big(\ln \frac{m+1}{n+1}\big)^{\lambda/2}}.$$

Then we have

$$h_{m,n} \leqslant \lambda \Big(\ln \frac{m+1}{n+1} \Big)^{\frac{\lambda-2}{2}} \ln\Big(1 + \frac{1}{n+1} \Big).$$

Using the inequality $e^x \geqslant 1 + x$ for all $x \geqslant 0$ we obtain

$$h_{m,n} \leqslant \frac{\lambda}{n+1} \Big(\ln \frac{m+1}{n+1} \Big)^{\frac{\lambda-2}{2}}.$$

Hence from Theorem 2.2 we have the following result.

Corollary 2.4. *Assume that all the assumptions of Theorem 2.2 hold, except the condition* (2.21) *is replaced by*

$$\limsup_{m\to\infty}\frac{1}{\ln^{\lambda}(m+1)}\sum_{n=0}^{m-1}\Big[\kappa\Big(\ln\frac{m+1}{n+1}\Big)^{\lambda}\varrho_n q_{n+1}-B_{m,n}\Big]=\infty, \tag{27}$$

where

$$B_{m,n}=\frac{p_{n+1}^{1/\gamma}\varrho_{n+1}^2}{4M^{(\gamma-1)/\gamma}\varrho_n}\bigg(\frac{\lambda}{n+1}\Big(\ln\frac{m+1}{n+1}\Big)^{\frac{\lambda-2}{2}}-\frac{\Delta\varrho_n}{\varrho_{n+1}}\sqrt{\Big(\ln\frac{m+1}{n+1}\Big)^{\lambda}}\bigg)^2$$

for every positive number M and some $\lambda\geqslant 1$. Then every solution of (1) *oscillates.*

Other $H_{m,n}$ may be chosen as

$$H_{m,n}=\varphi(m-n),\quad m\geqslant n\geqslant 0,$$

or

$$H_{m,n}=(m-n)^{(\lambda)},\quad \lambda>2,\ m\geqslant n\geqslant 0,$$

where $\varphi\colon[0,\infty)\to[0,\infty)$ is a continuously differentiable function which satisfies $\varphi(0)=0$ and $\varphi(u)>0$, $\varphi'(u)\geqslant 0$ for $u>0$, and $(m-n)^{(\lambda)}=(m-n)\times(m-n+1)\dots(m-n+\lambda-1)$ and

$$\Delta_2(m-n)^{(\lambda)}=(m-n-1)^{(\lambda)}-(m-n)^{(\lambda)}=-\lambda(m-n)^{(\lambda-1)}.$$

Corresponding corollaries can also be stated.

3. The case when (H2) holds

Next, we consider the case when (H2) holds.

Theorem 3.1. *Assume that* (H2), (H3) *and* (H4) *hold. Furthermore, assume that there exist positive sequences $\{\varrho_n\}_{n=0}^{\infty}$ such that* (4) *holds for every positive number M and*

$$\sum_{n=n_0}^{\infty}\Big(\frac{1}{p_n}\sum_{i=n_0}^{n-1}q_{i+1}\Big)^{\frac{1}{\gamma}}=\infty. \tag{28}$$

Then every solution of (1) *oscillates or converges to zero.*

P r o o f. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. From (1) we have

$$\Delta(p_n(\Delta x_n)^\gamma) = -q_{n+1} f(x_{n+1}) \leqslant 0, \tag{29}$$

and so $\{p_n(\Delta x_n)^\gamma\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, $\{\Delta x_n\}$ is either eventually negative or eventually positive.

If $\{\Delta x_n\}$ is eventually positive, we are then back to the case when (5) holds. Thus the proof of Theorem 2.1 goes through, and we may conclude that $\{x_n\}$ cannot be eventually positive, which is not possible.

If $\{\Delta x_n\}$ is eventually negative, then $\lim\limits_{n\to\infty} x_n = b \geqslant 0$. We assert that $b = 0$. If not, then $f(x_{n+1}) \to f(b) > 0$ as $n \to \infty$. Since $f(x)$ is nondecreasing, there exists $n_2 > 0$ such that $f(x_{n+1}) \geqslant f(b)$ for $n \geqslant n_2$. Therefore from (2.29) we have

$$\Delta(p_n(\Delta x_n)^\gamma) \leqslant -q_{n+1} f(b).$$

Define a sequence $u_n = p_n(\Delta x_n)^\gamma$ for $n \geqslant n_2$. Then we have

$$\Delta u_n \leqslant -f(b) q_{n+1}.$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \leqslant u_{n_2} - f(b) \sum_{s=n_2}^{n-1} q_{s+1} < -b\kappa \sum_{s=n_2}^{n-1} q_{s+1},$$

where u_{n_2} is nonpositive. Summing the last inequality from n_2 to n we obtain

$$x_{n+1} \leqslant x_{n_2} - (f(b))^{1/\gamma} \sum_{s=n_2}^{n} \left(\frac{1}{p_s} \sum_{i=n_2}^{s-1} q_{i+1} \right)^{1/\gamma}.$$

Condition (28) implies that $\{x_n\}$ is eventually negative, which is a contradiction. Thus $\{x_n\}$ converges to zero. The proof is complete. □

As an example, consider the linear difference equation

$$\Delta(n^2 \Delta x_n) + \mu x_{n+1} = 0, \quad n \geqslant 1 \tag{30}$$

where $\mu > 1/4$. Then $p_n = n^2$, $\gamma = 1$. If we take $\varrho_n = n + 1$ and $\beta_n = 1$, then one can easily see that (28) holds and

$$\begin{aligned} \sum_{s=n_0}^{n} \left[(s+1) q_s - \frac{p_{s+1}((s+2)-(s+1))^2}{4(s+1)} \right] &= \sum_{s=1}^{n} \left[\mu(s+1) - \frac{(s+1)^2}{4(s+1)} \right] \\ &= \sum_{s=1}^{n} \frac{(4\mu - 1)}{4}(s+1) \to \infty, \end{aligned}$$

as $n \to \infty$. Thus, Theorem 3.1 asserts that every solution of (30) oscillates or $x_n \to 0$ as $n \to \infty$. Note that none of the above mentioned papers can be applied to (30).

By choosing $\{\varrho_n\}_{n=0}^{\infty}$ in appropriate manners, we may obtain different oscillation criteria. For instance, let $\varrho_n = n^{\lambda}$ for $n \geqslant 0$ and $\lambda > 1$. Then we have the following result.

Corollary 3.1. *Assume that all the assumptions of Theorem* 3.1 *hold, except that the condition* (4) *is replaced by* (18). *Then every solution of* (1) *oscillates or converges to zero.*

Theorem 3.2. *Assume that* (H2), (H3), (H4) *and* (28) *hold. Furthermore, let* $\{\varrho_n\}_{n=0}^{\infty}$ *be a positive sequence and assume that there exists a double sequence* $\{H_{m,n}\colon\ m \geqslant n \geqslant 0\}$ *as defined in Theorem* 2.2 *and* (21) *holds. Then every solution of* (1) *oscillates or converges to zero.*

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1). Then as seen in the proof of Theorem 2.1, either $\{\Delta x_n\}$ is eventually positive or it is eventually negative. In the former case, we may follow the proof of Theorem 2.2 and obtain a contradiction. If $\{\Delta x_n\}$ is eventually negative, then we may follow the proof of Theorem 3.1 to show that $\{x_n\}$ converges to zero.

By choosing $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1) when (H2) holds. For instance, let us consider double sequences $\{H_{m,n}\}$ defined again by (24) or (26).

Corollary 3.2. *Assume that all the assumptions of Theorem* 3.2 *hold, except that the condition* (21) *is replaced by* (25). *Then every solution of* (1) *oscillates or converges to zero.*

Corollary 3.3. *Assume that all the assumptions of Theorem* 3.2 *hold, except that the condition* (21) *is replaced by* (27). *Then every solution of* (1) *oscillates or converges to zero.*

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Authors' addresses: Samir H. Saker, Department of Mathematics, Mansoura Univeristy, Mansoura 35516, Egypt, e-mail: `shsaker@mum.mans.eun.eg`; Sui Sun Cheng, Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China.