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COEXISTENCE OF SMALL AND LARGE AMPLITUDE LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE FOUR

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Abstract. A class of degree four differential systems that have an invariant conic $x^2 + Cy^2 = 1$, $C \in \mathbb{R}$, is examined. We show the coexistence of small amplitude limit cycles, large amplitude limit cycles, and invariant algebraic curves under perturbations of the coefficients of the systems.

Keywords: stability, limit cycle, center, bifurcation

MSC 2000: 92D25, 58F14, 58F21

1. Introduction

A polynomial differential system is a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities

$$(1)\quad \dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij}x^i y^j, \quad \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij}x^i y^j, \quad \text{with } a_{ij}, b_{ij} \in \mathbb{R}.$$

The problem of the analysis of limit cycles (isolated periodic solutions) in polynomial systems was first discussed by H. Poincaré [23]. Then, in the second part of the 16th problem from the famous list of 23 mathematical problems stated in 1900, D. Hilbert [12] asked to find an upper bound for the number of limit cycles for the nth degree polynomial systems, in terms of this degree n, and to obtain possible relative configurations.

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Let S_n be a collection of the systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{2}$$

where P and Q are polynomials of degree at most n. Denoting the system (2) by (P, Q), let $\pi(P, Q)$ be the number of limit cycles of (2), and we define the so-called Hilbert numbers as $H_n = \sup\{\pi(P, Q);\ (P, Q) \in S_n\}$.

Among the results obtained for H_n in the first decades of the last century we mention the work of I. Bendixson [2], H. Dulac [11], B. van der Pol [30], A. Liénard [16] and L. S. Pontryagin [24]. Starting from the fifties, a lot of research has been done by Russian and Chinese mathematicians, and during the last two decades the limit cycle problem has attracted the attention of several pure and applied mathematicians in the western countries as well.

For the quadratic case the following results are known: N. Bautin [1] proved that $H_2 \geqslant 3$, Shi [27], Chen and Wang [7] proved that $H_2 \geqslant 4$. (For a complete reference list, see [25].)

For the cubic case, the best results that we know are given by P. Yu and M. Han [34] (they showed that $H_3 \geqslant 12$), and H. Zoladek [35], J. Li [15] (they proved that $H_3 \geqslant 11$ with different configurations).

There are two closely related questions of interest. The first one is the coexistence of algebraic invariant curves, and large and small amplitude limit cycles. The second one is the derivation of necessary and sufficient conditions for a critical point to be a center.

Recall that a *center* is a critical point in the neighborhood of which all orbits are closed. A *limit cycle* is an isolated closed orbit.

Let $n = \max(\partial P, \partial Q)$, where the symbol ∂ denotes 'degree of'. A function h is said to be *invariant* with respect to (2) if there is a polynomial $k(x, y)$, called the cofactor, with $\partial k < n$ such that $\dot{h} = hk$. Here $\dot{h} = h_x P + h_y Q$ is the rate of change of h along orbits.

It is interesting to note that the existence of algebraic trajectories has been known to have a strong influence on the behavior of polynomial systems. For instance, quadratic systems ($n = 2$) with an invariant ellipse, hyperbola, or a pair of straight lines can have no limit cycles other than the possible ellipse itself. Moreover, if there is an invariant line, there can not be more than one limit cycle (see [3], [8]). The case of parabola was considered in [9].

The cubic case with an invariant ellipse is similar to the quadratic one. Furthermore, there exist different classes of cubic systems in which there may coexist an invariant hyperbola or straight lines with limit cycles (see [4], [5], [6], [10], [14], [17], [18], [19], [26], [29], [32], [33]).

In this paper we are concerned with the limit cycle problem and the center problem for a class of degree four polynomial differential systems that have an invariant conic $x^2 + Cy^2 = 1$, $C \in \mathbb{R}$, and we prove the coexistence of large elliptic limit cycle that contains at least four small amplitude limit cycles generated by Hopf bifurcations.

2. Limit cycles and center conditions

We suppose that the origin is a critical point of (2), and let us transform the system to the canonical form

$$\dot{x} = \lambda x - y + p(x, y), \quad \dot{y} = x + \lambda y + q(x, y),$$

where p, q are polynomials without linear terms. We must have $\lambda = 0$ for the origin to be a center. If $\lambda = 0$, and the origin is not a center, then it is said to be a *fine focus.*

The necessary conditions for a center are obtained by computing the *focal values.* They are polynomials in the coefficients of P and Q, which are defined as follows. There is a function V, in a neighborhood of the origin, such that the rate of change along orbits, $\dot{V}$, is of the form $\eta_2 r^2 + \eta_4 r^4 + \ldots$, where $r^2 = x^2 + y^2$. The coefficients η_{2k} are the focal values, and the origin is the center if and only if all focal values vanish. However, since they are polynomials, the ideal they generate has a finite basis, so there is M, such that $\eta_{2\ell} = 0$, for $\ell \leqslant M$. This implies that $\eta_{2\ell} = 0$ for all ℓ. The value of M is not known *a priori*, so it is not clear in advance how many focal values should be calculated.

The software Mathematica [30] is used to calculate the first few focal values. Then they become 'reduced' in the sense that each of them is computed modulo the ideal generated by the previous ones. That is, the relations $\eta_2 = \eta_4 = \ldots = \eta_{2k} = 0$ are used to eliminate some of the variables in η_{2k+2}. The reduced focal value η_{2k+2}, with removed strictly positive factors, is known as the *Liapunov quantity* and denoted as $L(k)$. Common factors of the reduced focal values are removed and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield the necessary center conditions. The origin is a fine focus of *order k* if $L(i) = 0$ for $i = 0, 1, \ldots, k - 1$ and $L(k) \neq 0$. At most k limit cycles can bifurcate out of a fine focus of order k; they are called *small amplitude* limit cycles.

Various methods are used to prove the sufficiency of possible center conditions. The symmetry and the construction of the first analytic integral are of particular interest in this paper.

3. Main results

For $\theta = (\lambda, B, C, D, e) \in \mathbb{R}^5$, let us consider the vector field $X_\theta = P\partial/\partial x + Q\partial/\partial y$. More explicitly

$$(3)\quad X_\theta(x,y) = h\Big[(y-\lambda x+Bxy+ey^2)\frac{\partial}{\partial x} - (x+\lambda y - Cxy)\frac{\partial}{\partial y}\Big] - \frac{D}{2}(x^2+2Cy^2)X_h,$$

where $h(x,y) = x^2+Cy^2-1$ is the Hamiltonian function and $X_h(x,y) = -2cy\partial/\partial x + 2x\partial/\partial y$ is the vector field generated by h.

Lemma 1. *For all $\theta \in \mathbb{R}^5$, the conic $x^2 + Cy^2 - 1 = 0$ is an invariant algebraic curve of the system* (3)*. In particular, if $C > 0$ and $D \neq 0$, this conic is an elliptic hyperbolic limit cycle, attracting if $\lambda > 0$ and a repelling if $\lambda < 0$.*

Proof. Let us consider $h(x,y) = x^2 + Cy^2 - 1$. It is easy to verify that for all $\theta \in \mathbb{R}^5$

$$\dot h = h_x\dot x + h_y \dot y = h(x,y)k(x,y),$$

where the cofactor is $k(x,y) = -(\lambda x^2) + (1-C)xy - C\lambda y^2 + Bx^2y + (C^2+e)xy^2$. Then the conic is an invariant curve of (3).

In order to study the common roots of P, Q, h, as polynomials of y, we consider the following *Resultant*

$$\begin{aligned}\text{Resul}(P,h)(x) &= C^5D^2(-1+x)(1+x)(-2+x^2)^2,\\ \text{Resul}(Q,h)(x) &= C^3D^2x^2(-2+x^2)^2.\end{aligned}$$

As $C > 0$ and $D \neq 0$, the possible roots will be the roots of $x^2 = 2$. If $x^2 = 2$ we have $Cy^2 = -1$ and we conclude that $P^{-1}(0) \cap Q^{-1}(0) \cap h^{-1}(0) = \emptyset$ and the conic is a periodic orbit.

In the neighborhood of the ellipse $h^{-1}(0)$ let us consider Π as the Poincaré map or the first return map, and let $h(t)$ be a periodic solution of period $T > 0$ of (3). As $X_\theta(-x,-y)|_h = -X_\theta(x,y)|_h$, $\operatorname{div} X_\theta(x,y)|_{h^{-1}(0)} = 2[-\lambda + xy(1-C+CD) + Bx^2y + (C^2+e)xy^2]$, and by the fact that

$$\int_h xy\,\mathrm{d}t = \int_h x^2y\,\mathrm{d}t = \int_h xy^2\,\mathrm{d}t = 0,$$

the Poincaré map for $h(t)$ at the point $(1,0)$ is given by

$$\Pi'(1,0) = \exp\left(\int_0^T \operatorname{div} X_\theta(h(t))\,\mathrm{d}t\right) = \mathrm{e}^{-2\lambda T}.$$

Thus, for $\lambda > 0$, we have a large attracting limit cycle, and for $\lambda < 0$, we have a large repelling limit cycle (elliptic limit cycle). $\square$

Lemma 2. *In* (3), *let* $\lambda = 0$, $C = \frac{1}{\sqrt{2}}c$, $B = b\sqrt{\frac{\mp 7\sqrt{2}+36}{2}}$ *and* $D = \frac{\mp 7+9\sqrt{2}}{6}d$ *with* $bcd \neq 0$. *Then there exists* $b_\pm \in \mathbb{R}$, ($b_\pm \approx \pm 1.03177$), *such that the origin is a weak focus of the following type:*

$$\text{attracting of order:}\begin{cases} \text{one}, & \text{if } be < 0,\\ \text{two}, & \text{if } e = 0,\ c(c^2-1) > 0,\\ \text{three}, & \text{if } e = 0,\ c^2 - 1 = 0,\ c(1+b^2) < d,\\ \text{four}, & \text{if } e = 0, \text{ and } \begin{cases} c = 1,\ -1-b^2-d = 0,\ bd < 0,\\ c = -1,\ 1+b^2-d = 0,\\ \qquad b_- < b < b_+, \end{cases}\\ \text{five}, & \text{if } e = 0,\ c = -1,\ 1+b^2-d = 0,\ b = b_+, \end{cases}$$

$$\text{repelling of order:}\begin{cases} \text{one}, & \text{if } be > 0,\\ \text{two}, & \text{if } e = 0,\ c(c^2-1) < 0,\\ \text{three}, & \text{if } e = 0,\ c^2 - 1 = 0,\ c(1+b^2) > d,\\ \text{four}, & \text{if } e = 0, \text{ and } \begin{cases} c = 1,\ -1-b^2-d = 0,\ bd > 0,\\ c = -1,\ 1+b^2-d = 0,\ b > b_+,\\ \qquad b < b_-, \end{cases}\\ \text{five}, & \text{if } e = 0,\ c = -1,\ 1+b^2-d = 0,\ b = b_-. \end{cases}$$

Proof. The first focal value is given by $L(1) = be$, and for $e = 0$, we have $L(2) = bcd(1-c)(1+c)$. So, for $c = \pm 1$, we have $L(3) = bd(\mp 1 \mp b^2 - d)$. If $d = \mp 1 \mp b^2$, then $L(4) = \mp bdp_\pm(b)$, where

$$p_\pm(b) = \pm 230715 - 163568\sqrt{2} \pm 706740b^2$$
$$-495683\sqrt{2}b^2 \mp 880017b^4 + 601093\sqrt{2}b^4.$$

As $p_+(b) < 0$ for all b, we have $L(4) \neq 0$ for $bd \neq 0$, and $\operatorname{sng}(L(4)) = \operatorname{sng}(bd)$. Then the vector field (3) has a repelling or an attracting weak focus of order four at the singularity $(0,0)$ if $bd > 0$ or $bd < 0$, respectively.

Moreover, $p_-(b) = 0$ has only two real roots, namely $\{b_\pm \approx \pm 1.03177\}$, and with the Mathematica Software we have

$$L(5)|_{p_-(b)=0} = -db(218676597 + 154305925\sqrt{2} + 490154266b^2$$
$$+ 348614207\sqrt{2}b^2 - 6495006941b^4 - 4603775662\sqrt{2}b^4$$
$$+ 5510480032b^6 + 3886752910b^6).$$

As $b = b_\mp$, we have $L(5)|_{p_-(b_-)} > 0$ and $L(5)|_{p_-(b_+)} < 0$. Then the vector field (3) has a repelling or an attracting weak focus at the singularity $(0,0)$ of order five, respectively. □

Lemma 3. *In* (3), *let* $\lambda = 0$, $C = \frac{1}{\sqrt{2}}c$, $B = b\sqrt{\frac{\mp 7\sqrt{2}+36}{2}}$ *and* $D = \frac{\mp 7+9\sqrt{2}}{6}d$. *Then the critical point* $(0,0)$ *is a center if and only if one of the following conditions is satisfied*

(i) $b = 0$,

(ii) $e = 0$, $d = 0$,

(iii) $e = 0$, $c = 0$.

P r o o f. If (3) has a center in the origin, then the Liapunov quantity $L(i) = 0$ vanishes for all $i \in \mathbb{N} \cup \{0\}$.

If $L(1) = be = 0$ then $b = 0$ or $e = 0$. If $b = 0$, then we have (i). Furthermore for $b \neq 0, e = 0$, we have in this case $L(2) = bcd(1-c)(1+c) = 0$ and this implies $c = 0$ or $d = 0$ or $c = \pm 1$. If $d = 0$, then we have (ii) and for $d \neq 0$ we have $c = 0$ or $c = \pm 1$. If $c = \pm 1$, by Lemma 2 the system (3) does not have a center at the origin. Hence $c = 0$, and we have the necessary condition (iii).

To prove these necessary conditions to be also sufficient for $(0,0)$ to be a center, we must use symmetries or elementary integration.

If $b = 0$, then by Lemma 2, $B = 0$. The system (3) is given by

$$\begin{aligned}\dot{x} &= -y - ey^2 + (1 - CD)x^2y + C(1 - 2CD)y^3 + ex^2y^2 + Cey^4,\\ \dot{y} &= x - Cxy + (-1 + D)x^3 + C(-1 + 2D)xy^2 + Cx^3y + C^2xy^3.\end{aligned}$$

As $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$, the system is symmetric with respect to the y-axis. Hence, $(0,0)$ is a center of the system (2).

If $e = 0$, $d = 0$, then by Lemma 2, $D = 0$ and the system (3) is given by

$$\begin{aligned}\dot{x} &= y(1 + Bx)(-1 + x^2 + Cy^2),\\ \dot{y} &= x(-1 + Cy)(-1 + x^2 + Cy^2).\end{aligned}$$

By elementary integration, the above system in $\Omega = \{(x,y) \colon x^2 + Cy^2 < 1\}$ is topologically equivalent to the system

$$\begin{aligned}\dot{x} &= y(1 + Bx),\\ \dot{y} &= x(-1 + Cy),\end{aligned}$$

where $H(x,y) = BC(By - Cx) + B^2\ln(1 - Cy) + C^2\ln(1 + Bx)$ is the first analytic integral in a neighborhood of $(0,0)$ small enough and $(0,0)$ is a local maximum of H. Then, in this neighborhood the orbits of the system lie on the closed level curves of H, and this proves the existence of the center.

If $e = 0$, $c = 0$, then the system (3) is given by

$$\dot{x} = y(-1 + x)(1 + x)(1 + Bx),$$
$$\dot{y} = x(1 - x^2(1 + d)).$$

As $P(x, -y) = -P(x, y)$ and $Q(x, -y) = Q(x, y)$, the system is symmetric in the x-axis. Hence, $(0, 0)$ is a center for (3). □

Theorem. *In the parameter space $\mathbb{R}^5$, there exist three sub-manifolds $\mathscr{V}_i$, $i = 1, 2, 3$ of dimension four such that:*

i) *If $\theta \in \mathscr{V}_1$ and $C = 0$, then the system (3) has two real invariant straight lines and at least one small amplitude limit cycle, repelling if $Be > 0$ and attracting if $Be < 0$ (see Fig. 1).*

ii) *If $\theta \in \mathscr{V}_2$ and $C = -\frac{1}{\sqrt{2}}$, the system (3) has an invariant hyperbola with at least* **five** *small amplitude limit cycles (see Fig. 1).*

iii) *If $\theta \in \mathscr{V}_3$ and $C = \frac{1}{\sqrt{2}}$, the system (3) has a large elliptic limit cycle which enclosures at least* **four** *small amplitude limit cycles (see Fig. 1).*

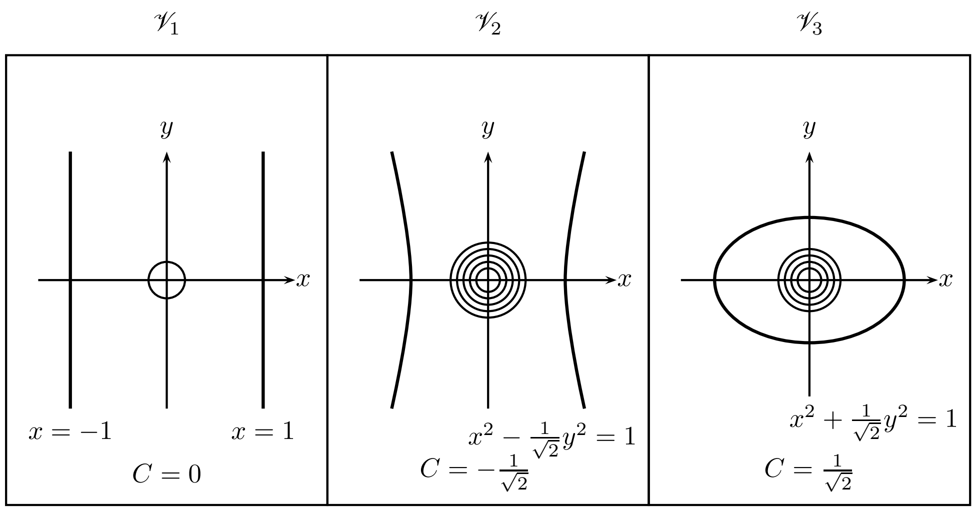

Figure 1. Coexistence Diagram.

P r o o f. i) For $C = 0$ the system (3) becomes

$$\dot{x} = (-1 + x)(1 + x)(-(\lambda x) + y + Bxy + ey^2),$$
$$\dot{y} = x - x^3 + Dx^3 + \lambda y - \lambda x^2 y.$$

By Lemma 1, $x = \pm 1$ are two real invariant straight lines of (3). If $\lambda = 0$, then the first focal values are given by $L(0) = 0$ and $L(1) = Be$. Hence then the singularity at the origin is a weak focus of order one. First, we perturb λ, so that $L(0)$ becomes non-zero and of opposite sign to $L(1)$. Then there exists a sub-manifold $\mathscr{V}_1$ of dimension four, such that if $\theta \in \mathscr{V}_1$, then the system (3) has one hyperbolic small amplitude limit cycle created by the Hopf bifurcations (see Fig. 1).

ii) Let $C = -\frac{1}{\sqrt{2}}$. By Lemma 1, the conic $x^2 - \frac{1}{\sqrt{2}}y^2 = 1$ is an invariant hyperbola for (3). Let $C = c\frac{1}{\sqrt{2}}$, $B = b\sqrt{\frac{\mp 7\sqrt{2}+36}{2}}$, and $D = \frac{\mp 7+9\sqrt{2}}{6}d$. By Lemma 2, there exist $b = b_{\mp}$, such that at the origin the system (3) has a weak focus of order **five** repelling if $e = 0$, $c = -1$, $1 + b^2 - d = 0$, and $b = b_-$, or attracting if $e = 0$, $c = -1$, $1 + b^2 - d = 0$ and $b = b_+$.

Considering the Liapunov quantities $L(k)$ as in Lemma 2 and perturbing the parameters turn by turn, at each stage ensuring that $L(k)L(k+1) < 0$, $k = 1, 2, 3, 4$, and $L(j) = 0$ for $j = 0, 1, \dots, k-1$. We finally perturb λ so that $L(0)$ becomes non-zero and of opposite sign to $L(1)$. There exists a sub-manifold $\mathscr{V}_2$ of dimension four such that if $\theta \in \mathscr{V}_2$, then the system (3) has **five** hyperbolic small amplitude limit cycles created by the Hopf bifurcations (see Fig. 1).

iii) Let $C = \frac{1}{\sqrt{2}}$. By Lemma 1, the conic $x^2 + \frac{1}{\sqrt{2}}y^2 = 1$ is an invariant ellipse of (3). Let $C = c\frac{1}{\sqrt{2}}$, $B = b\sqrt{\frac{\mp 7\sqrt{2}+36}{2}}$ and $D = \frac{\mp 7+9\sqrt{2}}{6}d$. By Lemma 2, the origin is a weak focus of order **four**, repelling if $e = 0$, $c = 1$, $1 + b^2 + d = 0$ and $bd > 0$, or attracting, if $e = 0$, $c = 1$, $1 + b^2 + d = 0$ and $bd < 0$. Perturbing the focal values as in ii), there exists a sub-manifold $\mathscr{V}_3$ of dimension four, such that if $\theta \in \mathscr{V}_3$, then the system (3) has a large elliptical limit cycle that contains **four** hyperbolic small amplitude limit cycles created by Hopf bifurcations (see Fig. 1). □

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