Czechoslovak Mathematical Journal

Jan W. Cholewa; Tomasz Dłotko

Cauchy problems in weighted Lebesgue spaces

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 4, 991–1013

Persistent URL: http://dml.cz/dmlcz/127947

Czechoslovak Mathematical Journal, 54 (129) (2004), 991–1013

CAUCHY PROBLEMS IN WEIGHTED LEBESGUE SPACES

Jan W. Cholewa and Tomasz Dlotko, Katowice

(Received February 7, 2002)

Abstract. Global solvability and asymptotics of semilinear parabolic Cauchy problems in $\mathbb{R}^n$ are considered. Following the approach of A. Mielke [15] these problems are investigated in weighted Sobolev spaces. The paper provides also a theory of second order elliptic operators in such spaces considered over $\mathbb{R}^n$, $n \in \mathbb{N}$. In particular, the generation of analytic semigroups and the embeddings for the domains of fractional powers of elliptic operators are discussed.

Keywords: Cauchy problem, parabolic equation, global existence, asymptotic behavior of solutions

MSC 2000: 35K15, 35B40

Motivation and introductory notes

It is well known that the studies of compactness of trajectories and asymptotic behavior of solutions to semilinear parabolic equations are more difficult when the space variable x belongs to the whole $\mathbb{R}^n$ or, at least, to an unbounded domain $\Omega \subset \mathbb{R}^n$. When dealing with the typical Cauchy problem of that form,

$$\begin{cases} u_t = \Delta u + u - u^3,\ t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

known as the *bi-stable* reaction diffusion equation, we are facing simultaneously the incorrectness of the Poincaré inequality and the incomparability of the $L^2(\mathbb{R}^n)$ and $L^4(\mathbb{R}^n)$ norms of the solution. There is thus no term in (1) allowing to

Supported by the State Committee for Scientific Research (KBN) Grant # 2PO3A03518.

bound the solutions uniformly on the time interval $[0,+\infty)$ in the usual Sobolev spaces [14]. Hence, studying such problems we are forced to work in *weighted spaces* that are not, at a first view, so common and natural as the usual Sobolev spaces (see [5], [7], [10], [16], [15]; also [9] for the most recent result on this subject).

In this paper we want to extend our result [7] using some ideas of analytic semigroups introduced in [3], [16] and [15]. Considering weight functions $\varrho\colon \mathbb{R}^n \to (0,+\infty)$ we denote by $L^p_\varrho(\mathbb{R}^n)$, $p>1$, the Banach weighted space consisting of all $\varphi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ having a finite norm

$$\|\varphi\|_{L^p_\varrho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\varphi(x)|^p \varrho(x)\,\mathrm{d}x\right)^{1/p}.$$

Throughout the paper we require of the weight function ϱ that

$$\text{(2)}\qquad \begin{cases} \varrho\colon \mathbb{R}^n \to (0,+\infty) \text{ is integrable on } \mathbb{R}^n, \text{ belongs to } C^2(\mathbb{R}^n), \text{ and satisfies} \\ \left|\dfrac{\partial \varrho}{\partial x_j}\right| \leqslant \varrho_0 \varrho \text{ and } \left|\dfrac{\partial^2 \varrho}{\partial x_j \partial x_k}\right| \leqslant C\varrho, \quad j,k=1,\dots,n, \end{cases}$$

where ϱ_0, C are positive constants.

Remark 1. Thanks to the first condition in (2) the following property of ϱ holds:

$$\text{(3)}\qquad \forall y \in \mathbb{R}^n \ \sup_{x\in\mathbb{R}^n} \frac{\varrho(x)}{\varrho(x-y)} < \infty.$$

Particular examples of weight functions are

$$\text{(4)}\qquad \varrho(x) = (1+|\varepsilon x|^2)^{-n} \quad \text{or, for } n=1, \quad \text{also } \varrho_1(x) = (\cosh \varepsilon x)^{-1},\ \varepsilon>0.$$

Unfortunately, the spaces $L^p_\varrho(\mathbb{R}^n)$ are not yet fully satisfactory from the point of view of the local solvability and the asymptotics of problems like (1). For example, although the Laplacian generates a strongly continuous analytic semigroup on $L^p_\varrho(\mathbb{R}^n)$ the smoothing action in this space is not as good as in the usual Sobolev spaces. Simple examples show that $W^{1,2}_\varrho(\mathbb{R}^n)$ is not included in $L^6_\varrho(\mathbb{R}^n)$.[1] Therefore, nonlinearity in (1) does not take $W^{1,2}_\varrho(\mathbb{R}^n)$ into $L^2_\varrho(\mathbb{R}^n)$, although it is Lipschitz continuous from $W^{1,2}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. What was said above makes it clear that one can hardly build the semigroup on weighted spaces $L^p_\varrho(\mathbb{R}^n)$ using the standard approach of [12], that is, checking the Lipschitz condition between a certain fractional power space and a base space $L^p_\varrho(\mathbb{R}^n)$. Thus, our main concern is to find a suitable

[1] Indeed, specifying $n=3$, $\varrho(x)=(1+|x|^2)^{-5}$ and $f(x)=1+|x|^2$ one have a counterexample.

base space where Henry's technique [12] would work. Our choice will be a Banach space borrowed from [15] (see also [10])

$$L^p_{\varrho,\infty}(\mathbb{R}^n) = \Big\{\varphi \in \bigcap_{y\in\mathbb{R}^n} L^p_{\tau_y\varrho}(\mathbb{R}^n)\colon\ \|\varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} \|\varphi\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} < \infty\Big\}, \tag{5}$$

$p \in [1,\infty)$, where $\{\tau_y\colon\ y \in \mathbb{R}^n\}$ is the group of translations

$$\tau_y\varrho(x) = \varrho(x-y), \quad x \in \mathbb{R}^n. \tag{6}$$

Also more specific spaces $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ are necessary in our studies. These are subspaces of $L^p_{\varrho,\infty}(\mathbb{R}^n)$, where our elliptic operator will enjoy a *dense domain*, that is it will generate a *strongly continuous analytic semigroup*. Since these are rather nonstandard spaces, their definitions and properties as well as a description of second order elliptic operators in these spaces will be discussed in detail in Part 2. In this way we fill up a gap that can be observed in the literature connected with such spaces.

There are thus two main achievements of this paper. First, we prove that second order elliptic operators (27) are negative generators of strongly continuous analytic semigroups on both weighted and locally uniform spaces (Theorems 5 and 6) under rather general assumptions on the coefficients. Our second task is to describe the asymptotic behavior of solutions to the Cauchy problem (8) in terms of globally attracting sets (Theorems 1 and 4). We show in particular that the convergence to the attractor may be viewed, besides $L^p_{\tau_y\varrho}(\mathbb{R}^n)$ (as in [16]), also in the topology of almost uniform convergence in $\mathbb{R}^n$.

In Part I the Cauchy problem in $\mathbb{R}^n$, generalizing (1), is considered. We study its global solvability and describe the asymptotics. We remark that our assumptions, made precise at the beginning of Part 1, are similar as in bounded domains and weaker than in [5], [8]. Although we start with the global smooth solutions (in the sense of [12]) we next extend them, constructing the semigroup in the whole of $L^p_\varrho(\mathbb{R}^n)$. Our main result of Part I will be the following.

Theorem 1. *Suppose that $f\colon\ \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function with growth restricted in* (10), (20) *and satisfying the dissipativeness assumption* (9). *Also, let ϱ fulfil* (2). *Then the problem* (8) *defines on $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, a C^0 semigroup of weak global solutions such that*

- *orbits of bounded sets are bounded, and*
- *there exists an absorbing set.*

Furthermore, this semigroup possesses an invariant set $\mathscr{M}$ which is bounded and closed in $L^p_{\varrho,\infty}(\mathbb{R}^n)$ and has the following properties:

- *$\mathscr{M}$ attracts bounded subsets of $\dot{W}^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$,*

- $\mathscr{M}$ *is compact in* $L^p_{\tau_y \varrho}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$,
- $\mathscr{M}$ *is invariant with respect to the group of translations in* $\mathbb{R}^n$.

Theorem 1 follows directly from the technical Theorem 4 and Proposition 1. The reader may notice that $\mathscr{M}$ is an $(L^p_{\varrho,\infty}(\mathbb{R}^n), L^p_{\tau_y\varrho}(\mathbb{R}^n))$-attractor for the mentioned semigroup in $\tilde{\mathscr{C}} \subset L^p_{\varrho,\infty}(\mathbb{R}^n)$ in the sense of [16, Definition 2.1], which is a special case of the notion of an attractor appearing in [6]. Both (8) and the Complex Ginzburg-Landau equation may be treated within a technique similar to that shown in [15]. Our considerations below, although inspired by a nice series of articles [16, 15], give however independent and precise results concerning second order problems in $\mathbb{R}^n$ for arbitrary $n \in \mathbb{N}$.

Part I

In this part we will study the Cauchy problem in $\mathbb{R}^n$, extending (1). Of course we may work in the usual Sobolev spaces studying some particular Cauchy problems for which the dissipation mechanism is strong (see [5]). However, dealing with problems like (1), we would not be able to obtain satisfactory global in time estimates of the solutions in these spaces. In particular, we would be unable to investigate the asymptotics of such solutions. This is why we need and will work in weighted spaces. To preserve the specific properties of the problem like (1) these weighted spaces should include constant stationary solutions (e.g. $u \equiv \pm 1$ for the problem (1)) as well as possible *travelling wave* solutions. In connection with the last remark recall the nice example of [12, p. 136].

Example 1. A onedimensional problem

$$\begin{cases} u_t = u_{xx} + u - bu^3, \ t > 0, \ x \in \mathbb{R}, \ b > 0, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{7}$$

has been reported in [12, §5.4, Exercise 6]. For any velocity $v \neq 0$ the problem (7) has exactly two nontrivial bounded solutions (one being the negative of the other), having the form of a *travelling wave*; $u(t,x) = \varphi(x + vt)$. They start from the unstable solution 0 at $x \to -\infty$ and go to $\pm\sqrt{b}$ when $x \to +\infty$ (when $v > 0$).

Since we would also like to use the semigroup approach to parabolic problems originated by [12], the topology of weighted space should be simultaneously strong enough to allow sufficiently nice Sobolev embeddings for the *fractional power spaces* (i.e. domains of the fractional powers of elliptic operators). Such spaces and their properties are described in detail in Part II of the paper. In the present part, without special explanation, we just borrow information from Part II.

1. Cauchy problem for a dissipative second order equation

We shall consider the Cauchy problem in $\mathbb{R}^n$, including (1):

$$\begin{cases} u_t = \Delta u + f(u), \ t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{8}$$

assuming that $f\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying the *dissipativeness condition*

$$\exists C_1, C_2 > 0\colon \ f(s)s \leqslant -C_1 s^2 + C_2, \tag{9}$$

or equivalently,

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0.$$

For the global solvability of (8) in the space[2] $X^{\alpha}_{p,\varrho,\infty}$ we shall also assume the polynomial growth condition

$$|f(s)| \leqslant c(1 + |s|^r), \ s \in \mathbb{R}, \tag{10}$$

with a certain (arbitrary, but finite) $r \in (1, \infty)$.

In this part of the paper we will consider real solutions to (8), also all function spaces considered in this part are real.

1.1. Smooth solutions to (8).

Problem (8) will be studied as an abstract parabolic equation in the Banach space $X = \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$, where ϱ satisfies (2), with a sectorial and densely defined operator $A = -\Delta\colon \ D(-\Delta) \subset \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n) \to L^p_{\varrho,\infty}(\mathbb{R}^n)$, $D(-\Delta) = \dot{W}^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$.

The existence of *smooth solutions* that belong to a class made precise in (19) will not be proved here in detail. This is because such results are simple and follow the standard scheme [8]. To be more specific:

- the existence of local $X^{\alpha}_{p,\varrho,\infty}$-solutions ($2\alpha - n/p > 0$, $\alpha \in [0,1)$, $p \in (1,\infty)$) is a consequence of the embedding $X^{\alpha}_{p,\varrho,\infty} \subset L^{\infty}(\mathbb{R}^n)$ in Lemma 6 and the local Lipschitz continuity of f,
- the proof that the problem (8) with $u_0 \in X^{\alpha}_{p,\varrho,\infty}$ has a unique global solution is based on the continuation method and uses (10) together with the "introductory" estimate below.

[2] For the definition of this and other function spaces see Part II.

$L^q_{\tau_y\varrho}(\mathbb{R}^n)$ **estimate**. Here we assume that ϱ satisfies the condition (2). Multiplying (8) by $u|u|^{q-2}\tau_y\varrho$ ($q \geqslant 2$, $\tau_y\varrho(x) = \varrho(x-y)$) we find

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x = \int_{\mathbb{R}^n}\Delta u u|u|^{q-2}\tau_y\varrho\,\mathrm{d}x + \int_{\mathbb{R}^n}f(u)u|u|^{q-2}\tau_y\varrho\,\mathrm{d}x. \tag{11}$$

Next we estimate the above terms as follows. For the first right hand side term we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Delta u u|u|^{q-2}\tau_y\varrho\,\mathrm{d}x &= -(q-1)\int_{\mathbb{R}^n}|\nabla u|^2|u|^{q-2}\tau_y\varrho\,\mathrm{d}x - \int_{\mathbb{R}^n}|u|^{q-2}u\nabla u\cdot\nabla\tau_y\varrho\,\mathrm{d}x\\ &\leqslant \int_{\mathbb{R}^n}|u|^{q-2}\left(-(q-1)|\nabla u|^2+\varrho_0|\nabla u||u|\right)\tau_y\varrho\,\mathrm{d}x =: \mathcal{H}.\end{aligned}$$

Maximizing the integrand with respect to $|\nabla u|$ (see [15]) we next find that

$$\mathcal{H} \leqslant \frac{1}{4(q-1)}\varrho_0^2\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x. \tag{12}$$

Transforming the term involving nonlinearity we get

$$\begin{aligned}\int_{\mathbb{R}^n}f(u)u|u|^{q-2}\tau_y\varrho\,\mathrm{d}x &\leqslant -C_1\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x + C_2\int_{\mathbb{R}^n}|u|^{q-2}\tau_y\varrho\,\mathrm{d}x\\ &\leqslant (-C_1+\delta)\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x + C_\delta,\end{aligned} \tag{13}$$

where (2) and (9) together with the Hölder and Young inequalities were used. Collecting these estimates we obtain

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x \leqslant \left(\frac{\varrho_0^2}{4(q-1)} - C_1 + \delta\right)\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x + C_\delta. \tag{14}$$

Choosing q large and δ small so that $C_{q,\delta} := -\left(\frac{1}{4}(q-1)^{-1}\varrho_0^2 - C_1 + \delta\right) > 0$ we thus get an estimate

$$\int_{\mathbb{R}^n}|u|^q\tau_y\varrho\,\mathrm{d}x \leqslant \left[\int_{\mathbb{R}^n}|u_0|^q\tau_y\varrho\,\mathrm{d}x\right]\mathrm{e}^{-qC_{q,\delta}t} + \frac{C_\delta}{C_{q,\delta}}\left[1-\mathrm{e}^{-qC_{q,\delta}t}\right]. \tag{15}$$

This leads successively to the bounds

$$\|u(t,u_0)\|_{L^q_{\tau_y\varrho}(\mathbb{R}^n)} \leqslant \|u_0\|_{L^q_{\tau_y\varrho}(\mathbb{R}^n)} + \left(\frac{C_\delta}{C_{q,\delta}}\right)^{1/q}, \tag{16}$$

$$\limsup_{t\to+\infty}\|u(t,u_0)\|_{L^q_{\tau_y\varrho}(\mathbb{R}^n)} \leqslant \left(\frac{C_\delta}{C_{q,\delta}}\right)^{1/q}.$$

Remark 2. Note that the constants appearing in estimate (15) are independent of the parameter y varying in $\mathbb{R}^n$. They are also uniform when u_0 varies in any bounded subset $B \subset L^p_{\varrho,\infty}(\mathbb{R}^n)$. Therefore, we can sharpen (16) to the estimates

$$\sup_{u_0\in B} \|u(t,u_0)\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \sup_{u_0\in B} \|u_0\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} + \left(\frac{C_\delta}{C_{q,\delta}}\right)^{1/q}, \tag{17}$$

$$\limsup_{t\to+\infty} \left(\sup_{u_0\in B} \|u(t,u_0)\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)}\right) \leqslant \left(\frac{C_\delta}{C_{q,\delta}}\right)^{1/q}.$$

Remark 3. If $C_1 > \frac{1}{4}\varrho_0^2$, then the estimates (16), (17) are true for any $q \geqslant 2$. Positivity of $C_{q,\delta}$ can also be achieved by decreasing the value of ϱ_0. In particular, for the weight function given in (4) we have $\varrho_0 = C\varepsilon$ and the last property is obvious provided $\varepsilon > 0$ is chosen small enough.

Theorem 2. *Let* (9) *and* (10) *hold with* $f\colon \mathbb{R} \to \mathbb{R}$ *locally Lipschitz continuous. Then problem* (8) *defines on* $X^\alpha_{p,\varrho,\infty}$ ($\alpha \in (\frac{1}{2}n/p, 1)$, $2p > n$, ϱ *as in* (2) *a* C^0 *semigroup* $\{T(t)\}$ *of global smooth solutions which is bounded dissipative and has bounded orbits of bounded sets.*

Proof. Note that as a consequence of the growth rate (10) and the estimate (17) we have

$$\|f(u(t,u_0))\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \text{const.}\bigl[1 + \|u(t,u_0)\|^r_{L^{rp}_{\varrho,\infty}(\mathbb{R}^n)}\bigr]. \tag{18}$$

Now (18), (17) with $q = rp$ together with [12, §3.3, Excercise 1] or [8, p. 70] justify global solvability of (8) in $X^\alpha_{p,\varrho,\infty}$, $\alpha \in (\frac{1}{2}n/p, 1)$, $2p > n$. Boundedness of the orbits and bounded dissipativeness in $X^\alpha_{p,\varrho,\infty}$ follows from (17) and the Cauchy integral formula (see [8, Corollary 4.1.3]). □

1.2. Weak solutions to (8).

So far we have considered smooth global solutions to (8), which were elements of the class (see [12], [8])

$$C([0,\infty), X^\alpha_{p,\varrho,\infty}) \cap C^1((0,\infty), X^{1^-}_{p,\varrho,\infty}) \cap C((0,\infty), X^1_{p,\varrho,\infty}). \tag{19}$$

Our next concern will be *weak solutions* to this problem.

Definition 1. Function $u \in C([0,\infty), L^p_{\varrho,\infty})$ is called a *weak* $L^p_{\varrho,\infty}(\mathbb{R}^n)$ *global solution* to (8) iff there is a sequence $\{u_n\}$ of smooth global solutions to (8) convergent to u in $C([0,\tau], L^p_{\varrho,\infty}(\mathbb{R}^n))$ on each compact interval $[0,\tau]$.

A notion of a *weak $L^p_\varrho(\mathbb{R}^n)$ global solution* to (8) may be introduced in a similar way.

It will be important in further studies to require the following condition on f:

$$f'(s) \leqslant C \quad \text{for a.e. } s \in \mathbb{R}. \tag{20}$$

It is evident that condition (20) is satisfied by the nonlinear term in our sample problem (1). Existence, uniqueness of the weak solutions and their continuous dependence with respect to the initial data will follow from Theorem 3 below.

Theorem 3. *Under the assumptions of Theorem 2 and (20), the problem (8) defines on each of the phase spaces*

(i) $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$, $2p > n$, *or*

(ii) $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $2p > n$,

a C^0 semigroup of weak global solutions which is bounded dissipative and has bounded orbits of bounded sets. These semigroups are extensions of $\{T(t)\}$ defined in Theorem 2 (which will not be marked in the notation).

Proof. The proof proceeds in two steps.

Step 1. We start with an $L^q_{\varrho,\infty}$-estimate of the difference $w = u(\cdot, u_0) - u(\cdot, v_0)$ of two smooth solutions to (8) having initial values u_0 and v_0, respectively. Evidently w is a smooth solution to

$$\begin{cases} w_t = \Delta w + f(u(\cdot, u_0)) - f(u(\cdot, v_0)), \ t > 0, \ x \in \mathbb{R}^n, \\ w(0, x) = u(\cdot, u_0) - u(\cdot, v_0) =: w_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{21}$$

where $[f(u(\cdot, u_0)) - f(u(\cdot, v_0))]w \leqslant Cw^2$ (thanks to (20)). We thus multiply (21) by $|w|^{q-2}w\tau_y\varrho$ ($q \geqslant 2$, $\tau_y\varrho(x) = \varrho(x-y)$) and obtain for w exactly the same estimates as those written in (11)–(12) for u. Because of (20), instead of (13) we now have

$$\int_{\mathbb{R}^n} [f(u(\cdot, u_0)) - f(u(\cdot, v_0))]w|w|^{q-2}\tau_y\varrho \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} |w|^q \tau_y\varrho \,\mathrm{d}x \tag{22}$$

so that w fulfils the relation

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |w|^q\tau_y\varrho \,\mathrm{d}x \leqslant \left(\frac{\varrho_0^2}{4(q-1)} + C\right)\int_{\mathbb{R}^n} |w|^q\tau_y\varrho \,\mathrm{d}x. \tag{23}$$

Writing $\tilde{C} = \frac{1}{4}(q-1)^{-1}\varrho_0^2 + C$ and solving the above differential inequality we get the bound

$$\sup_{t\in[0,\tau]} \|w\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \|w_0\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} \mathrm{e}^{\tilde{C}\tau}, \quad \tau > 0. \tag{24}$$

Step 2. As a consequence of (24) and the properties of continuous functions having values in Banach space, the following implication holds.

• If $\{u_{0n}\} \subset X^1_{p,\varrho\infty}$ and $u_{0n} \to u_0$ in $L^p_{\varrho,\infty}(\mathbb{R}^n)$, then there exists an element of $C([0,\infty), \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n))$, denoted further by $u(\cdot,u_0)$, such that

$$u(\cdot,u_{0n}) \to u(\cdot,u_0) \quad \text{in } \ C([0,\tau], \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)) \quad \text{for each } \tau > 0.$$

If $u(\cdot,u_0)$ and $u(\cdot,v_0)$ are two global weak solutions to (8), we may apply (24) to the difference of a pair of approximate sequences $u(\cdot,u_{0n})$ and $u(\cdot,v_{0k})$ and obtain the estimate

$$\sup_{t\in[0,\tau]} \|u(\cdot,u_0) - u(\cdot,v_0)\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \|u_0 - v_0\|_{L^q_{\varrho,\infty}(\mathbb{R}^n)} \mathrm{e}^{\tilde{C}\tau}, \quad \tau > 0.$$

This justifies that $u(\cdot,u_0)$ is not only continuous with respect to t but also with respect to $u_0 \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$, uniformly for t varying on bounded subintervals of $[0,\infty)$.

The above considerations and the density of $X^1_{p,\varrho,\infty}$ in $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ ensure the existence of a C^0 semigroup of weak global solutions to (8) on $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ $(2p > n)$, which is the extension of the semigroup $\{T(t)\}$ from Theorem 2. Since the estimates obtained in (17) hold for such an extension as well, the proof is complete in part related to the $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ spaces.

The proof concerning the phase spaces $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, is quite similar. We only remark that $L^p_{\tau_y\varrho}(\mathbb{R}^n)$ contains a dense subset $C_0^\infty(\mathbb{R}^n)$ which is also a subset of $X^1_{p,\varrho,\infty}$. □

1.3. Asymptotics.

We shall now describe the stability properties of the solutions to (8). First we will collect some properties of the semigroup $\{T(t)\}$ defined in Theorem 2.

Lemma 1. *The following conditions hold.*

(i) *There is a $\{T(t)\}$ positively invariant set $\mathscr{C}$, bounded in $X^1_{p,\varrho,\infty}$, and absorbing bounded subsets of $X^\alpha_{p,\varrho,\infty}$ under $\{T(t)\}$.*

(ii) *$\mathscr{C}$ is a bounded subset of $C_{bd}(\mathbb{R}^n)$ which is precompact in the topology of almost uniform convergence on $\mathbb{R}^n$.*

(iii) *$\mathscr{C}$ is precompact in $L^p_{\tau_y\varrho}(\mathbb{R}^n)$ for each $y \in \mathbb{R}^n$.*

(iv) *$cl_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\mathscr{C}$ does not depend on $y \in \mathbb{R}^n$ and*

$$\tilde{\mathscr{C}} := \mathrm{cl}_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\mathscr{C} \subset L^p_{\varrho,\infty}(\mathbb{R}^n), \quad y \in \mathbb{R}^n.$$

(v) *$\tilde{\mathscr{C}}$ is bounded and closed in $L^p_{\varrho,\infty}(\mathbb{R}^n)$.*

P r o o f. By Theorem 2 there exists a bounded set $\mathscr{B} \subset X^{\alpha}_{p,\varrho,\infty}$ which absorbs bounded subsets of $X^{\alpha}_{p,\varrho,\infty}$ under $\{T(t)\}$. It is now obvious that the positive orbit $\gamma^+(\mathscr{B})$, as well as any image $T(t)\gamma^+(\mathscr{B})$, $t \geqslant 0$, possess the same properties. Therefore, item (i) is a consequence of the smoothing action of the sectorial equation (see [8, Lemma 3.2.1]), which ensures that there exists $t_{\mathscr{B}} > 0$ such that $\mathscr{C} := T(t_{\mathscr{B}})\gamma^+(\mathscr{B})$ is bounded in the norm of $X^1_{p,\varrho,\infty}$.

Concerning (ii), first note that the elements of $\mathscr{C}$ and their first order partial derivatives are bounded in the $W^{1,p}_{\varrho,\infty}(\mathbb{R}^n)$ norm. This ensures, via Sobolev embeddings (see Corollary 1), that $\mathscr{C}$, as a family of functions with values in $\mathbb{R}^n$, is uniformly bounded and equicontinuous. It is then purely technical to see that $\mathscr{C}$ is precompact with respect to the topology of almost uniform convergence in $\mathbb{R}^n$.

Since any sequence chosen from $\mathscr{C}$ contains a subsequence uniformly bounded in $\mathbb{R}^n$ and almost uniformly convergent in $\mathbb{R}^n$, we obtain easily (iii) as well as the first assertion in (iv). To complete the proof of (iv) we only need to justify that $\mathrm{cl}_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\mathscr{C} \subset L^p_{\varrho,\infty}(\mathbb{R}^n)$. If $u_0 \in \mathrm{cl}_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\mathscr{C}$, then without lack of generality we may assume that u_0 is a limit of a sequence $\{u_{0m}\}$ almost uniformly convergent in $\mathbb{R}^n$ which is simultaneously bounded both in $C_{bd}(\mathbb{R}^n)$ and in $L^p_{\varrho,\infty}(\mathbb{R}^n)$. Consequently, we obtain

$$\|u_0\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} \leqslant \|u_{0m} - u_0\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} + M,\ y \in \mathbb{R}^n, \tag{25}$$

where M does not depend on $y \in \mathbb{R}^n$. Letting m tend to ∞ in (25) we see that $u_0 \in L^p_{\varrho,\infty}(\mathbb{R}^n)$.

Finally, to justify (v) note that since $\mathscr{C}$ is bounded in $X^1_{p,\varrho,\infty}$, the constant M appearing in (25) does not depend on the choice of $u_0 \in \tilde{\mathscr{C}}$. Therefore, $\tilde{\mathscr{C}}$ is bounded in $L^p_{\varrho,\infty}(\mathbb{R}^n)$. Since $\tilde{\mathscr{C}} \subset L^p_{\varrho,\infty}(\mathbb{R}^n)$ is closed in a weaker topology of $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, hence $\tilde{\mathscr{C}}$ is also closed in $L^p_{\varrho,\infty}(\mathbb{R}^n)$. Lemma 1 is thus proved. □

Existence of an attractor. Recalling Example 1, we observe that the semigroup in Theorem 3 cannot be asymptotically compact. Therefore, one cannot expect the existence of a compact global attractor (in the sense of [11]) for $\{T(t)\}$ in $X^{\alpha}_{p,\varrho,\infty}$. Indeed, considering the travelling wave φ described in Example 1 we observe that a sequence $\{\varphi(x + vt_n)\}$, $t_n \to +\infty$, cannot be convergent in $L^p_{\varrho,\infty}(\mathbb{R})$ (the Cauchy condition is violated). Hence (7) with $u_0 = \varphi$ has the empty ω-limit set in $L^p_{\varrho,\infty}(\mathbb{R})$.

Having the C^0 semigroup $\{T(t)\}$ extended onto $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, in Theorem 3 (ii) and the property in Lemma 1 (iv) we may define a C^0 semigroup $\{T(t)_{|\tilde{\mathscr{C}}}\}$, where $T(t)_{|\tilde{\mathscr{C}}}$ is the restriction of $T(t)$ to $\tilde{\mathscr{C}}$.

Theorem 4. *Let $2p > n$ and let ϱ be as in condition* (2)*. For each $y \in \mathbb{R}^n$ the semigroup $T(t)_{|\tilde{\mathscr{C}}}$ has a compact global attractor $\mathscr{M}_y$ (in the sense of* [11]*) in a metric subspace $\tilde{\mathscr{C}}$ of $L^p_{\tau_y\varrho}(\mathbb{R}^n)$. Furthermore,*

(i) *$\mathscr{M}_y$ attracts bounded subsets of $X^\alpha_{p,\varrho,\infty}$, $\alpha > \frac{1}{2}n/p$, in $L^p_{\tau_y\varrho}(\mathbb{R}^n)$,*

(ii) *$\mathscr{M}_y = \mathscr{M}$, $y \in \mathbb{R}^n$, and $\mathscr{M}$ is bounded and closed in $L^p_{\varrho,\infty}(\mathbb{R}^n)$.*

Proof. By Lemma 1 (iii), $T(t)_{|\tilde{\mathscr{C}}}$ is a semigroup on a compact set. Therefore,

$$\mathscr{M}_y = \bigcap_{s\geqslant 0} \mathrm{cl}_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} \bigcup_{t\geqslant s} T(t)\tilde{\mathscr{C}}$$

is a compact global attractor in $\tilde{\mathscr{C}} \subset L^p_{\tau_y\varrho}(\mathbb{R}^n)$.

Next, if B is bounded in $X^\alpha_{p,\varrho,\infty}$, then B is absorbed by $\mathscr{C}$ and hence B is attracted by $\mathscr{M}_y$ in the topology of $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, which proves (i).

Finally, (ii) follows easily from Lemma 1 (v) and the properties of the compact global attractor. The proof is complete. □

1.4. Invariance of solutions with respect to translations.

An interesting feature of the problem (8) is that the semigroup of global solutions corresponding to (8) commutes with the group of translations; i.e.

$$\tau_z T(t) = T(t)\tau_z, \quad z \in \mathbb{R}^n,\ t \geqslant 0. \tag{26}$$

Indeed, it is immediate from (18) that if we take a smooth solution originating at u_0 and shift its argument by $-z$ we obtain the solution originating at $\tau_z u_0$. Since weak solutions are obtained from smooth solutions by passing to a limit in $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ or $L^p_{\tau_y\varrho}(\mathbb{R}^n)$, we observe that the property (26) holds also for the extensions of $\{T(t)\}$ defined in Theorem 3.

Proposition 1. *The set $\mathscr{M}$ defined in Theorem* 4 *is invariant with respect to the group of translations $\{\tau_z\colon\ z \in \mathbb{R}^n\}$.*

Proof. From (26) we obtain that $\tau_z\mathscr{M}$ is an invariant set for $\{T(t)\}$. Recalling that $\mathscr{M}$ is bounded in the $L^p_{\varrho,\infty}(\mathbb{R}^n)$ norm we observe that $\tau_z\mathscr{M}$ is bounded in each $L^p_{\tau_y\varrho}(\mathbb{R}^n)$ norm. Since $\mathscr{C}$ is bounded in $X^1_{p,\varrho,\infty}$ and $\mathscr{C}$ absorbs bounded sets of $X^1_{p,\varrho,\infty}$, we see that $T(t) \bigcup_{y\in\mathbb{R}^n} \tau_y\mathscr{C} \subset \mathscr{C}$ for t sufficiently large and, consequently, $\tau_z\mathscr{M} = T(t)\tau_z\mathscr{M} \subset \tilde{\mathscr{C}}$. Therefore $\mathscr{M}$, as the maximal bounded invariant set for $\{T(t)_{|\tilde{\mathscr{C}}}\}$ in $\tilde{\mathscr{C}} \subset L^p_{\tau_z\varrho}(\mathbb{R}^n)$, must contain $\tau_z\mathscr{M}$. Since $z \in \mathbb{R}^n$ is arbitrary, the latter implies that $\tau_{-z}\mathscr{M} \subset \mathscr{M}$. Thus $\mathscr{M} \subset \tau_z\mathscr{M}$, which completes the proof. □

Part II

In this part of the paper we want to collect basic properties of the weight function spaces that are useful in the studies of solvability and asymptotics of parabolic Cauchy problems. Our main task will be the description of properties of elliptic second order operators in these spaces, including the description of their domains, sectoriality, Calderon-Zygmund type estimates and embeddings among the weighted and the usual Sobolev spaces. Some, mostly one dimensional, studies of such type may be found in a series of articles [16], [15]. The n-dimensional case presented here should be useful in the future studies of other Cauchy problems for semilinear parabolic equations.

2. Weighted spaces and elliptic operators

2.1. Second order elliptic operators on $\mathbb{R}^n$.

All function spaces appearing in this part will be complex spaces. We shall consider a linear differential operator

$$A := \sum_{k,l=1}^{n} a_{kl} D_k D_l + \sum_{j=1}^{n} b_j D_j + c, \quad D_j := -\mathrm{i}\frac{\partial}{\partial x_j}, \tag{27}$$

with coefficients a_{kl}, b_j and c being *bounded and uniformly continuous* complex valued functions. We abbreviate the *principal symbol* of A by

$$A_0(x,\xi) = \sum_{k,l=1}^{n} a_{kl}(x)\xi_k\xi_l, \quad x,\xi \in \mathbb{R}^n$$

and, following [3, §7], impose on A the *ellipticity condition*:

- *there exist $M > 0$ and $\theta_0 \in (0, \frac{1}{2}\pi)$ such that $|A_0^{-1}(x,\xi)| \leqslant M$ and $\sigma(A_0(x,\xi)) \subset \{z \in C\colon\ |\arg z| \leqslant \theta_0\}$ for each x, $\xi \in \mathbb{R}^n$, $|\xi| = 1$.*

As known from [3, Corollary 9.5], under the above assumptions we have

Proposition 2. *$-A$ is a generator of a strongly continuous analytic semigroup on $L^p(\mathbb{R}^n)$, $p \in (1, +\infty)$.*

We remark that the ellipticity condition above is satisfied, for example, by the negative Laplacian $-\Delta = D_1^2 + \ldots + D_n^2$, or more generally, by a general second order operator (27) having real valued top order coefficients a_{kl}, $k,l = 1,\ldots,n$, and such that, for certain $c > 0$,

$$A_0(x,\xi) \geqslant c > 0, \quad x,\xi \in \mathbb{R}^n,\ |\xi| = 1.$$

It is also true that in [3] less regularity for b_j and c is required. This is a matter we will not pursue here. Our concern will be to use Proposition 2 to obtain a similar result for A acting in weighted spaces $W^{s,p}_{\varrho}(\mathbb{R}^n)$ instead of the usual Sobolev-Slobodeckii spaces $W^{s,p}(\mathbb{R}^n)$ [19].

We define the *weighted norms* as

$$\|\varphi\|_{W^{k,p}_{\varrho}(\mathbb{R}^n)} = \sum_{|\alpha|\leqslant k} \left(\int_{\mathbb{R}^n} |D^{\alpha}\varphi(x)|^p \varrho(x)\,\mathrm{d}x \right)^{1/p}, \quad k \in \mathbb{N},\ p \in (1,\infty).$$

Usually, it is convenient to choose ϱ in the form

$$\varrho(x) = (1+|\varepsilon x|^2)^{\gamma}, \quad x \in \mathbb{R}^n,\ \varepsilon > 0$$

(see [5]); this function for $\gamma < -n/2$ satisfies requirements (2).

2.2. Analytic semigroups on weighted spaces.

The operator A will be studied here both in $L^p_{\varrho}(\mathbb{R}^n)$ and in the spaces $L^p_{\varrho,\infty}(\mathbb{R}^n)$, $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ defined below. In applications the semigroup will be constructed in the stronger topology of $L^p_{\varrho,\infty}(\mathbb{R}^n)$ while the global attractor will be compact in a weaker sense.

The space $L^p_{\varrho}(\mathbb{R}^n)$. First we will prove the following generalization of Proposition 2.

Theorem 5. *For any $p \in (1,+\infty)$ and ϱ with the properties* (2)*, the operator $-A$ given in* (27) *with the domain $W^{2,p}_{\varrho}(\mathbb{R}^n)$ generates a strongly continuous analytic semigroup in $L^p_{\varrho}(\mathbb{R}^n)$.*

P r o o f. The idea will be to "transfer" our considerations from the weighted spaces to the usual $W^{2,p}(\mathbb{R}^n)$-$L^p(\mathbb{R}^n)$ spaces, where Proposition 2 holds. After that, we shall come back to the weighted spaces saving the estimates for the resolvent obtained in the $W^{2,p}(\mathbb{R}^n)$-$L^p(\mathbb{R}^n)$ setting.

Since $C_0^{\infty}(\mathbb{R}^n) \subset W^{2,p}_{\varrho}(\mathbb{R}^n) \subset L^p_{\varrho}(\mathbb{R}^n)$, the domain of A is dense in $L^p_{\varrho}(\mathbb{R}^n)$. Consider the resolvent equation

$$\lambda v - Av = f \in L^p_{\varrho}(\mathbb{R}^n) \tag{28}$$

and define $\tilde{\varrho} := \varrho^{1/p}$. It is easy to see that $\tilde{\varrho}$ is still a C^2 function possessing the properties listed in (2). The same remains also true for $\tilde{\tilde{\varrho}} = \varrho^{-1/p}$.

Simple calculations show the validity of the following equivalence:

• *$v \in W^{2,p}_\varrho(\mathbb{R}^n)$ is a solution to (28) if and only if $w = \tilde\varrho v \in W^{2,p}(\mathbb{R}^n)$ is a solution to*

$$\lambda w - \tilde A_\varrho w = \tilde f \in L^p(\mathbb{R}^n) \tag{29}$$

where $\tilde f = \tilde\varrho f$ and

$$\tilde A_\varrho w = \sum_{k,l=1}^n \tilde a_{kl} D_k D_l w + \sum_{j=1}^n \tilde b_j D_j w + \tilde c w \tag{30}$$

with the coefficients

$$\begin{aligned} &\tilde a_{kl} = a_{kl}, \quad k,l = 1,\dots,n, \\ &\tilde b_j = b_j - \frac{1}{p}\Bigg[\sum_{k=1}^n a_{kj}\frac{D_k\varrho}{\varrho} + \sum_{l=1}^n a_{jl}\frac{D_l\varrho}{\varrho}\Bigg], \quad j = 1,\dots,n, \\ &\tilde c = c + \frac{1}{p}\sum_{k,l=1}^n a_{kl}\left[\Big(1+\frac{1}{p}\Big)\frac{D_k\varrho D_l\varrho}{\varrho^2} - \frac{D_k D_l\varrho}{\varrho}\right] - \frac{1}{p}\sum_{j=1}^n b_j\frac{D_j\varrho}{\varrho}. \end{aligned} \tag{31}$$

By our assumptions, Proposition 2 is applicable to the operator $\tilde A_\varrho$. This implies that, for certain $a \in \mathbb{R}$, $\tilde K > 0$ and $\theta \in (0, \frac{1}{2}\pi)$,

$$\|w\|_{L^p(\mathbb{R}^n)} \leqslant \frac{\tilde K}{|\lambda - a|}\|\tilde f\|_{L^p(\mathbb{R}^n)} \tag{32}$$

for each λ from the sector

$$\mathscr{S}_{a,\theta} := \{z \in C\colon\ \theta \leqslant |\arg(z-a)| \leqslant \pi, z \neq a\}$$

($\mathscr{S}_{a,\theta}$ being contained in the resolvent set of $\tilde A$). What was said above ensures that for any $\lambda \in \mathscr{S}_{a,\theta}$ and any $f \in L^p_\varrho(\mathbb{R}^n)$ equation (28) has a unique solution $v = \tilde\varrho^{-1} w = \tilde{\tilde\varrho} w \in W^{2,p}_\varrho(\mathbb{R}^n)$ and, thanks to (32),

$$\|v\|_{L^p_\varrho(\mathbb{R}^n)} = \|\tilde\varrho v\|_{L^p(\mathbb{R}^n)} \leqslant \frac{\tilde K}{|\lambda - a|}\|\tilde f\|_{L^p(\mathbb{R}^n)} = \frac{\tilde K}{|\lambda - a|}\|f\|_{L^p_\varrho(\mathbb{R}^n)}. \tag{33}$$

The main condition in the definition of a sectorial operator [12] is thus verified.

We remark finally that A is closed in $L^p_\varrho(\mathbb{R}^n)$. Indeed, for μ as in [3, Theorem 9.4] $(\mu I - A)^{-1}$ is bounded and defined on the whole of $L^p_\varrho(\mathbb{R}^n)$. Hence, $(\mu I - A)^{-1}$ is closed and so is A. The proof of Theorem 5 is complete. □

Our next concern will be the Banach space

$$L^p_{\varrho,\infty}(\mathbb{R}^n) = \Big\{\varphi \in \bigcap_{y\in\mathbb{R}^n} L^p_{\tau_y\varrho}(\mathbb{R}^n)\colon\ \|\varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} \|\varphi\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} < \infty\Big\},$$

$p \in [1,\infty)$, where $\{\tau_y\colon\ y \in \mathbb{R}^n\}$ is the group of translations defined in (6).

Remark 4. Note that the completeness of $L^p_{\varrho,\infty}(\mathbb{R}^n)$ is a consequence of the completeness of the space

$$B(\mathbb{R}^n, L^p(\mathbb{R}^n)) = \Big\{\varphi\colon\ \mathbb{R}^n \to L^p(\mathbb{R}^n),\ \sup_{y\in\mathbb{R}^n} \|\varphi(y)\|_{L^p(\mathbb{R}^n)} < \infty\Big\},$$

see [2, p. 40]. We also remark that, for ϱ integrable and bounded, $L^p_{\varrho,\infty}(\mathbb{R}^n)$ will contain $L^p(\mathbb{R}^n)$ as well as $L^\infty(\mathbb{R}^n)$.

In a similar way, for $k = 1, 2, \ldots$ and $p \in [1, +\infty)$, we define Banach spaces

$$W^{k,p}_{\varrho,\infty}(\mathbb{R}^n) = \Big\{\varphi \in \bigcap_{y\in\mathbb{R}^n} W^{k,p}_{\tau_y\varrho}(\mathbb{R}^n)\colon\ \|\varphi\|_{W^{k,p}_{\varrho,\infty}(\mathbb{R}^n)} = \sum_{|\alpha|\leqslant k} \sup_{y\in\mathbb{R}^n} \|D^\alpha\varphi\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} < \infty\Big\}.$$

In Theorem 6 it will be shown that $-A$ generates *an analytic semigroup* on $L^p_{\varrho,\infty}(\mathbb{R}^n)$ in the sense of [13]; that is, the domain of A need not be dense in the $L^p_{\varrho,\infty}(\mathbb{R}^n)$. Furthermore, the spaces $W^{k,p}_{\varrho,\infty}(\mathbb{R}^n)$, as well as the fractional powers of A above $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ (see Definition 2), will have "nice" embedding properties (65).

The space $L^p_{\varrho,\infty}(\mathbb{R}^n)$. As before, we will essentially use the "calculus for elliptic operators" developed in [3, Section 9].

Theorem 6. *Let $p \in (1, +\infty)$ and let ϱ satisfy* (2)*. Then $-A$ given in* (27) *with the domain $W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$ generates an analytic semigroup in $L^p_{\varrho,\infty}(\mathbb{R}^n)$.*

Proof. We proceed as in the proof of Theorem 5 replacing ϱ with $\tau_y\varrho$ ($y \in \mathbb{R}^n$) and letting $f \in L^p_{\varrho,\infty}(\mathbb{R}^n)$ on the right hand side of the equation (28). We next conclude that for any $\lambda \in \mathscr{S}_{a,\theta}$ and any $y \in \mathbb{R}^n$ there is a unique solution $w_y \in W^{2,p}(\mathbb{R}^n)$ to

$$\lambda w_y - \tilde{A}_{\tau_y\varrho} w_y = f(\tau_y\varrho)^{1/p} \in L^p(\mathbb{R}^n), \tag{34}$$

and that $v = (\tau_y\varrho)^{-1/p} w_y$ is independent of $y \in \mathbb{R}^n$. Indeed, if $(\tau_y\varrho)^{-1/p} w_y$ depended on y, then (35) below would have two different solutions $v_{y_1} \in W^{2,p}_{\tau_{y_1}\varrho}(\mathbb{R}^n)$, $v_{y_2} \in W^{2,p}_{\tau_{y_2}\varrho}(\mathbb{R}^n)$. But, since $\varrho/(\tau_{y_1}\varrho)$ and $\varrho/(\tau_{y_2}\varrho)$ are bounded functions of $x \in \mathbb{R}^n$ (see (3)), we would have $w_{y_1} = (\varrho)^{1/p} v_{y_1}$, $w_{y_2} = \varrho^{1/p} v_{y_2} \in W^{2,p}(\mathbb{R}^n)$ and they would be two different solutions to $\lambda w - \tilde{A}_\varrho w = f\varrho^{1/p}$, which is not possible. Therefore, $v \in \bigcap\limits_{y\in\mathbb{R}^n} W^{2,p}_{\tau_y\varrho}(\mathbb{R}^n)$ solves uniquely the equation

$$\lambda v - Av = f \in L^p_{\varrho,\infty}(\mathbb{R}^n). \tag{35}$$

Next, it is crucial to observe that, under our assumptions on A and ϱ, the constant $\tilde{K}$ in the estimates of Theorem 5 will remain independent of $y \in \mathbb{R}^n$. Such a property is expressed explicitly in [3, Theorem 9.4] (as well as in the remarks just below [3, Corollary 9.5]). For this we only need to have the coefficients of $\tilde{A}_{\tau_y\varrho}$ bounded uniformly for $y \in \mathbb{R}^n$, which may be easily seen from (31).

We are thus allowed to rewrite (33) in the form

$$\|v\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} = \|(\tau_y\varrho)^{1/p}v\|_{L^p(\mathbb{R}^n)} = \|w_y\|_{L^p(\mathbb{R}^n)} \leqslant \frac{\tilde{K}}{|\lambda-a|}\|f\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} \tag{36}$$

and to take lower upper bound on both sides of (36) to get

$$\|v\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \frac{\tilde{K}}{|\lambda-a|}\|f\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}. \tag{37}$$

As previously one may easily notice that A is closed in $L^p_{\varrho,\infty}(\mathbb{R}^n)$. The proof is complete. $\square$

Let us continue for a while the considerations of the above theorem.

Lemma 2. *Under the assumptions of Theorem 6 the following versions of the Calderon-Zygmund estimate hold:*

$$\begin{aligned} &\|(\lambda-A)^{-1}\|_{\mathscr{L}(L^p_{\tau_y\varrho}(\mathbb{R}^n),W^{2,p}_{\tau_y\varrho}(\mathbb{R}^n))} \leqslant \text{const.}(\lambda,\varrho), \quad \lambda\in S_{a,\theta},\ y\in\mathbb{R}^n,\\ &\|(\lambda-A)^{-1}\|_{\mathscr{L}(L^p_{\varrho,\infty}(\mathbb{R}^n),W^{2,p}_{\varrho,\infty}(\mathbb{R}^n))} \leqslant \text{const.}(\lambda,\varrho), \quad \lambda\in S_{a,\theta},\end{aligned} \tag{38}$$

and

$$\begin{aligned} &\|\lambda-A\|_{\mathscr{L}(W^{2,p}_{\tau_y\varrho}(\mathbb{R}^n),L^p_{\tau_y\varrho}(\mathbb{R}^n))} \leqslant \text{const.}(\lambda,\varrho), \quad \lambda\in S_{a,\theta},\ y\in\mathbb{R}^n,\\ &\|\lambda-A\|_{\mathscr{L}(W^{2,p}_{\varrho,\infty}(\mathbb{R}^n),L^p_{\varrho,\infty}(\mathbb{R}^n))} \leqslant \text{const.}(\lambda,\varrho), \quad \lambda\in S_{a,\theta}.\end{aligned} \tag{39}$$

Proof. Indeed, for $\mu > 0$ as introduced in [3, Theorem 9.4] we have

$$\begin{aligned}\|w_y\|_{W^{2,p}(\mathbb{R}^n)} &= C\|(\lambda-\tilde{A}_{\tau_y\varrho})^{-1}f(\tau_y\varrho)^{1/p}\|_{W^{2,p}(\mathbb{R}^n)}\\ &\leqslant C\|(\mu+\tilde{A}_{\tau_y\varrho})(\lambda-\tilde{A}_{\tau_y\varrho})^{-1}f(\tau_y\varrho)^{1/p}\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C|\mu+\lambda|\|(\lambda-\tilde{A}_{\tau_y\varrho})^{-1}f(\tau_y\varrho)^{1/p}\|_{L^p(\mathbb{R}^n)} + C\|f(\tau_y\varrho)^{1/p}\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C|\mu+\lambda|\|w_y\|_{L^p(\mathbb{R}^n)} + C\|f\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\end{aligned}$$

which, for $v = (\tau_y\varrho)_y^{-1p/w}$, gives the estimate

$$\|v(\tau_y\varrho)^{1/p}\|_{W^{2,p}(\mathbb{R}^n)} \leqslant C|\mu+\lambda|\|v\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} + C\|f\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}. \tag{40}$$

Obvious calculations show that

$$\|v\|_{W^{2,p}_{\tau_y\varrho}(\mathbb{R}^n)} \leqslant \text{const.}\|v(\tau_y\varrho)^{1/p}\|_{W^{2,p}(\mathbb{R}^n)}. \tag{41}$$

Joining (40) and (41) we get the inequality

$$\|v\|_{W^{2,p}_{\tau_y\varrho}(\mathbb{R}^n)} \leqslant \text{const.}(\lambda,\varrho)\left(\|v\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)} + \|f\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}\right), \quad y\in\mathbb{R}^n, \tag{42}$$

which extends to

$$\|v\|_{W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \text{const.}(\lambda,\varrho)\left(\|v\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} + \|f\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}\right). \tag{43}$$

Therefore (38) follows from (42), (36) and (43), (37) respectively.

The bounds in (39) are immediate consequences of the boundedness of the coefficients a_{ij}, b_j and c assumed in Subsection 2.1. The proof is complete. $\square$

Remark 5. In [15] one may find a direct proof that $-\Delta$ is sectorial in the weighted spaces. The above proof is different. It adapts the results known for a class of second order operators in the usual Sobolev spaces to the case of weighted spaces. A similar remark concerns the proof of the embeddings of weighted spaces and fractional power spaces in Subsection 2.3.

Density of the domain of A in weighted spaces. It is well known that the semigroup in a Banach space will be strongly continuous only if the domain of its infinitesimal generator is dense. While this property is not necessary to discuss the solutions to semilinear abstract parabolic equations (see [13]), we would like to keep it in our considerations. That is why our next concern will be the base space $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ (see definition below), in which an analytic semigroup generated by A will be *strongly continuous*.

Definition 2. $\dot{W}^{k,p}_{\varrho,\infty}(\mathbb{R}^n)$, $k\in\mathbb{N}$, $p\in[1,+\infty)$, denotes a Banach subspace of $W^{k,p}_{\varrho,\infty}(\mathbb{R}^n)$ consisting of all elements $\varphi\in W^{k,p}_{\varrho,\infty}(\mathbb{R}^n)$ having the following translation continuity property:

$$\tau_z\varphi\to\varphi \quad \text{in } W^{k,p}_{\varrho,\infty}(\mathbb{R}^n) \quad \text{as } |z|\to 0. \tag{44}$$

Remark 6. As shown in [16, Lemma 3.1 (d)], $W^{2,2}_{\varrho,\infty}(\mathbb{R}^1)$ is not densely embedded in $L^2_{\varrho,\infty}(\mathbb{R}^1)$, although this is the case if we consider the inclusion

$$\dot{W}^{2,2}_{\varrho,\infty}(\mathbb{R}^1)\subset\dot{L}^2_{\varrho,\infty}(\mathbb{R}^1)$$

(see [16, Lemma 3.1 (c)]). We shall prove below such a property in the general case, showing in particular that $C^\infty_{bd}(\mathbb{R}^n)$ is densely embedded in $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$.

Lemma 3. *For each $p \in [1,\infty)$ the set $\bigcap\limits_{k\in\mathbb{N}} \dot{W}^{k,p}_{\varrho,\infty}(\mathbb{R}^n)$ is dense in $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$.*

Proof. The proof based on approximation by *mollifiers* (see [1]) is different then in the onedimensional case [16, Lemma 3.1]. Since the elements of $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$ are locally integrable and $J_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, the convolution

$$J_\varepsilon * \varphi(x) = \int_{\mathbb{R}^n} J_\varepsilon(z)\varphi(x-z)\,\mathrm{d}z = \int_{\{|x-z|<\varepsilon\}} J_\varepsilon(x-z)\varphi(z)\,\mathrm{d}z$$

is well defined for $\varphi \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$. We then have

$$\begin{aligned} |J_\varepsilon * \varphi(x)\tau_y\varrho^{1/p}(x)| &= \left|\int_{\mathbb{R}^n} J_\varepsilon^{(p-1)/p}(z)J_\varepsilon^{1/p}(z)\varphi(x-z)\tau_y\varrho^{1/p}(x)\,\mathrm{d}z\right| \\ &\leqslant \left(\int_{\mathbb{R}^n} J_\varepsilon(z)|\varphi(x-z)|^p\tau_y\varrho(x)\,\mathrm{d}z\right)^{1/p}, \end{aligned} \tag{45}$$

which implies that

$$\begin{aligned} \int_{\mathbb{R}^n} |J_\varepsilon * \varphi(x)\tau_y\varrho^{1/p}(x)|^p\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} J_\varepsilon(z)|\varphi(x-z)|^p\tau_y\varrho(x)\,\mathrm{d}z\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} J_\varepsilon(z)\left(\int_{\mathbb{R}^n}|\varphi(x-z)|^p\tau_y\varrho(x)\,\mathrm{d}x\right)\mathrm{d}z \leqslant \|\varphi\|^p_{L^p_{\varrho,\infty}(\mathbb{R}^n)}. \end{aligned} \tag{46}$$

From (46) we thus get

$$\|J_\varepsilon * \varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \|\varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \tag{47}$$

and consequently

$$\|J_\varepsilon * \varphi - \tau_z(J_\varepsilon * \varphi)\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} = \|J_\varepsilon * (\varphi - \tau_z\varphi)\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \|\varphi - \tau_z\varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}.$$

The above condition ensures that

$$J_\varepsilon * \varphi \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n) \quad \text{whenever } \varphi \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n). \tag{48}$$

Next, using the *generalized Minkowski inequality* (see [17, Chapter 18]), we obtain

$$\begin{aligned} &\|J_\varepsilon * \varphi - \varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)} \\ &\leqslant \sup_{y\in\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} J_\varepsilon(z)|\varphi(x-z)-\varphi(x)|\tau_y\varrho^{1/p}(x)\,\mathrm{d}z\right)^p\mathrm{d}x\right]^{1/p} \\ &\leqslant \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n} J_\varepsilon(z)\left[\int_{\mathbb{R}^n}|\varphi(x-z)-\varphi(x)|^p\tau_y\varrho(x)\,\mathrm{d}x\right]^{1/p}\mathrm{d}z \\ &\leqslant \sup_{|z|<\varepsilon}\|\tau_z\varphi - \varphi\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}. \end{aligned} \tag{49}$$

Since $\varphi \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$, the right hand side of (49) becomes arbitrarily small as $\varepsilon \to 0^+$ (see (44)). Therefore,

$$J_\varepsilon * \varphi \to \varphi \quad \text{in } L^p_{\varrho,\infty}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0^+. \tag{50}$$

Note that the estimate (49) indicates why $W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$ need not be dense in $L^p_{\varrho,\infty}(\mathbb{R}^n)$ and why we need to work in translation continuous subspaces.

To consider partial derivatives of $J_\varepsilon * \varphi$ let us recall that $D^\sigma J_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\sigma \in \mathbb{N}^n$, and

$$D^\sigma[J_\varepsilon * \varphi](x) = [D^\sigma J_\varepsilon] * \varphi(x), \quad x \in \mathbb{R}^n. \tag{51}$$

As a consequence, after calculations similar to those in (45)–(48) but with J_ε replaced by $D^\sigma J_\varepsilon$, one may easily verify that (48) can be strengthened to the condition

$$D^\sigma[J_\varepsilon * \varphi] \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n) \quad \text{for arbitrary } \sigma \in \mathbb{N}^n \quad \text{whenever } \varphi \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n). \tag{52}$$

This justifies that

$$\text{a map } \varphi \longrightarrow J_\varepsilon * \varphi \text{ takes } \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n) \text{ into } \bigcap_{k \in \mathbb{N}} \dot{W}^{k,p}_{\varrho,\infty}(\mathbb{R}^n). \tag{53}$$

Since also (50) holds, the proof is complete. □

Remark 7. A note should be made that if $\varphi \in \dot{W}^{l,p}_{\varrho,\infty}(\mathbb{R}^n)$, then

$$D^\sigma[J_\varepsilon * \varphi] = J_\varepsilon * D^\sigma \varphi, \quad |\sigma| \leqslant l \tag{54}$$

(see [1, p. 52]). We may thus repeat calculations of (49), but with $D^\sigma \varphi$ instead of φ, and obtain the following stronger version of (50):

$$D^\sigma[J_\varepsilon * \varphi] \to D^\sigma \varphi \quad \text{in } L^p_{\varrho,\infty}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0^+ \quad \text{for each } |\sigma| \leqslant l. \tag{55}$$

Conditions (55) and (53) ensure that

$$\bigcap_{k \in \mathbb{N}} \dot{W}^{k,p}_{\varrho,\infty}(\mathbb{R}^n) \text{ is dense in } \dot{W}^{l,p}_{\varrho,\infty}(\mathbb{R}^n) \text{ for each } l \in \mathbb{N}. \tag{56}$$

Denote by $C^k_{bd}(\mathbb{R}^n)$ the space of complex valued functions having bounded and continuous all partial derivatives up to the order k. While the elements of $C_{bd}(\mathbb{R}^n)$

are not translation continuous (counter-example; $\sin x^2$), the elements of $C^1_{bd}(\mathbb{R}^n)$ have yet such a property since

$$\left(\int_{\mathbb{R}^n} |\varphi(x+z)-\varphi(x)|^p \tau_y \varrho \,\mathrm{d}x\right)^{1/p} \leqslant \sup_{x\in\mathbb{R}^n} |\nabla\varphi||z| \left(\int_{\mathbb{R}^n} \tau_y\varrho\,\mathrm{d}x\right)^{1/p}.$$

Due to the Sobolev embedding $W^{1,p}_{\varrho,\infty}(\mathbb{R}^n) \subset C_{bd}(\mathbb{R}^n)$ (see Corollary 1 below), we thus have the inclusions

$$\dot W^{2,p}_{\varrho,\infty}(\mathbb{R}^n) \subset C^1_{bd}(\mathbb{R}^n) \subset \dot L^p_{\varrho,\infty}(\mathbb{R}^n),$$

which together with (53), (55) justifies the following result.

Lemma 4. *For any $l \in \mathbb{N}$ and $p \in [1,\infty)$, $C^\infty_{bd}(\mathbb{R}^n)$ is dense in $\dot W^{l,p}_{\varrho,\infty}(\mathbb{R}^n)$.*

The base space $\dot L^p_{\varrho,\infty}(\mathbb{R}^n)$. Recalling Lemma 2 and assuming that the coefficients of A are bounded and uniformly continuous, we observe that the operator $\lambda - A$ ($\lambda \in S_{a,\theta}$) takes $\dot W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$ into $\dot L^p_{\varrho,\infty}(\mathbb{R}^n)$ and is a one-to-one closed map between these spaces. From Lemma 3 we also know that $A\colon \dot W^{2,p}_{\varrho,\infty}(\mathbb{R}^n) \subset \dot L^p_{\varrho,\infty}(\mathbb{R}^n) \to \dot L^p_{\varrho,\infty}(\mathbb{R}^n)$ is densely defined. Although it is true that (see (37))

$$\|(\lambda - A)^{-1} f\|_{\dot L^p_{\varrho,\infty}(\mathbb{R}^n)} \leqslant \frac{\tilde K}{|\lambda - a|} \|f\|_{\dot L^p_{\varrho,\infty}(\mathbb{R}^n)}, \tag{57}$$

we cannot yet infer that λ is in the resolvent set of the operator $A\colon \dot W^{2,p}_{\varrho,\infty}(\mathbb{R}^n) \subset \dot L^p_{\varrho,\infty}(\mathbb{R}^n) \to \dot L^p_{\varrho,\infty}(\mathbb{R}^n)$ unless we verify that

$$\text{the image } (\lambda - A)(\dot W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)) \text{ is equal to } \dot L^p_{\varrho,\infty}(\mathbb{R}^n). \tag{58}$$

Note that (58) follows easily from Lemma 2 when A has constant coefficients. Below we will prove that this is also true when the coefficients are bounded and uniformly continuous. Thanks to (38) we know that $(\lambda - A)^{-1}\colon L^p_{\varrho,\infty}(\mathbb{R}^n) \to W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$. Take $f \in \dot L^p_{\varrho,\infty}(\mathbb{R}^n)$, then $(\lambda - A)^{-1} f = v \in W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$ and observe that

$$\begin{aligned}(\lambda - A)[\tau_z v - v] = [\tau_z f - f] &- \sum_{k,l=1}^n [a_{kl} - \tau_z a_{kl}] D_k D_l \tau_z v \\ &- \sum_{j=1}^n [b_j - \tau_z b_j] D_j \tau_z v - [c - \tau_z c]\tau_z v.\end{aligned} \tag{59}$$

Equality (59) together with (38) imply the estimate

$$\begin{aligned}
(60)\qquad \frac{1}{\text{const.}(\lambda,\varrho)}\|\tau_z v - v\|_{W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)} &\leqslant \|(\lambda - A)[\tau_z v - v]\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}\\
&\leqslant \|\tau_z f - f\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}\\
&\quad + \sum_{k,l=1}^{n} \sup_{x\in\mathbb{R}^n} |a_{kl}(x) - \tau_z a_{kl}(x)| \|D_k D_l \tau_z v\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}\\
&\quad + \sum_{j=1}^{n} \sup_{x\in\mathbb{R}^n} |b_j(x) - \tau_z b_j(x)| \|D_j \tau_z v\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}\\
&\quad + \sup_{x\in\mathbb{R}^n} |c(x) - \tau_z c(x)| \|\tau_z v\|_{L^p_{\varrho,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Since $f \in \dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$, $v \in W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$ and the coefficients of A are uniformly continuous, the right hand side of (60) tends to zero as $z \to 0$. Consequently, $v \in \dot{W}^{2,p}_{\varrho,\infty}(\mathbb{R}^n)$, which ensures (58).

The above considerations may be then summarized as follows.

Theorem 7. *Let the coefficients of A be bounded and uniformly continuous, let ϱ satisfy* (2), *and let $p \in (1,+\infty)$. Then the operator $-A$ generates a strongly continuous analytic semigroup in $\dot{L}^p_{\varrho,\infty}(\mathbb{R}^n)$.*

2.3. Embeddings of the domains of fractional powers.

In this subsection we prove the embeddings of weighted spaces in $C_{bd}(\mathbb{R}^n)$, which are useful in applications. We start with the following auxiliary inequality.

Lemma 5. *Suppose that ϱ satisfies* (2). *Then*

$$(61)\qquad \|\varphi\|_{C_{bd}(\mathbb{R}^n)} \leqslant c\|\varphi\|^{\theta}_{W^{2,p}_{\varrho,\infty}(\mathbb{R}^n)} \|\varphi\|^{(1-\theta)}_{L^p_{\varrho,\infty}(\mathbb{R}^n)}, \quad p > \tfrac{1}{2}n,\ \theta \in (\tfrac{1}{2}n/p, 1].$$

P r o o f. For the prescribed range of parameters, by the usual Sobolev embeddings and the interpolation inequality, we get

$$\begin{aligned}
(62)\qquad [\varrho(0)]^{1/p}|\varphi(y)| &\leqslant \|\varphi(\tau_y\varrho)^{1/p}\|_{C_{bd}(\mathbb{R}^n)} \leqslant c\|\varphi(\tau_y\varrho)^{1/p}\|_{H^s_p(\mathbb{R}^n)}\\
&\leqslant c\|\varphi(\tau_y\varrho)^{1/p}\|^{s/2}_{W^{2,p}(\mathbb{R}^n)} \|\varphi(\tau_y\varrho)^{1/p}\|^{1-s/2}_{L^p(\mathbb{R}^n)}\\
&\leqslant c\|\varphi(\tau_y\varrho)^{1/p}\|^{s/2}_{W^{2,p}(\mathbb{R}^n)} \|\varphi\|^{1-s/2}_{L^p_{\tau_y\varrho}(\mathbb{R}^n)}, \quad s \in (n/p, 2]
\end{aligned}$$

(see [19, §2.4.2 (11), §2.8.1 (16)]). Since conditions in (2) are valid for $\tau_y\varrho$, $y \in \mathbb{R}^n$ and the bounds in (2) are independent of $y \in \mathbb{R}^n$, we then have

$$(63)\qquad \|\varphi(\tau_y\varrho)^{1/p}\|_{W^{2,p}(\mathbb{R}^n)} \leqslant \text{const.} \sum_{|\alpha|\leqslant 2} \|D^\alpha\varphi\|_{L^p_{\tau_y\varrho}(\mathbb{R}^n)},$$

where const. depends on ϱ_0 and C (as in (2)) but not on $y \in \mathbb{R}^n$. Substituting (63) into the right hand side of (62) and taking the lower upper bound of both sides we obtain (61). The proof is complete. $\square$

Based on the well known Henry's result reported in [12, §1.4, Exercise 11], Lemma 5 can be next easily extended to the following embedding theorem.

Lemma 6. *Let* $X^\alpha_{p,\varrho,\infty} = D\left((A + \omega I)^\alpha\right)$, $\alpha \geqslant 0$, *where* A *is the sectorial operator from Theorem* 7 *and* $\omega > 0$ *is sufficiently large. Then*

$$X^\alpha_{p,\varrho,\infty} \subset C_{bd}(\mathbb{R}^n) \quad \text{whenever } 2p > n,\ \alpha > \tfrac{1}{2}n/p. \tag{64}$$

As a particular conclusion, from the proof of Lemma 5, we also get

Corollary 1. *If* ϱ *satisfies* (2), *then*

$$\begin{aligned} W^{1,p}_{\varrho,\infty}(\mathbb{R}^n) &\subset C_{bd}(\mathbb{R}^n), \quad p > n, \\ X^\alpha_{p,\varrho,\infty} &\subset \dot{W}^{1,p}_{\varrho,\infty}(\mathbb{R}^n), \quad p > \tfrac{1}{2}n,\ \alpha > \tfrac{1}{2}. \end{aligned} \tag{65}$$

Remark 8. Note that the second embedding in (65) requires the assumptions on A as in Theorem 7, whereas the first expresses the property of $W^{1,p}_{\varrho,\infty}(\mathbb{R}^n)$ itself.

Acknowledgements. We are grateful to Professor A. Rodriguez-Bernal for pointing our attention to difficulties appearing in the studies of problem (1). We are also grateful to the referee for his remarks improving original version of the paper.

References

[1] *R. A. Adams*: Sobolev Spaces. Academic Press, New York, 1975.
[2] *H. Amann*: Linear and Quasilinear Parabolic Problems. Birkhäuser, Basel, 1995.
[3] *H. Amann, M. Hieber and G. Simonett*: Bounded H_∞-calculus for elliptic operators. Diff. Int. Equations *3* (1994), 613–653.
[4] *J. M. Arrieta, J. W. Cholewa, T. Dlotko and A. Rodriguez-Bernal*: Asymptotic behavior and attractors for reaction diffusion equations in unbounded domains. Nonlinear Anal. *TMA56* (2004), 515–554.
[5] *A. V. Babin and M. I. Vishik*: Attractors of partial differential evolution equations in unbounded domain. Proc. Roy. Soc. Edinburgh *116* (1990), 221–243.
[6] *A. V. Babin and M. I. Vishik*: Attractors of Evolution Equations. North-Holland, Amsterdam, 1992.
[7] *J. W. Cholewa and T. Dlotko*: Cauchy problem with subcritical nonlinearity. J. Math. Anal. Appl. *210* (1997), 531–548.
[8] *J. W. Cholewa and T. Dlotko*: Global Attractors in Abstract Parabolic Problems. Cambridge University Press, Cambridge, 2000.

[9] *M. A. Efendiev and S. V. Zelik*: The attractor for a nonlinear reaction-diffusion system in an unbounded domain. Comm. Pure Appl. Math. *54* (2001), 625–688.
[10] *E. Feireisl*: Bounded, locally compact global attractors for semilinear damped wave equations on R^N. Differential Integral Equations *9* (1996), 1147–1156.
[11] *J. K. Hale*: Asymptotic Behavior of Dissipative Systems. AMS, Providence, RI, 1988.
[12] *D. Henry*: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics 840. Springer, Berlin, 1981.
[13] *A. Lunardi*: Analytic Semigroup and Optimal Regularity in Parabolic Problems. Birkhäuser, Berlin, 1995.
[14] *S. Merino*: On the existence of the compact global attractor for semilinear reaction diffusion systems on R^N. J. Differential Equations *132* (1996), 87–106.
[15] *A. Mielke*: The complex Ginzburg-Landau equation on large and unbounded domains: sharper bounds and attractors. Nonlinearity *10* (1997), 199–222.
[16] *A. Mielke and G. Schneider*: Attractors for modulation equations on unbounded domains–existence and comparison. Nonlinearity *8* (1995), 743–768.
[17] *S. M. Nikolsky*: A Course of Mathematical Analysis. Mir Publishers, Moscow, 1987.
[18] *R. Temam*: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York, 1988.
[19] *H. Triebel*: Interpolation Theory, Function Spaces, Differential Operators. VEB Deutscher, Berlin, 1978.

Authors' address: Institute of Mathematics, Silesian University, 40-007 Katowice, Poland, e-mails: `jcholewa@ux2.math.us.edu.pl`, `tdlotko@ux2.math.us.edu.pl`.