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A NOTE ON THE INDEPENDENT DOMINATION NUMBER OF SUBSET GRAPH

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Abstract. The independent domination number $i(G)$ (independent number $\beta(G)$) is the minimum (maximum) cardinality among all maximal independent sets of G. Haviland (1995) conjectured that any connected regular graph G of order n and degree $\delta \leqslant \frac{1}{2}n$ satisfies $i(G) \leqslant \lceil 2n/3\delta \rceil \frac{1}{2}\delta$. For $1 \leqslant k \leqslant l \leqslant m$, the subset graph $S_m(k,l)$ is the bipartite graph whose vertices are the k- and l-subsets of an m element ground set where two vertices are adjacent if and only if one subset is contained in the other. In this paper, we give a sharp upper bound for $i(S_m(k,l))$ and prove that if $k+l=m$ then Haviland's conjecture holds for the subset graph $S_m(k,l)$. Furthermore, we give the exact value of $\beta(S_m(k,l))$.

Keywords: independent domination number, independent number, subset graph

MSC 2000: 05C69, 05C35

1. Introduction

Let $G = (V, E)$ be a simple graph of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. Let $\varepsilon(S, V-S)$ denote the number of edges between S and $V-S$.

For integers $1 \leqslant k \leqslant l \leqslant m$, we define the subset graph $S_m(k,l)$ to be the bipartite graph (χ, E, ψ) where the vertices of χ are the k-subsets of $[m] = \{1, 2, \ldots, m\}$, the vertices of ψ are the l-subsets of $[m]$, and for $X \in \chi$ and $Y \in \psi$, X is adjacent to Y

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if and only if $X \subseteq Y$. Notice that the subset graph $S_m(k,k)$ is a matching with $\binom{m}{k}$ edges, and if $k+l=m$ then $S_m(k,l)$ is a regular graph.

An independent set is a set of pairwise nonadjacent vertices of G. The independent domination number $i(G)$ is the minimum cardinality of a maximal independent set of G, while the maximum cardinality of an independent set of vertices of G is the independent number of G and is denoted by $\beta(G)$.

A set of vertices is a dominating set if $N[S]=V$. The domination number of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G, and the upper domination number $\Gamma(G)$ is the maximum cardinality of a dominating set in G.

For $x \in X \subseteq V$, if $N[x]-N[X-\{x\}]=\emptyset$, then x is said to be redundant in X. A set X containing no redundant vertex is called irredundant. The irredundant number of G, denoted by $ir(G)$, is the minimum cardinality taken over all maximal irredundant sets of G. The upper irredundant number of G, denoted by $\mathrm{IR}(G)$, is the maximum cardinality of an irredundant set of G. Let $\mathrm{EPN}(v,X)=\{u \in V-X\colon\ u$ is only adjacent to v but to no other vertex of $X\}$.

The parameter $i(G)$ was introduced by Cockayne and Hedetniemi in [1] and some results on it can be found in [1]–[7]. Favaron [2] and Haviland [3] established upper bounds for $i(G)$ in terms of n and δ. For regular graphs of degree different from zero, we can prove that $i(G) \leqslant \frac{1}{2}n$. However, for most values of δ, this is far from the best possible. In [2], it was shown that for any graph with $\frac{1}{2}n \leqslant \delta \leqslant n$, we have $i(G) \leqslant n-\delta$, and this bound could be attained only by complete multipartite graphs with vertex classes all of the same order. By adapting the arguments from [3], the following results can readily be proved (see [4]).

Proposition 1.1. *Let G be a regular graph. If $\frac{1}{4}n \leqslant \delta \leqslant \frac{1}{2}(3-\sqrt{5})n$, then $i(G) \leqslant n-\sqrt{n\delta}$, and if $\frac{1}{2}(3-\sqrt{5})n \leqslant \delta \leqslant \frac{1}{2}n$, then $i(G) \leqslant \delta$.*

If $n=2m\delta$, then $i(mK_{\delta,\delta})=\frac{1}{2}n$ and $mK_{\delta,\delta}$ is disconnected for $m>1$. Haviland [3] thought that if G was connected then the upper bound for $i(G)$ could be a function of n and δ. She also stated the following conjecture in [4].

Conjecture 1.2. *If G is a connected r-regular graph with $r=\delta \leqslant \frac{1}{2}n$, then $i(G) \leqslant \lceil 2n/3\delta \rceil \frac{1}{2}\delta$.*

However, Pear Che Bor Lam et al. [7] provided counterexamples to Conjecture 1.2.

In this paper, we give a sharp upper bound for $i(S_m(k,l))$ and prove that if $k+l=m$ then Haviland's conjecture holds for the subset graph $S_m(k,l)$. Furthermore, we give the exact value of $\beta(S_m(k,l))$.

2. Main results

By the definition of the subset graph, it is easy to prove the following two lemmas.

Lemma 1. *If $k = l$, then $i(S_m(k,l)) = \beta(S_m(k,l)) = \binom{m}{k}$.*

Lemma 2. *If $1 \leqslant k < l = m$, then $i(S_m(k,l)) = 1$ and $\beta(S_m(k,l)) = \binom{m}{k}$.*

Now, we give the main results of this paper.

Theorem 1. *If $1 \leqslant k \leqslant l \leqslant m$, then $i(S_m(k,l)) \leqslant \binom{m-l+k}{k}$ and the bound is sharp.*

Proof. Let $d = l - k$. By Lemma 1 and Lemma 2, if $k = l$ or $l = m$, Theorem 1 holds. So, we only consider $1 \leqslant k < l < m$. Let

$$A = \{X \in \chi\colon\ i \notin X \ \text{ for } \ m - d \leqslant i \leqslant m\}$$

and

$$B = \{Y \in \psi\colon\ i \in Y \ \text{ for } \ m - d \leqslant i \leqslant m\}.$$

We have the following claims.

Claim 1. *$A \cup B$ is an independent set of $S_m(k,l)$.*

Let t_1 and t_2 be arbitrary two vertices of $A \cup B$. If $t_1, t_2 \in A$ or $t_1, t_2 \in B$, then it is obvious that t_1 is not adjacent to t_2. Without loss of generality, we assume that $t_1 \in A$ and $t_2 \in B$. Let $t_1 = \{x_1, x_2, \ldots, x_k\}$ and $t_2 = \{y_1, y_2, \ldots, y_l\}$ where $1 \leqslant x_1 < x_2 < \ldots < x_k < m$ and $1 \leqslant y_1 < y_2 < \ldots < y_l < m$. Since $i \notin t_1$ and $i \in t_2$ for $m - d \leqslant i \leqslant m$, $\{y_1, y_2, \ldots, y_l\}$ has at most $l - (d+1)$ elements which are identical to elements of $\{x_1, x_2, \ldots, x_k\}$. Since $l - (d+1) = k - 1 < k$, it follows that $\{x_1, x_2, \ldots, x_k\} \nsubseteq \{y_1, y_2, \ldots, y_l\}$. Hence, t_1 is not adjacent to t_2. Since t_1 and t_2 are arbitrary two vertices of $A \cup B$, $A \cup B$ is an independent set of $S_m(k,l)$.

Claim 2. *$A \cup B$ is a dominating set of $S_m(k,l)$.*

For an arbitrary vertex $t \in (V(S_m(k,l)) - (A \cup B))$, we prove that t is dominated by at least one vertex of $A \cup B$.

Case 1: $t \in (\chi - A)$. Let $t = \{x_1, x_2, \ldots, x_k\}$ where $1 \leqslant x_1 < x_2 < \ldots < x_k < m$. Then there exists a x_i such that $x_i \in \{m - d, m - d + 1, \ldots, m\}$. Without loss of generality, we assume that x_s is the first number such that $x_s \in \{m - d, m - d + 1, \ldots, m\}$. Let $C = \{x_1, x_2, \ldots, x_{s-1}, x_{m-d}, \ldots, x_m\}$. Since $k \geqslant s$, $|C| = s - 1 + d +$

$1 = s + d = s + l - k = l - (k - s) \leqslant l$. So there exists a vertex $Y \in \psi \cap B$ such that $\{x_1, x_2, \ldots, x_k\} \subseteq \{x_1, x_2, \ldots, x_{s-1}, x_{m-d}, \ldots, x_m\} \subseteq Y$. Hence t is adjacent to Y.

Case 2: $t \in (\psi - B)$. Let $t = \{y_1, y_2, \ldots, y_l\}$ where $1 \leqslant y_1 < y_2 < \ldots < y_l < m$. Then there exists an $i \in \{m - d, \ldots, m\}$ such that $y_j \neq i$ for $1 \leqslant j \leqslant l$. Let y_s be the first number that belongs to $\{m - d, \ldots, m\}$ and let $C = \{y_1, y_2, \ldots y_{s-1}\}$. Since $l - (s - 1) < d + 1 = l - k + 1$, it follows that $k < s$ and $|C| \geqslant k$. Hence if X is a k-subset of C, then $X \subseteq A$ and X is adjacent to t. Since t is an arbitrary vertex, by Case 1 and Case 2, it follows that $A \cup B$ is a dominating set of $S_m(k, l)$. By Claim 1 and Claim 2, $A \cup B$ is an independent dominating set of $S_m(k, l)$. Hence,

$$\begin{aligned} i(S_m(k,l)) &\leqslant |A \cup B| = |A| + |B| = \binom{m-(d+1)}{k} + \binom{m-(d+1)}{l-(d+1)} \\ &= \binom{m-(d+1)}{k} + \binom{m-(d+1)}{k-1} = \binom{m-l+k}{k}. \end{aligned}$$

Corollary 1. *If $1 \leqslant k \leqslant l < m$ and $k + l = m$, then $i(S_m(k, l)) \leqslant \binom{2k}{k}$ and the bound is sharp.*

The sharpness of Theorem 1 and Corollary 1 can be seen from the following result.

Theorem 2. *If $1 < l < m$ and $l + 1 = m$, then $i(S_m(1, l)) = 2 = \binom{2}{1} = \binom{2k}{k}$.*

Proof. Since $1 < l < m$, it follows that $m \geqslant 3$ and $S_m(1, l)$ is not a star. Hence, $\gamma(S_m(1, l)) \geqslant 2$. Since $2 \leqslant \gamma(S_m(1, l)) \leqslant i(S_m(1, l)) \leqslant 2$, it follows that $i(S_m(1, l)) = 2$. □

Theorem 3. $i(S_5(2, 3)) = 6 = \binom{4}{2} = \binom{2k}{k}$.

Proof. Since $S_5(2, 3)$ is a 3-regular graph,

$$i(S_5(2,3)) \geqslant \gamma(S_5(2,3)) \geqslant \frac{|V(S_5(2,3))|}{\Delta(S_5(2,3)) + 1} = \frac{2\binom{5}{2}}{4} = 5.$$

If $\gamma(S_5(2, 3)) = 5$, then let I be a dominating set of $S_5(2, 3)$ with cardinality 5. We have the following claims.

Claim 1. *I is an independent set of $S_5(2, 3)$.*

Otherwise, if I is not an independent set, then there exists at least one edge in $\langle I \rangle$. Hence, $\varepsilon(I, V - I) \leqslant \sum_{v \in I} d(v) - 2 = 3 \times 5 - 2 = 13 < 15 = |V(S_5(2, 3)) - I|$. So there exists a vertex $v \in V - I$ such that v is not dominated by I, which is a contradiction.

Claim 2. *For each vertex $v \in I$, $|\mathrm{EPN}(v, I)| = 3$.*

By Claim 1, I is an independent dominating set of $S_5(2,3)$. If there exists a vertex $v \in I$ such that $|\mathrm{EPN}(v, I)| < 3$, then I dominates at most $\sum_{v\in I} d(v) - 1 = 3 \times 5 - 1 = 14 < 15 = |V(S_5(2,3)) - I|$ vertices, which is a contradiction.

Let $A = \chi \cap I$ and $B = \psi \cap I$. Since $|I| = 5$, without loss of generality, we assume that $|A| \geqslant 3$. It is obvious that $|A| \leqslant 4$. So, $3 \leqslant |A| \leqslant 4$.

Case 1: If $|A| = 4$, then by Claim 1 and Claim 2 the set A dominates 12 vertices of ψ, which is a contradiction since ψ has 10 vertices.

Case 2: If $|A| = 3$, then by Claim 1 and Claim 2 the set A dominates 9 vertices of ψ. So, there is only one vertex of ψ that belongs to I, which is a contradiction with $|B| = 2$.

Hence, $\gamma(S_5(2,3)) \geqslant 6$. Since $6 \leqslant \gamma(S_5(2,3)) \leqslant i(S_5(2,3)) \leqslant 6$, it follows that $i(S_5(2,3)) = 6$.

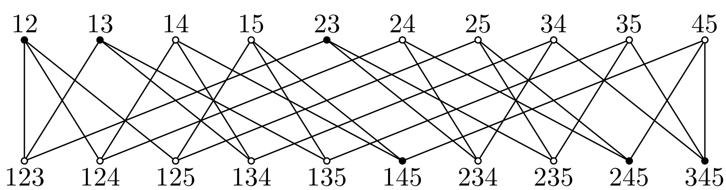

Figure 1.

By Figure 1, it is easy to see that the black vertices form an independent dominating set of $S_5(2,3)$ with cardinality 6.

The following theorem proves that conjecture 1.2 holds for the subset graph $S_m(k,l)$ if $1 \leqslant k < l < m$ and $k + l = m$.

Theorem 4. *If $1 \leqslant k < l < m$ and $k + l = m$, then Conjecture 1.2 holds for the subset graph $S_m(k,l)$.*

Proof. Let $d = l - k$. If $1 \leqslant k < l < m$ and $k + l = m$ then $S_m(k,l)$ is a connected regular graph with $n = |V(S_m(k,l))| = 2\binom{m}{k}$ and $\delta \leqslant \frac{1}{2}n$. By Corollary 1, $i(S_m(k,l)) \leqslant \binom{2k}{k}$. It follows that

$$\frac{i(S_m(k,l))}{|V(S_m(k,l))|} \leqslant \frac{\binom{2k}{k}}{2\binom{m}{k}} = \frac{\binom{2k}{k}}{2\binom{2k+d}{k}} \leqslant \frac{\binom{2k}{k}}{2\binom{2k+1}{k}} = \frac{(2k)!}{2k!k!}\frac{k!(k+1)!}{(2k+1)!} = \frac{k+1}{2(2k+1)} \leqslant \frac{1}{3}.$$

Hence,

$$i(S_m(k,l)) \leqslant \frac{|V(S_m(k,l))|}{3} = \frac{n}{3} \leqslant \frac{2n}{3\delta}\frac{\delta}{2} \leqslant \left\lceil \frac{2n}{3\delta} \right\rceil \frac{\delta}{2}.$$

The exact value of $\beta(S_m(k,l))$ is given by the following result.

Lemma 3 [5]**.** *For every r-regular graph $G=(V,E)$ of order n, $\mathrm{IR}(G)\leqslant\frac{1}{2}n$.*

Lemma 4 [6]**.** *If G is bipartite, then $\beta=\Gamma=\mathrm{IR}$.*

Theorem 5. *If $1\leqslant k<l\leqslant m$, then $\beta(S_m(k,l))=\Gamma(S_m(k,l))=\mathrm{IR}(S_m(k,l))=\max\{\binom{m}{k},\binom{m}{l}\}$.*

Proof. Since $S_m(k,l)$ is the bipartite graph, then by Lemma 4, $\beta(S_m(k,l))=\Gamma(S_m(k,l))=\mathrm{IR}(S_m(k,l))$.

Case 1: If $k+l=m$, then $S_m(k,l)$ is a regular graph. By Lemma 3, $\mathrm{IR}(G)\leqslant\frac{1}{2}|V(S_m(k,l))|=\binom{m}{k}$. Since $\beta(S_m(k,l))\geqslant\binom{m}{k}$, it follows that $\beta(S_m(k,l))=\Gamma(S_m(k,l))=\mathrm{IR}(S_m(k,l))=\max\{\binom{m}{k},\binom{m}{l}\}$.

Case 2: If $k+l\neq m$, then $\beta(S_m(k,l))\geqslant\max\{\binom{m}{k},\binom{m}{l}\}$. Without loss of generality, assume $\binom{m}{k}=\max\{\binom{m}{k},\binom{m}{l}\}$. That is to say $|\chi|>|\psi|$ and $\beta(S_m(k,l))\geqslant\binom{m}{k}$. For arbitrary vertices $X\in\chi$ and $Y\in\psi$, $d(X)<d(Y)$. If $\beta(S_m(k,l))>\binom{m}{k}=|\chi|$, then let I be a maximal independent set with cardinality $\beta(S_m(k,l))$. Hence, I must contain some vertices of χ and some vertices of ψ. Let $X_1=I\cap\chi$ and $Y_1=I\cap\psi$. Let $X_2=\chi-X_1$ and $Y_2=\psi-Y_1$. So $X_i\neq\emptyset$ and $Y_i\neq\emptyset$ for $i=1,2$. Since

$$\varepsilon(Y_1,V-I)=\varepsilon(Y_1,X_2)=\sum_{Y\in Y_1}d(Y)=d(Y)|Y_1|\leqslant\sum_{X\in X_2}d(X)=d(X)|X_2|$$

and $d(X)<d(Y)$, it follows that $|Y_1|<|X_2|$. Hence $|I|=|X_1|+|Y_1|<|X_1|+|X_2|=|X|$, which is a contradiction. Hence, $\beta(S_m(k,l))=\binom{m}{k}$. So, $\beta(S_m(k,l))=\Gamma(S_m(k,l))=\mathrm{IR}(S_m(k,l))=\max\{\binom{m}{k},\binom{m}{l}\}$.

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