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NEW SUFFICIENT CONVERGENCE CONDITIONS FOR THE SECANT METHOD

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Abstract. We provide new sufficient conditions for the convergence of the secant method to a locally unique solution of a nonlinear equation in a Banach space. Our new idea uses "Lipschitz-type" and center-"Lipschitz-type" instead of just "Lipschitz-type" conditions on the divided difference of the operator involved. It turns out that this way our error bounds are more precise than the earlier ones and under our convergence hypotheses we can cover cases where the earlier conditions are violated.

Keywords: secant method, Banach space, majorizing sequence, divided difference, Fréchet-derivative

MSC 2000: 65H10, 65B05, 65G99, 65N30, 47H17, 49M15

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations [4], [7], [16]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ (for some suitable operator Q), where x is the state. Then the equilibrium states are determined by solving equation (1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We consider the secant method in the form

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad (n \geqslant 0), \tag{2}$$

where $\delta F(x, y) \in L(X, Y)$ $(x, y \in D)$ is a consistent approximation of the Fréchet-derivative of F [3], [4], [10]. Bosarge and Falb [4], Dennis [5], Potra [11], Argyros [1], [2], Gutiérrez [6], [7], and others [8], [10], [3], [12], have provided sufficient convergence conditions for the secant method based on "Lipschitz-type" conditions on δF. Here using "Lipschitz-type" and center-"Lipschitz-type" conditions we provide a semilocal convergence analysis for (2). It turns out that our error bounds are more precise and our convergence conditions hold in cases where the corresponding hypotheses in the earlier references mentioned above are violated.

2. Semilocal convergence analysis for the secant method

We need the following result on majorizing sequences.

Lemma 1. *Assume there exist non-negative parameters l, l_0, η, c, and $a \in [0, 1]$,*

$$\delta \in \begin{cases} \left[0, \dfrac{-1 + \sqrt{1 + 4a}}{2a}\right], & a \neq 0 \\ [0, 1), & a = 0 \end{cases} \tag{3}$$

such that:

$$(l + \delta l_0)(c + \eta) \leqslant \delta, \tag{4}$$

$$\eta \leqslant \delta c, \tag{5}$$

and

$$l_0 \leqslant al. \tag{6}$$

Then,

(a) *the iteration* $\{t_n\}$ $(n \geqslant -1)$ *given by*

$$(7) \qquad \begin{gathered} t_{-1}=0, \quad t_0=c, \quad t_1=c+\eta, \\ t_{n+2}=t_{n+1}+\frac{l(t_{n+1}-t_{n-1})}{1-l_0[t_{n+1}-t_0+t_n]}(t_{n+1}-t_n) \quad (n \geqslant 0) \end{gathered}$$

is non-decreasing, bounded above by

$$(8) \qquad t^{**}=\frac{\eta}{1-\delta}+c,$$

and converges to some t^* *such that*

$$(9) \qquad 0 \leqslant t^* \leqslant t^{**}.$$

Moreover, the following error bounds hold for all $n \geqslant 0$

$$(10) \qquad 0 \leqslant t_{n+2}-t_{n+1} \leqslant \delta(t_{n+1}-t_n) \leqslant \delta^{n+1}\eta.$$

(b) *The iteration* $\{s_n\}$ $(n \geqslant 0)$ *given by*

$$(11) \qquad \begin{gathered} s_{-1}-s_0=c, \quad s_0-s_1=\eta, \\ s_{n+1}-s_{n+2}=\frac{l(s_{n-1}-s_{n+1})}{1-l_0[(s_0+s_{-1})-(s_n+s_{n+1})]}(s_n-s_{n+1}) \quad (n \geqslant 0) \end{gathered}$$

for s_{-1}, s_0, $s_1 \geqslant 0$ *is non-increasing, bounded below by*

$$(12) \qquad s^{**}=s_0-\frac{\eta}{1-\delta},$$

and converges to some s^* *such that*

$$(13) \qquad 0 \leqslant s^{**} \leqslant s^*.$$

Moreover, the following error bounds hold for all $n \geqslant 0$

$$(14) \qquad 0 \leqslant s_{n+1}-s_{n+2} \leqslant \delta(s_n-s_{n+1}) \leqslant \delta^{n+1}\eta.$$

Proof. (a) The result clearly holds if $\delta=0$ or $l=0$ or $c=0$. Let us assume that $\delta \neq 0$, $l \neq 0$ and $c \neq 0$. We must show that for all $k \geqslant 0$:

$$(15) \qquad \begin{gathered} l(t_{k+1}-t_{k-1})+\delta l_0[(t_{k+1}-t_0)+t_k] \leqslant \delta, \\ 1-l_0[(t_{k+1}-t_0)+t_k]>0. \end{gathered}$$

The inequalities (15) hold for $k=0$ by the initial conditions. But then (7) gives

$$0 \leqslant t_2 - t_1 \leqslant \delta(t_1 - t_0).$$

Let us assume (10) and (15) hold for all $k \leqslant n+1$. By the induction hypotheses we have in turn:

$$\begin{aligned}(16)\quad & l(t_{k+2} - t_k) + \delta l_0[(t_{k+2} - t_0) + t_{k+1}] \\ & \leqslant l[(t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)] + \delta l_0\left[\frac{1-\delta^{k+2}}{1-\delta} + \frac{1-\delta^{k+1}}{1-\delta}\right]\eta + \delta l_0 c \\ & \leqslant l(\delta^{k+1} + \delta^k)\eta + \frac{\delta l_0}{1-\delta}(2 - \delta^{k+1} - \delta^{k+2})\eta + \delta l_0 c.\end{aligned}$$

We must show that δ is the upper bound in (16). Instead by (5) we can show that

$$l\delta^k(1+\delta)\eta + \frac{\delta l_0}{1-\delta}(2 - \delta^{k+2} - \delta^{k+1})\eta + \delta l_0 c \leqslant (l + \delta l_0)(c + \eta)$$

or

$$\delta l_0\left[\frac{2 - \delta^{k+2} - \delta^{k+1}}{1-\delta} - 1\right]\eta \leqslant l[c + \eta - \delta^k(1+\delta)\eta]$$

or

$$a\delta l\frac{1 + \delta - \delta^{k+1}(1+\delta)}{1-\delta}\eta \leqslant l\left[\frac{\eta}{\delta} + \eta - \delta^k(1+\delta)\eta\right]$$

or

$$a\delta^2(1+\delta)(1-\delta^{k+1}) \leqslant (1-\delta)(1+\delta)(1-\delta^{k+1})$$

or

$$a\delta^2 + \delta - 1 \leqslant 0,$$

which is true by the choice of δ. Moreover, by (5) and (10)

$$\begin{aligned}(17)\qquad \delta l_0[(t_{k+2} - t_0) + t_{k+1}] & \leqslant \frac{\delta l_0}{1-\delta}(2 - \delta^{k+2} - \delta^{k+1})\eta + \delta l_0 c \\ & < (l + \delta l_0)(c + \eta) \leqslant \delta,\end{aligned}$$

which proves the second inequality in (15). We must also show that

$$(18)\qquad t_k \leqslant t^{**} \quad (k \geqslant -1).$$

For $k = -1, 0, 1, 2$ we have $t_{-1} = 0 \leqslant t^{**}$, $t_0 = \eta \leqslant t^{**}$, $t_1 = \eta + c \leqslant t^{**}$ by (8), and $t_2 = c + \eta + \delta\eta = c + (1+\delta)\eta \leqslant t^{**}$ by the choice of δ. Assume (18) holds for all $k \leqslant n+1$. It follows from (10) that

$$\begin{aligned} t_{k+2} &\leqslant t_{k+1} + \delta(t_{k+1} - t_k) \leqslant t_k + \delta(t_k - t_{k-1}) + \delta(t_{k+1} - t_k) \\ &\leqslant \ldots \leqslant t_1 + \delta(t_1 - t_0) + \ldots + \delta(t_{k+1} - t_k) \\ &\leqslant c + \eta + \delta\eta + \ldots + \delta^{k+1}\eta \\ &= c + \frac{1 - \delta^{k+2}}{1-\delta}\eta < \frac{\eta}{1-\delta} + c = t^{**}. \end{aligned}$$

That is, $\{t_n\}$ $(n \geqslant -1)$ is bounded above by t^{**}. It also follows from (7) and (15) that it is also non-decreasing and as such it converges to some t^* satisfying (9).

(b) We proceed as in part (a) but we show that $\{s_n\}$ $(n \geqslant -1)$ is non-increasing and bounded below by s^{**}. Note that the inequality corresponding to (16) is

$$l(s_k - s_{k+2}) \leqslant \delta[1 - \beta(s_0 + s_{-1}) + \beta(s_{k+1} + s_{k+2})],$$

or

$$\begin{aligned} &l[\delta^k(s_0 - s_1) + \delta^{k+1}(s_0 - s_1)] \\ &\leqslant \delta\left[1 - l_0(s_0 + s_{-1}) + l_0\left(s_0 - \frac{1-\delta^{k+1}}{1-\delta}(s_0 - s_1)\right) + l\left(s_0 - \frac{1-\delta^{k+2}}{1-\delta}(s_0 - s_1)\right)\right], \end{aligned}$$

or

$$l\delta^k(1+\delta)\eta + \delta l_0\left[\frac{2 - \delta^{k+1} - \delta^{k+2}}{1-\delta}\eta + c\right]$$

must be bounded above by δ, which was shown in part (a).

That completes the proof of Lemma 1. □

Remark 1. It follows from (16) and (17) that the conclusions of Lemma 1 hold if (3), (5), (6) are replaced by the weaker conditions: for all $n \geqslant 0$ there exists $\delta \in [0, 1)$ such that

$$l\delta^n(1+\delta)\eta + \frac{\delta l_0}{1-\delta}(2 - \delta^{n+2} - \delta^{n+1})\eta + \delta l_0 c \leqslant \delta,$$

and

$$\frac{\delta l_0}{1-\delta}(2 - \delta^{n+2} - \delta^{n+1})\eta + \delta l_0 c < 1.$$

The above conditions hold in many cases for all $n \geqslant 0$. One such stronger case is:

$$l(1+\delta)\eta + \frac{2\delta l_0 \eta}{1-\delta} + \delta l_0 c \leqslant \delta,$$

and

$$\frac{2\delta l_0 \eta}{1-\delta} + \delta l_0 c < 1.$$

We shall study the iterative procedure (2) for triplets (F, x_{-1}, x_0) belonging to the class $C(l, l_0, \eta, c)$ defined as follows:

Definition 1. Let l, l_0, η, c be non-negative parameters satisfying the hypotheses of Lemma 1 or Remark 1 (including (4)).

We say that a triplet (F, x_{-1}, x_0) belongs to the class $C(l, l_0, \eta, c)$ if:

c_1) F is a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y;

c_2) x_{-1} and x_0 are two points belonging to the interior D^0 of D and satisfying the inequality

$$\|x_0 - x_{-1}\| \leqslant c; \tag{19}$$

c_3) F is Fréchet-differentiable on D^0 and there exists an operator $\delta F\colon\, D^0 \times D^0 \to L(X, Y)$ such that:
the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded and

$$\|A^{-1} F(x_0)\| \leqslant \eta;$$
$$\|A[\delta F(x,y) - F'(z)]\| \leqslant l(\|x - z\| + \|y - z\|),$$
$$\|A[\delta F(x,y) - F'(x_0)]\| \leqslant l_0(\|x - x_0\| + \|y - x_0\|)$$

for all $x, y, z \in D$.

c_4) the set $D_c = \{x \in D;\ F \text{ is continuous at } x\}$ contains the closed ball $\overline{U}(x_0, s^*) = \{x \in X \mid \|x - x_0\| \leqslant s^*\}$ where s^* is given in Lemma 1.

We present the following semilocal convergence theorem for secant method (2).

Theorem 1. *If $(F, x_{-1}, x_0) \in C(l, l_0, \eta, c)$ then the sequence $\{x_n\}$ $(n \geqslant -1)$ generated by the secant method (2) is well defined, remains in $\overline{U}(x_0, s^*)$ for all $n \geqslant 0$ and converges to a solution $x^* \in \overline{U}(x_0, s^*)$ of the equation $F(x) = 0$.*

Moreover the following estimates hold for all $n \geqslant 0$

$$\|x_{n+2} - x_{n+1}\| \leqslant s_{n+1} - s_{n+2}, \tag{23}$$

$$\|x_n - x^*\| \leqslant \alpha_n \tag{24}$$

and

$$\|x_n - x^*\| \geqslant \beta_n \tag{25}$$

where,

$$s_{-1} = \frac{1 + l_0 c}{2l_0}, \qquad s_0 = \frac{1 - l_0 c}{2l_0} \quad \text{for } l_0 \neq 0, \tag{26}$$

the sequence $\{s_n\}$ $(n \geqslant 0)$ *is given by* (11), *and* α_n, β_n *are respectively the non-negative solutions of the equations*

$$\begin{aligned} &l_0 t^2 - 2l_0(s_0 - \|x_n - x_0\|)t \\ &\quad - l(\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\| = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &lt^2 + [l\|x_n - x_{n-1}\| + 1 - l_0(\|x_n - x_0\| + \|x_{n-1} - x_0\| + c)]t \\ &\qquad + [l_0(\|x_n - x_0\| + \|x_{n-1} - x_0\| + c) - 1]\|x_{n+1} - x_n\| = 0. \end{aligned} \tag{28}$$

Proof. We first show that the operator $L = \delta F(u, v)$ is invertible for all $u, v \in D^0$ with

$$\|u - x_0\| + \|v - x_0\| < 2s_0. \tag{29}$$

It follows from (22) and (29) that

$$\begin{aligned} \|I - A^{-1}L\| = \|A^{-1}(L - A)\| &\leqslant \|A^{-1}(L - F'(x_0))\| + \|A^{-1}(F'(x_0) - A)\| \\ &\leqslant l_0(\|u - x_0\| + \|v - x_0\| + \|x_0 - x_{-1}\|) < 1. \end{aligned} \tag{30}$$

According to the Banach Lemma on invertible operators [9] and (30) L is invertible and

$$\|L^{-1}A\| \leqslant [1 - l_0(\|u - x_0\| + \|v - x_0\| + c)]^{-1}. \tag{31}$$

Condition (21) implies the Lipschitz condition for F'

$$\|A^{-1}(F'(u) - F'(v))\| \leqslant 2l\|u - v\|, \quad u, v \in D^0. \tag{32}$$

By the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x-y))\,\mathrm{d}t(x-y) \tag{33}$$

we get

$$\|A_0^{-1}[F(x) - F(y) - F'(u)(x-y)]\| \leqslant l(\|x-u\| + \|y-u\|)\|x-y\| \tag{34}$$

and

$$\|A_0^{-1}[F(x) - F(y) - \delta F(u,v)(x-y)]\| \leqslant l(\|x-v\| + \|y-v\| + \|u-v\|)\|x-y\| \tag{35}$$

for all $x, y, u, v \in D^0$. By a continuity argument (33)–(35) remain valid if x and/or y belong to D_c.

We first show (23). If (23) holds for all $\eta \leqslant k$ and if $\{x_n\}$ $(n \geqslant 0)$ is well defined for $n = 0, 1, 2, \ldots, k$ then

$$\|x_0 - x_n\| \leqslant s_0 - s_n < s_0 - s^*, \quad n \leqslant k. \tag{36}$$

Hence (29) holds for $u = x_i$ and $v = x_j$ $(i, j \leqslant k)$. That is (2) is well defined for $n = k+1$. For $n = -1$ and $n = 0$ (23) reduces to $\|x_{-1} - x_0\| \leqslant c$ and $\|x_0 - x_1\| \leqslant \eta$, respectively. Suppose (23) holds for $n = -1, 0, 1, \ldots, k$ $(k \geqslant 0)$. Using (31), (35) and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)(x_{k+1} - x_k) \tag{37}$$

we obtain in turn

$$\begin{aligned}
\|x_{k+2} - x_{k+1}\| &= \|\delta F(x_k, x_{k+1})^{-1} F(x_{k+1})\| \\
&\leqslant \|\delta F(x_k, x_{k+1})^{-1} A\|\,\|A^{-1} F(x_{k+1})\| \\
&\leqslant \frac{l(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\|)}{1 - l_0[\|x_{k+1} - x_0\| + \|x_k - x_0\| + c]}\|x_{k+1} - x_k\| \\
&\leqslant \frac{l(s_k - s_{k+1} + s_{k-1} - s_k)}{1 - l_0[s_0 - s_{k+1} + s_0 - s_k + s_{-1} - s_0]}(s_k - s_{k+1}) \\
&= s_{k+1} - s_{k+2}.
\end{aligned} \tag{38}$$

The induction for (23) is now complete. It follows from (23) and Lemma 1 that the sequence $\{x_n\}$ $(n \geqslant -1)$ is Cauchy in the Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, s^*)$ (since $\overline{U}(x_0, s^*)$ is a closed set) so that

$$\|x_n - x^*\| \leqslant s_n - s^*. \tag{39}$$

By letting $k \to \infty$ in (38) we obtain $F(x^*) = 0$.

Set $x = x_n$ and $y = x^*$ in (33), $M = \int_0^1 F'(x^* + t(x_n - x^*))\,\mathrm{d}t$. Using (23) and (39) we get in turn

$$\begin{aligned} \|x_n - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\| &\leqslant 2\|x_n - x_0\| + \|x_n - x^*\| + c \\ &< 2(\|x_n - x_0\| + \|x_n - x^*\|) \leqslant 2(s_0 - s_n + s_n - s^*) + c \\ &\leqslant 2s_0 + c = \frac{1}{l_0}. \end{aligned} \tag{40}$$

By (40) and the Banach Lemma on invertible operators we get

$$\|M^{-1}A\| \leqslant [1 - l_0(2\|x_n - x_0\| + \|x_n - x^*\| + c)]^{-1}. \tag{41}$$

It follows from (2) and (41) that

$$\begin{aligned} \|x_n - x^*\| &\leqslant \|M^{-1}A\| \cdot \|A^{-1}F(x_n)\| \\ &\leqslant \frac{l[\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|]}{1 - l_0[2\|x_n - x_0\| + \|x_n - x^*\| + c]} \|x_n - x_{n-1}\|, \end{aligned} \tag{42}$$

which shows (24).

Using the approximation

$$\begin{aligned} x_{n+1} - x^* &= x^* - x_n \\ &\quad + [A\delta F(x_{n-1}, x_n)]^{-1} A[F(x^*) - F(x_n) - \delta F(x_{n-1}, x_n)(x^* - x_n)] \end{aligned} \tag{43}$$

and the estimates (30) and (35) we get

$$\|x_{n+1} - x_n\| \leqslant \frac{l[\|x^* - x_n\| + \|x_n - x_{n-1}\|]}{1 - l_0[\|x_n - x_0\| + \|x_{n-1} + x_0\| + c]} \|x_n - x^*\| + \|x_n - x^*\|, \tag{44}$$

which proves (25).

That completes the proof of Theorem 1. □

In the next result we examine the uniqueness of the solution x^*.

Theorem 2. *If $(F, x_{-1}, x_0) \in C(l, l_0, \eta, c)$, the equation* (1) *has a solution $x^* \in \overline{U}(x_0, s^*)$. This solution is unique in the set $U_1 = \{x \in D_c \mid \|x - x_0\| < s_0 + \gamma\}$ if $\gamma > 0$ or in the set $U_2 = \{x \in D_c \mid \|x - x_0\| \leqslant s_0\}$ if $\gamma = 0$.*

Proof. *Case 1*: $\gamma > 0$. Let $x^* \in \overline{U}(x_0, s^*)$ and $y^* \in U_1$ be solutions of the equation $F(x) = 0$. Set $P = \int_0^1 F'(y + t(x - y))\,\mathrm{d}t$. Using (22) we get

$$\begin{aligned}\|I - A^{-1}P\| = \|A^{-1}(A - P)\| &\leqslant l_0(\|y^* - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\|)\\ &< l_0(s_0 + \gamma + s_0 - \gamma + c) = 1.\end{aligned}$$

Hence, P is invertible and from (33) we get $x^* = y^*$.

Case 2: $\gamma = 0$. Consider the modified secant method

$$s_{n+1} = s_n - A^{-1}F(y_n) \quad (n \geqslant 0). \tag{45}$$

By Theorem 1 the sequence $\{y_n\}$ $(n \geqslant 0)$ converges to x^* and

$$\|x_n - x_{n+1}\| \leqslant \overline{s}_n - \overline{s}_{n+1} \tag{46}$$

where,

$$\overline{s}_0 = \sqrt{\frac{n}{l}}, \quad \overline{s}_{n+1} = \overline{s}_n - l s_n^2 \quad (n \geqslant 0), \quad \text{for } l > 0. \tag{47}$$

Using induction on $n \geqslant 0$ we get

$$\overline{s}_n \geqslant \frac{\sqrt{\eta/l}}{n+1} \quad (n \geqslant 0). \tag{48}$$

Let y^* be a solution of $F(x) = 0$. Set $P_n = \int_0^1 F'(y^* + t(x_n - y^*))\,\mathrm{d}t$. It follows from (22), (33), (45) and (48) that

$$\begin{aligned}\|x_{n+1} - y^*\| &= \|A^{-1}(A - P_n)(x_n - y^*)\|\\ &\leqslant l(\|y^* - x_0\| + \|x_n - x_0\| + \|x_0 - x_{-1}\|)\|x_n - y^*\|\\ &\leqslant (1 - l\overline{s}_n)\|x_n - y^*\| \leqslant \ldots \leqslant \prod_{i=1}^{n}(1 - l\overline{s}_i)\|x_1 - y^*\|.\end{aligned} \tag{49}$$

By (49), we get $\lim_{n\to\infty} \prod_{i=1}^{n}(1 - l\overline{s}_i) = 0$. Hence, we deduce that $x^* = y^*$.

That completes the proof of Theorem 2. □

Remark 2. The parameter s^* can be computed as the limit of the sequence $\{s_n\}$ ($n \geqslant -1$) using (11). Simply set

$$s^* = \lim_{n\to\infty} s_n. \tag{50}$$

Remark 3. A similar convergence analysis can be provided if the sequence $\{s_n\}$ is replaced by $\{t_n\}$. Indeed under the hypotheses of Theorem 1 we have for all $n \geqslant 0$

$$\|x_{n+2} - x_{n+1}\| \leqslant t_{n+2} - t_{n+1} \tag{51}$$

and

$$\|x^* - x_n\| \leqslant t^* - t_n. \tag{52}$$

In order for us to compare with earlier results we first need the definition:

Definition 2. Let l, η, c be three non-negative numbers satisfying the inequality

$$lc + 2\sqrt{l\eta} \leqslant 1. \tag{53}$$

We say that a triplet $(F, x_{-1}, x_0) \in C_1(l, \eta, c)$ $(l > 0)$ if the conditions (c_1)–(c_4) hold (excluding (22)). Define iteration $\{p_n\}$ ($n \geqslant -1$) by

$$p_{-1} = \frac{1+lc}{2l}, \quad p_0 = \frac{1-lc}{2l}, \quad p_{n+1} = p_n - \frac{p_n^2 - p^2}{p_n + p_{n-1}}, \tag{54}$$

where,

$$p = \frac{1}{2l}\sqrt{(1-lc)^2 - 4l\eta}. \tag{55}$$

The proof of the following semilocal convergence theorem can be found in [3], [9], [10]–[12].

Theorem 3. *If $(F, x_{-1}, x_0) \in C_1(l, \eta, c)$, the sequence $\{x_n\}$ ($n \geqslant -1$) generated by the secant method (2) is well defined, remains in $\overline{U}(x_0, p)$ for all $n \geqslant 0$ and converges to a unique solution $x^* \in \overline{U}(x_0, p)$ of the equation $F(x) = 0$.*

Moreover the following error bounds hold for all $n \geqslant 0$:

$$\|x_{n+1} - x_n\| \leqslant p_n - p_{n+1} \tag{56}$$

and

$$\|x_n - x^*\| \leqslant p_n - p. \tag{57}$$

Using induction on n we can easily show the following favorable comparison of error bounds between Theorems 1 and 3.

Proposition 1. *Under the hypotheses of Theorems* 1 *and* 3 *the following estimates hold for all* $n \geqslant 0$

$$p_n \leqslant s_n \tag{58}$$

$$s_n - s_{n+1} \leqslant p_n - p_{n+1} \tag{59}$$

and

$$s_n - s^* \leqslant p_n - p. \tag{60}$$

Remark 4. We cannot compare conditions (4) and (53) in general because of l_0. However in the special case $l = l_0 \neq 0$, we can set $a = 1$ to obtain $\delta = \frac{\sqrt{5}-1}{2}$. The condition (4) can be written as

$$lc + l\eta \leqslant \beta = \frac{\delta}{1+\delta} = .381966011.$$

It can then easily be seen that if

$$0 < lc < 2\sqrt{\beta} - 1 = .236067977,$$

then the condition (4) holds but (53) is violated. That is, even in the special case of $l = l_0$, our Theorem 1 can be applied in cases not covered by Theorem 3.

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