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EXISTENCE AND ITERATION OF POSITIVE SOLUTIONS FOR A SINGULAR TWO-POINT BOUNDARY VALUE PROBLEM WITH A p-LAPLACIAN OPERATOR

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Abstract. In the paper, we obtain the existence of symmetric or monotone positive solutions and establish a corresponding iterative scheme for the equation $(\varphi_p(u'))' + q(t)f(u) = 0$, $0 < t < 1$, where $\varphi_p(s) := |s|^{p-2}s$, $p > 1$, subject to nonlinear boundary condition. The main tool is the monotone iterative technique. Here, the coefficient $q(t)$ may be singular at $t = 0, 1$.

Keywords: iteration, symmetric and monotone positive solution, nonlinear boundary value problem, p-Laplacian

MSC 2000: 34B10, 34B15

1. Introduction

The purpose of this paper is to consider the existence of symmetric or monotone positive solutions and establish a corresponding iterative scheme for the nonlinear two-point singular boundary value problem (BVP) with a p-Laplacian operator

$$(\varphi_p(u'))' + q(t)f(u) = 0, \quad 0 < t < 1, \tag{1}$$

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subject to

$$
\text{(BC)} \qquad \begin{cases} \text{(a)} \quad u(0) - B_0(u'(0)) = 0, & u(1) + B_1(u'(1)) = 0, \\ \text{(b)} \quad u(0) - B(u'(0)) = 0, & u(1) + B(u'(1)) = 0, \\ \text{(c)} \quad u(0) - B(u'(0)) = 0, & u'(1) = 0, \\ \text{(d)} \quad u'(0) = 0, & u(1) + B(u'(1)) = 0, \end{cases}
$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, B_0, B_1 and B are both continuous functions defined on $(-\infty, +\infty)$, and the coefficient $q(t)$ may be singular at $t = 0, 1$.

Several papers have been devoted in the recent years to the study of (1) subject to different linear or nonlinear boundary conditions, see [1], [2], [4], [6], [7] and their references. Here, only positive solutions are meaningful. By using the fixed point theorem in cones due to Krasnoselskii [3], Wang [1] and Kong and Wang [2] established the existence of one positive solution for (1) subject to one of the following nonlinear boundary conditions:

$$
\text{(w1)} \qquad u(0) - g_1(u'(0)) = 0, \qquad u(1) + g_2(u'(1)) = 0,
$$

$$
\text{(w2)} \qquad u(0) - g_1(u'(0)) = 0, \qquad u'(1) = 0,
$$

$$
\text{(w3)} \qquad u'(0) = 0, \qquad u(1) + g_2(u'(1)) = 0.
$$

By applying a new twin fixed point theorem due to Avery and Henderson [5], He and Ge [4] obtained the existence of two positive solutions for (1) subject to (w1), (w2), (w3). By using the fixed point theorem in cones due to Krasnoselskii [3], R. P. Agarwal, Haishen Lü and D. O'Regan [6] studied the problem of eigenvalues of (1) subject to (BCa) when $B_0 = B_1 = 0$, they also obtained the existence of two positive solutions. By using an extension of the Leggett-Williams theorem, i.e., the fixed point theorem of five functionals, Guo and Ge [7] got the existence of three positive solutions for (1) subject to (w1), (w2), (w3), and

$$
\text{(g1)} \qquad u(0) = 0, \qquad u(1) = 0.
$$

We can see easily that all the results obtained in [1], [2], [4], [6], [7] are the existence of positive solutions. Seeing such a fact, we cannot but ask *"how can we find the solutions since the solutions exist definitely?"* Motivated by the above-mentioned results, in this paper, by improving the classical monotone iterative technique of Amann [8], we obtain not only the existence of positive solutions, but also give an iterative scheme for approximating the solutions. It is worth stating that the first term of our iterative scheme is a constant function or a simple function. Therefore, the iterative scheme is significant and feasible. At the same time, we give a way to find the solution which will be useful from an application viewpoint.

We consider the Banach space $E = C[0,1]$ equipped with norm $\|w\| = \max\limits_{0\leqslant t\leqslant 1} |w(t)|$. In this paper, a positive solution w^* of (1) and (BC) means a solution w^* of (1), (BC) satisfying $w^*(t) > 0$, $0 < t < 1$. A symmetric solution w^* of (1), (BC) means a solution w^* of (1), (BC) satisfying $w^*(t) = w^*(1-t)$, $0 \leqslant t \leqslant 1$. A monotone solution w^* of (1), (BC) means a solution w^* of (1), (BC) such that $w(t)$ is nondecreasing or $w(t)$ is nonincreasing.

We list the following conditions for convenience:

(H1) $f \in C([0,+\infty),[0,+\infty))$;

(H2) $q(t)$ is a nonnegative measurable function defined on $(0,1)$, and $q(t)$ is not identically zero on any compact subinterval of $(0,1)$. Furthermore, $q(t)$ satisfies

$$0 < \int_0^1 q(t)\,\mathrm{d}t < +\infty;$$

(H3a) $B_0(v)$ and $B_1(v)$ are both nondecreasing, continuous, odd functions defined on $(-\infty,+\infty)$ and at least one of them satisfies the condition that there exists $m > 0$ such that

$$0 \leqslant B_i(v) \leqslant mv \quad \text{for all } v \geqslant 0,\ \ i = 0 \text{ or } 1;$$

(H3b) $B(v)$ is a nondecreasing, continuous, odd function defined on $(-\infty,+\infty)$ and there exists $m > 0$ such that

$$0 \leqslant B(v) \leqslant mv \quad \text{for all } v \geqslant 0.$$

2. Some background definitions and two lemmas

In this section, we suppose that (H1), (H2) and (H3a) hold.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leqslant$, i.e.,

$$x \leqslant y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2.1. Given a cone P in E, a functional $\psi\colon P \to \mathbb{R}$ is said to be nondecreasing on P, provided $\psi(x) \leqslant \psi(y)$, for all $x, y \in P$ with $x \leqslant y$.

Definition 2.2. A functional $\psi\colon [0,1] \to \mathbb{R}$ is said to be concave on $[0,1]$, provided $\psi(tx + (1-t)y) \geqslant t\psi(x) + (1-t)\psi(y)$, for all $x, y \in [0,1]$ and $t \in [0,1]$.

Lemma 2.1. *For any fixed $k_1 \geqslant 0$, $k_2 \geqslant 0$, if $w_i(t) \in E$, $i = 1, 2$ satisfy*

$$\begin{cases} -(\varphi_p(w_2'(t)))' + (\varphi_p(w_1'(t)))' \geqslant 0, \quad 0 \leqslant t \leqslant 1, \\ k_1 w_2(0) - B_0 w_2'(0) = 0, \quad k_2 w_2(1) + B_1 w_2'(1) = 0, \\ k_1 w_1(0) - B_0 w_1'(0) = 0, \quad k_2 w_1(1) + B_1 w_1'(1) = 0, \end{cases}$$

then $w_2(t) \geqslant w_1(t)$, $0 \leqslant t \leqslant 1$.

Proof. Suppose not, then there exists $t_0 \in [0,1]$ such that $(w_2 - w_1)(t_0) = \min\limits_{0 \leqslant t \leqslant 1} (w_2 - w_1)(t) < 0$. We now show that $(w_2 - w_1)'(t_0) = 0$. In fact, if $t_0 = 0$, then from the definition of minimum, $(w_2 - w_1)'(0) \geqslant 0$, that is $w_2'(0) \geqslant w_1'(0)$. On the other hand, $B_0(w_2'(0)) - B_0(w_1'(0)) = k_1(w_2(0) - w_1(0)) \leqslant 0$, so $w_2'(0) \leqslant w_1'(0)$ since $B_0(v)$ is nondecreasing. Thus, $(w_2 - w_1)'(0) = 0$. If $t_0 = 1$, similarly to $t_0 = 0$, we can also prove that $(w_2 - w_1)'(1) = 0$. If $t_0 \in (0,1)$, then $(w_2 - w_1)'(t_0) = 0$ certainly.

Since $(\varphi_p(w_1') - \varphi_p(w_2'))' \geqslant 0$ and $\varphi_p(w_1'(t_0)) - \varphi_p(w_2'(t_0)) = 0$, we have

$$\varphi_p(w_1'(t)) - \varphi_p(w_2'(t)) \geqslant 0, \ \ t \in [t_0, 1] \quad \text{and} \quad \varphi_p(w_1'(t)) - \varphi_p(w_2'(t)) \leqslant 0, \ \ t \in [0, t_0].$$

So

$$w_1'(t) - w_2'(t) \geqslant 0, \ \ t \in [t_0, 1] \quad \text{and} \quad w_1'(t) - w_2'(t) \leqslant 0, \ \ t \in [0, t_0].$$

Thus, when $t \in [t_0, 1]$, $k_2(w_1 - w_2)(t) \leqslant k_2(w_1 - w_2)(1) = B_1(w_2'(1)) - B_1(w_1'(1)) \leqslant 0$, i.e., $w_1(t) \leqslant w_2(t)$ and when $t \in [0, t_0]$, $k_1(w_1 - w_2)(t) \leqslant k_1(w_1 - w_2)(0) = -B_0(w_2'(0)) + B_0(w_1'(0)) \leqslant 0$, i.e., $w_1(t) \leqslant w_2(t)$. So, for any $t \in [0,1]$, $w_2(t) - w_1(t) \geqslant 0$, which is a contradiction to the assumption. Therefore, $w_2(t) \geqslant w_1(t)$, $0 \leqslant t \leqslant 1$ and the Lemma is proved. $\square$

3. Existence and iteration of solution of (1), (BCa)

In this section, we suppose that (H1), (H2) and (H3a) hold.

Define

$$P_a = \{u \in E \colon \ u(t) \text{ is a nonnegative, concave function}\}.$$

Then P_a is an cone in E. For each $u \in P_a$, we define an operator $T \colon \ P_a \to E$ by

$$(2) \qquad (T_a u)(t) := \begin{cases} B_0 \varphi_p^{-1}\big(\int_0^\sigma q(s) f(u(s)) \,\mathrm{d}s\big) + \int_0^t \varphi_p^{-1}\big(\int_s^\sigma q(\tau) f(u(\tau)) \,\mathrm{d}\tau\big) \,\mathrm{d}s, \\ \quad 0 \leqslant t \leqslant \sigma, \\ B_1 \varphi_p^{-1}\big(\int_\sigma^1 q(s) f(u(s)) \,\mathrm{d}s\big) + \int_t^1 \varphi_p^{-1}\big(\int_\sigma^s q(\tau) f(u(\tau)) \,\mathrm{d}\tau\big) \,\mathrm{d}s, \\ \quad \sigma \leqslant t \leqslant 1, \end{cases}$$

where $\sigma = 0$ if $(Tu)'(0) = 0$, and $\sigma = 1$ if $(Tu)'(1) = 0$; otherwise, σ is a solution of the equation

$$z(x) := v_1(x) - v_2(x) = 0, \tag{3}$$

where

$$v_1(x) = B_0\varphi_p^{-1}\Bigl(\int_0^x q(s)f(u(s))\,\mathrm{d}s\Bigr) + \int_0^x \varphi_p^{-1}\Bigl(\int_s^x q(\tau)f(u(\tau))\,\mathrm{d}\tau\Bigr)\,\mathrm{d}s,\quad 0 \leqslant x \leqslant 1,$$

and

$$v_2(x) = B_1\varphi_p^{-1}\Bigl(\int_x^1 q(s)f(u(s))\,\mathrm{d}s\Bigr) + \int_x^1 \varphi_p^{-1}\Bigl(\int_x^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Bigr)\,\mathrm{d}s,\quad 0 \leqslant x \leqslant 1.$$

Note that $z(x)$ is a strictly increasing continuous function defined on $[0,1]$ with $z(0) < 0$ and $z(1) > 0$, and hence there exists a unique $\sigma \in (0,1)$ which satisfies (3). The operator T_a is thus well defined.

Just as proved in [1] or [6], a standard argument shows the following result.

Lemma 3.1. *$T\colon P_a \to P_a$ is continuous and compact. Furthermore, each fixed point of T_a is a nonnegative, concave solution of* (1) *and* (BCa)*.*

Lemma 3.2 ([6])**.** *For any $0 < \delta < \frac{1}{2}$, $u \in P_a$ has the following properties:*
(a) *$u(t) \geqslant \|u\|t(1-t)$ for all $t \in [0,1]$.*
(b) *$u(t) \geqslant \delta^2\|u\|$ for all $t \in [\delta, 1-\delta]$.*

For any $0 < \delta < \frac{1}{2}$, define

$$y(x) = \int_\delta^x \varphi_p^{-1}\Bigl(\int_s^x q(\tau)\,\mathrm{d}\tau\Bigr)\,\mathrm{d}s + \int_x^{1-\delta} \varphi_p^{-1}\Bigl(\int_x^s q(\tau)\,\mathrm{d}\tau\Bigr)\,\mathrm{d}s,\quad x \in [\delta, 1-\delta].$$

Then $y(x)$ is continuous and positive on $[\delta, 1-\delta]$.

Theorem 3.1. *Assume* (H1), (H2) *and* (H3a) *hold. If there exist $\delta \in (0, \frac{1}{2})$ and two positive numbers $b < a$ such that*

(C1) *$f\colon [0,a] \to [0,+\infty)$ is nondecreasing;*
(C2) *$f(\delta^2 b) \geqslant (bB)^{p-1}$, $f(a) \leqslant (aA)^{p-1}$, where $B = 2/\Gamma$ and $\Gamma = \min\limits_{x\in[\delta,1-\delta]} y(x) > 0$; $A = 1/(m+1)\varphi_p^{-1}(\int_0^1 q(\tau)\,\mathrm{d}\tau)$,*

then (1), (BCa) *has one positive solution* $w^* \in P_a$ *with* $b \leqslant \|w^*\| \leqslant a$ *and* $\lim\limits_{n\to+\infty} T_a^n w_0 = w^*$, *where* $w_0(t) = a$, $t \in [0,1]$.

P r o o f. We denote $P_a[b,a] = \{w \in P_a \colon b \leqslant \|w\| \leqslant a\}$. In what follows, we first prove that $T_a P_a[b,a] \subset P_a[b,a]$.

Let $w \in P_a[b,a]$, then $0 \leqslant w(t) \leqslant \max\limits_{t\in[0,1]} w(t) = \|w\| \leqslant a$. By Lemma 3.2, $\min\limits_{t\in[\delta,1-\delta]} w(t) \geqslant \delta^2\|w\| \geqslant \delta^2 b$. So, by the assumptions $(C1)$ and $(C2)$, we have

$$0 \leqslant f(w(t)) \leqslant f(a) \leqslant (aA)^{p-1}, \quad t \in [0,1], \tag{4}$$

$$f(w(t)) \geqslant f(\delta^2 b) \geqslant (bB)^{p-1}, \quad t \in [\delta, 1-\delta]. \tag{5}$$

Therefore, for $w(t) \in P_a[b,a]$, on the one hand, by (5) we have, when $\sigma \in [\delta, 1-\delta]$,

$$\begin{aligned}
2\|T_a w\| &= 2T_a w(\sigma) \\
&= B_0\varphi_p^{-1}\Big(\int_0^\sigma q(s)f(u(s))\,\mathrm{d}s\Big) + \int_0^\sigma \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\quad + B_1\varphi_p^{-1}\Big(\int_\sigma^1 q(s)f(u(s))\,\mathrm{d}s\Big) + \int_\sigma^1 \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant \int_0^\sigma \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_\sigma^1 \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant \int_\delta^\sigma \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_\sigma^{1-\delta} \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant bB\Big[\int_\delta^\sigma \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_\sigma^{1-\delta} \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big] \\
&= bBy(\sigma) \geqslant bB\Gamma = 2b,
\end{aligned}$$

thus, $\|T_a w\| \geqslant b$; when $\sigma \in [0,\delta]$,

$$\begin{aligned}
\|T_a w\| &\geqslant (T_a w)(\delta) \\
&= B_1\varphi_p^{-1}\Big(\int_\sigma^1 q(s)f(u(s))\,\mathrm{d}s\Big) + \int_\delta^1 \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant \int_\delta^1 \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \geqslant \int_\delta^{1-\delta} \varphi_p^{-1}\Big(\int_\delta^s q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= bBy(\delta) \geqslant bB\Gamma = 2b \geqslant b;
\end{aligned}$$

and when $\sigma \in [1-\delta, 1]$,

$$\begin{aligned}
\|T_a w\| &\geqslant (T_a w)(1-\delta) \\
&= B_0 \varphi_p^{-1}\Big(\int_0^\sigma q(s) f(u(s))\,\mathrm{d}s\Big) + \int_0^{1-\delta} \varphi_p^{-1}\Big(\int_s^\sigma q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant \int_0^{1-\delta} \varphi_p^{-1}\Big(\int_s^\sigma q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \geqslant \int_\delta^{1-\delta} \varphi_p^{-1}\Big(\int_s^{1-\delta} q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= bBy(1-\delta) \geqslant bB\Gamma = 2b \geqslant b.
\end{aligned}$$

On the other hand, by (H3a) and (4) we see that at least one of the following holds,

$$\begin{aligned}
\|T_a w\| &= \|T_a w(\sigma)\| \\
&= B_0 \varphi_p^{-1}\Big(\int_0^\sigma q(s) f(u(s))\,\mathrm{d}s\Big) + \int_0^\sigma \varphi_p^{-1}\Big(\int_s^\sigma q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\leqslant B_0 \varphi_p^{-1}\Big(\int_0^1 q(s) f(u(s))\,\mathrm{d}s\Big) + \varphi_p^{-1}\Big(\int_0^1 q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big) \\
&\leqslant (m+1) a A \varphi_p^{-1}\Big(\int_0^1 q(\tau)\,\mathrm{d}\tau\Big) = a
\end{aligned}$$

or

$$\begin{aligned}
\|T_a w\| &= \|T_a w(\sigma)\| \\
&= B_1 \varphi_p^{-1}\Big(\int_\sigma^1 q(s) f(u(s))\,\mathrm{d}s\Big) + \int_\sigma^1 \varphi_p^{-1}\Big(\int_\sigma^s q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\leqslant B_1 \varphi_p^{-1}\Big(\int_0^1 q(s) f(u(s))\,\mathrm{d}s\Big) + \varphi_p^{-1}\Big(\int_0^1 q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big) \\
&\leqslant (m+1) a A \varphi_p^{-1}\Big(\int_0^1 q(\tau)\,\mathrm{d}\tau\Big) = a.
\end{aligned}$$

Altogether, we get $b \leqslant \|T_a w\| \leqslant a$ for $w \in P_a[b,a]$, which means that $T_a P_a[b,a] \subset P_a[b,a]$.

Let $w_0(t) = a$, $t \in [0,1]$, then $w_0(t) \in P_a[b,a]$. Let $w_1 = T_a w_0$, then $w_1 \in P_a[b,a]$; we denote

$$w_{n+1} = T w_n = T^{n+1} w_0 \quad (n = 1, 2, \ldots). \tag{6}$$

Since $T_a P_a[b,a] \subset P_a[b,a]$, we have $w_n \in P_a[b,a]$, $(n = 0, 1, 2, \ldots)$. From Lemma 3.1, T_a is compact, so $\{w_n\}_{n=1}^\infty$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^\infty$ and there exists $w^* \in P_a[b,a]$, such that $w_{n_k} \to w^*$.

Now, since $w_1 \in P_a[b, a]$, we have

$$0 \leqslant w_1(t) \leqslant \|w_1\| \leqslant a = w_0(t);$$

by the definition of w_1 and w_2, we have

$$\begin{cases} (\varphi_p(w_1'))' + q(t)f(w_0(t)) = 0, & 0 < t < 1 \\ w_1(0) - B_0(w_1'(0)) = 0, & w_1(1) + B_1(w_1'(1)) = 0, \end{cases} \tag{7}$$

$$\begin{cases} (\varphi_p(w_2'))' + q(t)f(w_1(t)) = 0, & 0 < t < 1 \\ w_2(0) - B_0(w_2'(0)) = 0, & w_2(1) + B_1(w_2'(1)) = 0. \end{cases} \tag{8}$$

Combining (7), (8), the fact that $f\colon [0, a] \to [0, +\infty)$ is nondecreasing, and $w_1 \leqslant w_0$, we get

$$\begin{cases} -(\varphi_p(w_1'(t)))' + (\varphi_p(w_2'(t)))' \geqslant 0, & 0 \leqslant t \leqslant 1, \\ w_2(0) - B_0 w_2'(0) = 0, & w_2(1) + B_1 w_2'(1) = 0, \\ w_1(0) - B_0 w_1'(0) = 0, & w_1(1) + B_1 w_1'(1) = 0. \end{cases}$$

By Lemma 2.1 (let $k_1 = k_2 = 1$), $w_1(t) \geqslant w_2(t)$, $0 \leqslant t \leqslant 1$.

By induction, then $w_n(t) \geqslant w_{n+1}(t)$, $0 \leqslant t \leqslant 1$ ($n = 0, 1, 2, \ldots$). Hence, we see that $w_n \to w^*$. Letting $n \to \infty$ in (6), we obtain $T_a w^* = w^*$ since T_a is continuous. Since $\|w^*\| \geqslant b > 0$ and w^* is a nonnegative concave function on $[0, 1]$, we conclude that $w^*(t) > 0$, $t \in (0, 1)$. Therefore, w^* is a positive solution of (1) and (BCa). $\square$

Corollary 3.1. *Assume* (H1), (H2) *and* (H3a) *hold. If there exists a* $\delta \in (0, \frac{1}{2})$ *such that*

(C1′) $f\colon [0, +\infty) \to [0, +\infty)$ *is nondecreasing;*

(C2′) $\overline{\lim\limits_{l\to 0}} f(l)/l^{p-1} > (B/\delta^2)^{p-1}$ *and* $\underline{\lim\limits_{l\to+\infty}} f(l)/l^{p-1} < A^{p-1}$ *(in particular,* $\lim\limits_{l\to 0} f(l)/l^{p-1} = +\infty$ *and* $\lim\limits_{l\to+\infty} f(l)/l^{p-1} = 0$*), where* A, B *are defined as in Theorem* 3.1,

then there exist two constants $a > 0$ *and* $b > 0$ *such that* (1) *and* (BCa) *has one positive solution* $w^* \in P_a$ *with* $b \leqslant \|w^*\| \leqslant a$ *and* $\lim\limits_{n\to+\infty} (T_a)^n w_0 = w^*$, *where* $w_0(t) = a$, $t \in [0, 1]$.

If $B_0 = B_1$ in (BCa), then we have an even better result, which we will give in the next section.

4. Existence and iteration of symmetric solution of (1), (BCb)

In this section, we suppose that (H1), (H2), (H3b) hold, and $h(1-t) = h(t)$, $t \in (0,1)$.

Define

$$P_b = \{u \in E\colon\ u(t) \text{ is nonnegative, symmetric and concave on } [0,1]\}.$$

Then P_b is a cone in E.

Lemma 4.1. *For any $0 < \delta < \frac{1}{2}$, $u \in P_b$ has the following properties:*
(a) $u(t) \geqslant 2\|u\| \min\{t, 1-t\}$ *for all* $t \in [0,1]$.
(b) $u(t) \geqslant 2\delta\|u\|$ *for all* $t \in [\delta, 1-\delta]$.

Proof. By the concavity and symmetry of u, (a) is very easy to prove. (b) follows easily from (a). □

When $q(t) = q(1-t)$, $B_0 = B_1 = B$, for any $u \in P_b$, if we choose $x = \frac{1}{2}$ in (3), then

$$\begin{aligned}(9)\quad z\Big(\frac{1}{2}\Big) &= v_1\Big(\frac{1}{2}\Big) - v_2\Big(\frac{1}{2}\Big)\\ &= B\varphi_p^{-1}\Big(\int_0^{1/2} q(s)f(u(s))\,\mathrm{d}s\Big) + \int_0^{1/2} \varphi_p^{-1}\Big(\int_s^{1/2} q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ &\quad - B\varphi_p^{-1}\Big(\int_{1/2}^{1} q(s)f(u(s))\,\mathrm{d}s\Big) - \int_{1/2}^{1} \varphi_p^{-1}\Big(\int_{1/2}^{s} q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s = 0.\end{aligned}$$

Thus, for any $u \in E$, we may define

$$(10)\qquad (T_b u)(t) := \begin{cases} B\Big(\varphi_p^{-1}\Big(\displaystyle\int_0^{1/2} q(s)f(u(s))\,\mathrm{d}s\Big)\Big) \\ \quad + \displaystyle\int_0^{t} \varphi_p^{-1}\Big(\int_s^{1/2} q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & 0 \leqslant t \leqslant \dfrac{1}{2},\\ B\Big(\varphi_p^{-1}\Big(\displaystyle\int_{1/2}^{1} q(s)f(u(s))\,\mathrm{d}s\Big)\Big) \\ \quad + \displaystyle\int_t^{1} \varphi_p^{-1}\Big(\int_{1/2}^{s} q(\tau)f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & \dfrac{1}{2} \leqslant t \leqslant 1,\end{cases}$$

and we have

Lemma 4.2. *$T_b\colon P_b \to P_b$ is continuous and compact. Furthermore, each fixed point of T_b in P_b is a nonnegative, symmetric and concave solution of* (1) *and* (BCb).

Proof. By Lemma 3.1, we only need to prove that $T_b u$ is symmetric for any $u \in P_b$.

Indeed, for any $u \in P_b$, since $q(t)$ and $u(t)$ are symmetric, when $t \in [0, \frac{1}{2}]$,

$$\begin{aligned} T_b u(1-t) &= B\Big(\varphi_p^{-1}\Big(\int_{1/2}^{1} q(s) f(u(s))\,\mathrm{d}s\Big)\Big) + \int_{1-t}^{1} \varphi_p^{-1}\Big(\int_{1/2}^{s} q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &= B\Big(\varphi_p^{-1}\Big(\int_{0}^{1/2} q(s) f(u(s))\,\mathrm{d}s\Big)\Big) + \int_{0}^{t} \varphi_p^{-1}\Big(\int_{s}^{1/2} q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &= T_b u(t), \end{aligned}$$

and when $t \in [\frac{1}{2}, 1]$,

$$\begin{aligned} T_b u(1-t) &= B\Big(\varphi_p^{-1}\Big(\int_{0}^{1/2} q(s) f(u(s))\,\mathrm{d}s\Big)\Big) + \int_{0}^{1-t} \varphi_p^{-1}\Big(\int_{s}^{1/2} q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &= B\Big(\varphi_p^{-1}\Big(\int_{1/2}^{1} q(s) f(u(s))\,\mathrm{d}s\Big)\Big) + \int_{t}^{1} \varphi_p^{-1}\Big(\int_{1/2}^{s} q(\tau) f(u(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &= T_b u(t). \end{aligned}$$

Therefore T_b leaves invariant the cone P_b. □

On the other hand, (H2) implies that $0 < \int_0^{1/2} q(r)\,\mathrm{d}r < +\infty$ and for any $0 < \delta < \frac{1}{2}$,

$$\int_{\delta}^{1/2} \varphi_p^{-1}\Big(\int_{s}^{1/2} q(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s > 0.$$

Theorem 4.1. *Assume that* (H1), (H2), (H3b) *hold, and* $h(1-t) = h(t)$, $t \in (0,1)$. *If there exist $\delta \in (0, \frac{1}{2})$ and two positive numbers $b < a$ such that*

(C3) *$f\colon [0,a] \to [0, +\infty)$ is nondecreasing;*

(C4) *$f(2\delta b) \geqslant (bB)^{p-1}$, $f(a) \leqslant (aA)^{p-1}$,*
where $B = 1/\int_{\delta}^{1/2} \varphi_p^{-1}(\int_s^{1/2} q(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s$, $A = 1/(m+\frac{1}{2})\varphi_p^{-1}(\int_0^{1/2} q(r)\,\mathrm{d}r)$,

then (1), (BCb) *has at least two positive, concave and symmetric solutions $w^*, v^* \in P_b$ with*

$$b \leqslant \|w^*\| \leqslant a \ \text{ and } \lim_{n\to+\infty} (T_b)^n w_0 = w^*,$$

where $w_0(t) = a$, $t \in [0,1]$,

$$b \leqslant \|v^*\| \leqslant a \ \text{ and } \lim_{n\to+\infty} (T_b)^n v_0 = v^*,$$

where $v_0(t) = 2b\min\{t, 1-t\}$, $t \in [0,1]$.

P r o o f. We denote $P_b[b,a]=\{w\in P_b\colon\, b\leqslant\|w\|\leqslant a\}$. We first prove that $T_bP_b[b,a]\subset P_b[b,a]$.

Let $w\in P_b[b,a]$, then $0\leqslant w(t)\leqslant \max\limits_{t\in[0,1]} w(t)=\|w\|\leqslant a$. By Lemma 4.1, $\min\limits_{t\in[\delta,1-\delta]} w(t)\geqslant 2\delta\|w\|\geqslant 2\delta b$. So, by the assumptions $(C3)$ and $(C4)$, we have

$$0\leqslant f(w(t))\leqslant f(a)\leqslant (aA)^{p-1},\quad t\in[0,1], \tag{11}$$

$$f(w(t))\geqslant f(2\delta b)\geqslant (bB)^{p-1},\quad t\in[\delta,1-\delta]. \tag{12}$$

Thus, for any $w(t)\in P_b[b,a]$,

$$\begin{aligned}
\|T_bw\| &= T_bw\Big(\frac{1}{2}\Big)\\
&= B\varphi_p^{-1}\Big(\int_0^{1/2} q(s)f(w(s))\,\mathrm{d}s\Big)+\int_0^{1/2}\varphi_p^{-1}\Big(\int_s^{12} q(\tau)f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geqslant \int_\delta^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2} q(\tau)f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geqslant bB\Big[\int_\delta^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2} q(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big]=b,\\
\|T_bw\| &= T_bw\Big(\frac{1}{2}\Big)\\
&= B\Big(\varphi_p^{-1}\Big(\int_0^{1/2} q(s)f(w(s))\,\mathrm{d}s\Big)\Big)+\int_0^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2} q(\tau)f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\leqslant m\varphi_p^{-1}\Big(\int_0^{1/2} q(s)f(w(s))\,\mathrm{d}s\Big)+\frac{1}{2}\varphi_p^{-1}\Big(\int_0^{1/2} q(\tau)f(w(\tau))\,\mathrm{d}\tau\Big)\\
&= \Big(m+\frac{1}{2}\Big)\varphi_p^{-1}\Big(\int_0^{1/2} q(\tau)f(w(\tau))\,\mathrm{d}\tau\Big)\\
&\leqslant \Big(m+\frac{1}{2}\Big)aA\varphi_p^{-1}\Big(\int_0^{1/2} q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big)=a.
\end{aligned}$$

Altogether, we get $b\leqslant\|T_bw\|\leqslant a$ for $w\in P_b[b,a]$, which means that $T_bP_b[b,a]\subset P_b[b,a]$.

Let $w_0(t)=a$, $t\in[0,1]$, then $w_0(t)\in P_a[b,a]$. Let $w_1=T_bw_0$, then $w_1\in P_a[b,a]$; we denote

$$w_{n+1}=T_bw_n=T_b^{n+1}w_0\quad(n=1,2,\ldots). \tag{13}$$

Since $T_bP_a[b,a]\subset P_a[b,a]$, we have $w_n\in P_a[b,a]$, $(n=0,1,2,\ldots)$. From Lemma 4.1, T_b is compact, so $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^*\in P_b[b,a]$, such that $w_{n_k}\to w^*$.

Now, since $w_1 \in P_b[b,a]$, we have

$$0 \leqslant w_1(t) \leqslant \|w\| \leqslant a = w_0(t);$$

by the definition of w_1 and w_2, we have

$$\begin{cases} (\varphi_p(w_1'))' + q(t)f(w_0(t)) = 0, \quad 0 < t < 1 \\ w_1(0) - B(w_1'(0)) = 0, \quad w_1(1) + B(w_1'(1)) = 0, \end{cases} \tag{14}$$

$$\begin{cases} (\varphi_p(w_2'))' + q(t)f(w_1(t)) = 0, \quad 0 < t < 1 \\ w_2(0) - B(w_2'(0)) = 0, \quad w_2(1) + B(w_2'(1)) = 0. \end{cases} \tag{15}$$

Combining (14), (15), the fact that $f\colon\ [0,a] \to [0,+\infty)$ is nondecreasing, and $w_1 \leqslant w_0$, we get

$$\begin{cases} -(\varphi_p(w_1'(t)))' + (\varphi_p(w_2'(t)))' \geqslant 0, \quad 0 \leqslant t \leqslant 1, \\ w_2(0) - Bw_2'(0) = 0, \quad w_2(1) + Bw_2'(1) = 0, \\ w_1(0) - Bw_1'(0) = 0, \quad w_1(1) + Bw_1'(1) = 0. \end{cases}$$

By Lemma 2.1 (let $k_1 = k_2 = 1$, $B_0 = B_1 = B$), $w_1(t) \geqslant w_2(t)$, $0 \leqslant t \leqslant 1$.

By induction, then $w_n(t) \geqslant w_{n+1}(t)$, $0 \leqslant t \leqslant 1$ $(n = 0,1,2,\ldots)$. Hence, we see that $w_n \to w^*$. Letting $n \to \infty$ in (13), we obtain $T_b w^* = w^*$ since T_b is continuous. Since $\|w^*\| \geqslant b > 0$ and w^* is a nonnegative concave function on $[0,1]$, we conclude that $w^*(t) > 0$, $t \in (0,1)$. Therefore, w^* is a positive, symmetric solution of (1) and (BCb).

Let $v_0(t) = 2b\min\{t, 1-t\}$, $t \in [0,1]$, then $\|v_0\| = b$, and $v_0(t) \in P_b[b,a]$. Let $v_1 = (T_b)v_0$, then $v_1 \in P_a[b,a]$; we denote

$$v_{n+1} = T_b v_n = (T_b)^{n+1} v_0 \quad (n = 1,2,\ldots).$$

Similarly to $\{w_n\}_{n=1}^{\infty}$, we see that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in P_b[b,a]$ such that $v_{n_k} \to v^*$.

Now, since $v_1 \in P_b[b,a]$, we have by Lemma 4.1,

$$v_1(t) \geqslant 2\|v_1\|\min\{t,1-t\} \geqslant 2b\min\{t,1-t\} = v_0(t).$$

Similarly to $\{w_n\}_{n=1}^{\infty}$, we can show easily that $v_n(t) \leqslant v_{n+1}(t)$, $0 \leqslant t \leqslant 1$ $(n = 0,1,2,\ldots)$. Hence, we see that $v_n \to v^*$, $T_b v^* = v^*$ and $v^*(t) > 0$, $t \in (0,1)$. Therefore, v^* is a positive, symmetric solution of (1) and (BCb). $\square$

Remark 4.1. We can easily get that w_* and v^* are the maximal and the minimal solution of (1) and (BCb) in $P_b[b,a]$.

Corollary 4.1. *Assume that* (H1), (H2) *and* (H3b) *hold, and* $h(1-t)=h(t)$, $t\in(0,1)$. *If there exists* $\delta\in(0,\frac{1}{2})$ *such that*

(C3′) $f\colon [0,+\infty)\to[0,+\infty)$ *is nondecreasing;*

(C4′) $\overline{\lim\limits_{l\to 0}} f(l)/l^{p-1} > (\frac{1}{2}B/\delta)^{p-1}$ *and* $\underline{\lim\limits_{l\to+\infty}}\, f(l)/l^{p-1} < (aA)^{p-1}$ *(in particular,* $\lim\limits_{l\to 0} f(l)/l^{p-1}=+\infty$ *and* $\lim\limits_{l\to+\infty} f(l)/l^{p-1}=0$*), where* A*,* B *are defined as in Theorem* 4.1*,*

then there exist two constants $a>0$ *and* $b>0$ *such that* (1) *and* (BCb) *has two positive, concave and symmetric solutions* $w^*, v^*\in P_b$ *with*

$$b\leqslant\|w^*\|\leqslant a \ \text{ and } \lim_{n\to+\infty}(T_b)^n w_0 = w^*, \quad \text{where } w_0(t)=a,\ t\in[0,1],$$

$$b\leqslant\|v^*\|\leqslant a \ \text{ and } \lim_{n\to+\infty}(T_b)^n v_0 = v^*, \quad \text{where } v_0(t)=2b\min\{t,1-t\},\ t\in[0,1].$$

5. Existence and iteration of monotone solution of (1), (BCc) and (1), (BCd)

In this section, we need (H1), (H2) and (H3b) to hold.

Define

(16) $P_c=\{u\in E\colon u(t)$ is nonnegative, nondecreasing and concave on $[0,1]\}$,

(17) $P_d=\{u\in E\colon u(t)$ is nonnegative, nonincreasing and concave on $[0,1]\}$.

Then P_c and P_d are two cones in E.

Lemma 5.1. *If* $u\in P_c$*, then* $u(t)\geqslant t\|u\|$ *for all* $t\in[0,1]$.

For any $u\in E$, define

$$(T_c u)(t) = B\biggl(\varphi_p^{-1}\biggl(\int_0^1 q(s)f(u(s))\,\mathrm{d}s\biggr)\biggr) + \int_0^t \varphi_p^{-1}\biggl(\int_s^1 q(\tau)f(u(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \quad t\in[0,1]. \tag{18}$$

Lemma 5.2. $T_c\colon P_c\to P_c$ *is continuous and compact. Furthermore, each fixed point of* T_c *in* P_c *is a nonnegative, nondecreasing and concave solution of* (1) *and* (BCc).

Proof. By Lemma 3.2, we only need to prove that $T_c u$ is nondecreasing for any $u\in P_c$ which is obvious by the definition of $(T_c u)(t)$. □

Theorem 5.1. *Assume* (H1), (H2) *and* (H3b) *hold. If there exist* $\delta \in (0,1)$ *and two positive numbers* $b < a$ *such that*

(C5) $f\colon [0,a] \to [0,+\infty)$ *is nondecreasing;*

(C6) $f(\delta b) \geqslant (bB)^{p-1}$, $f(a) \leqslant (aA)^{p-1}$, *where* $B = 1/\int_\delta^1 \varphi_p^{-1}(\int_s^1 q(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s$, $A = 1/(m+1)\varphi_p^{-1}(\int_0^1 q(r)\,\mathrm{d}r)$, *then* (1), (BCc) *has at least two positive and nondecreasing solutions* $w^*, v^* \in P_c$ *with*

$$b \leqslant \|w^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_c)^n w_0 = w^*, \quad \text{where } w_0(t) = a, \ t \in [0,1],$$
$$b \leqslant \|v^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_c)^n v_0 = v^*, \quad \text{where } v_0(t) = bt, \ t \in [0,1].$$

Proof. We denote $P_c[b,a] = \{w \in P_c\colon b \leqslant \|w\| \leqslant a\}$. We first prove that $T_c P_c[b,a] \subset P_c[b,a]$.

Let $w \in P_c[b,a]$, then $0 \leqslant w(t) \leqslant \max_{t\in[0,1]} w(t) = \|w\| \leqslant a$. By Lemma 5.1, $\min_{t\in[\delta,1]} w(t) \geqslant \delta\|w\| \geqslant \delta b$. So, by the assumptions $(C5)$ and $(C6)$, we have

$$0 \leqslant f(w(t)) \leqslant f(a) \leqslant (aA)^{p-1}, \quad t \in [0,1], \tag{19}$$
$$f(w(t)) \geqslant f(\delta b) \geqslant (bB)^{p-1}, \quad t \in [\delta,1]. \tag{20}$$

Thus, for any $w(t) \in P_c[b,a]$,

$$\begin{aligned}
\|T_c w\| &= T_c w(1) \\
&= B\varphi_p^{-1}\Big(\int_0^1 q(s) f(w(s))\,\mathrm{d}s\Big) + \int_0^1 \varphi_p^{-1}\Big(\int_s^1 q(\tau) f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant \int_\delta^1 \varphi_p^{-1}\Big(\int_s^1 q(\tau) f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\geqslant bB\int_\delta^1 \varphi_p^{-1}\Big(\int_s^1 q(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = b, \\
\|T_c w\| &= T_c w(1) \\
&= B\Big(\varphi_p^{-1}\Big(\int_0^1 q(s) f(w(s))\,\mathrm{d}s\Big)\Big) + \int_0^1 \varphi_p^{-1}\Big(\int_s^1 q(\tau) f(w(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\leqslant m\varphi_p^{-1}\Big(\int_0^1 q(s) f(w(s))\,\mathrm{d}s\Big) + \varphi_p^{-1}\Big(\int_0^1 q(\tau) f(w(\tau))\,\mathrm{d}\tau\Big) \\
&= (m+1)\varphi_p^{-1}\Big(\int_0^1 q(\tau) f(w(\tau))\,\mathrm{d}\tau\Big) \\
&\leqslant (m+1)aA\varphi_p^{-1}\Big(\int_0^1 q(\tau)\,\mathrm{d}\tau\Big) = a.
\end{aligned}$$

Altogether, we get $b \leqslant \|T_c w\| \leqslant a$ for $w \in P_c[b,a]$, which means that $T_c P_c[b,a] \subset P_c[b,a]$.

Let $w_0(t) = a$, $t \in [0,1]$, then $w_0(t) \in P_c[b,a]$. Let $w_1 = (T_c)w_0$, then $w_1 \in P_c[b,a]$; we denote

$$w_{n+1} = T_c w_n = T_c^{n+1} w_0, \quad (n = 1, 2, \ldots). \tag{21}$$

Since $T_c P_c[b,a] \subset P_c[b,a]$ we have $w_n \in P_c[b,a]$, $(n = 0, 1, 2, \ldots)$. From Lemma 5.1, T_c is compact, so $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in P_c[b,a]$ such that $w_{n_k} \to w^*$.

Now, since $w_1 \in P_c[b,a]$, we have

$$0 \leqslant w_1(t) \leqslant \|w\| \leqslant a = w_0(t);$$

by the definition of w_1 and w_2, we have

$$\begin{cases} (\varphi_p(w_1'))' + q(t) f(w_0(t)) = 0, & 0 < t < 1 \\ w_1(0) - B(w_1'(0)) = 0, & w_1'(1) = 0, \end{cases} \tag{22}$$

$$\begin{cases} (\varphi_p(w_2'))' + q(t) f(w_1(t)) = 0, & 0 < t < 1 \\ w_2(0) - B(w_2'(0)) = 0, & w_2'(1) = 0. \end{cases} \tag{23}$$

Combining (22), (23), the fact that $f\colon [0,a] \to [0,+\infty)$ is nondecreasing, and $w_1 \leqslant w_0$, we get

$$\begin{cases} -(\varphi_p(w_1'(t)))' + (\varphi_p(w_2'(t)))' \geqslant 0, & 0 \leqslant t \leqslant 1, \\ w_2(0) - B w_2'(0) = 0, & w_2'(1) = 0, \\ w_1(0) - B w_1'(0) = 0, & w_1'(1) = 0. \end{cases}$$

By Lemma 2.1 (let $k_1 = 1$, $k_2 = 0$, $B_0(v) = B(v)$, $B_1(v) = v$), $w_1(t) \geqslant w_2(t)$, $0 \leqslant t \leqslant 1$.

By induction, then $w_n(t) \geqslant w_{n+1}(t)$, $0 \leqslant t \leqslant 1$ $(n = 0, 1, 2, \ldots)$. Hence, we see that $w_n \to w^*$. Letting $n \to \infty$ in (21), we obtain $Tw^* = w^*$ since T_c is continuous. Since $\|w^*\| \geqslant b > 0$ and w^* is a nonnegative concave function on $[0,1]$, we conclude that $w^*(t) > 0$, $t \in (0,1)$. Therefore, w^* is a positive, nondecreasing solution of (1), (BCc).

Let $v_0(t) = bt$, $t \in [0,1]$, then $\|v_0\| = b$, and $v_0(t) \in P_c[b,a]$. Let $v_1 = (T_c)v_0$, then $v_1 \in P_c[b,a]$; we denote

$$v_{n+1} = T_c v_n = (T_c)^{n+1} v_0 \quad (n = 1, 2, \ldots).$$

Similarly to $\{w_n\}_{n=1}^{\infty}$, we see that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in P_c[b,a]$, such that $v_{n_k} \to v^*$.

Now, since $v_1 \in P_b[b,a]$, we have by Lemma 5.1,

$$v_1(t) \geqslant t\|v_1\| \geqslant bt = v_0(t).$$

Similarly to $\{w_n\}_{n=1}^{\infty}$, we can show easily that $v_n(t) \leqslant v_{n+1}(t)$, $0 \leqslant t \leqslant 1$ ($n = 0, 1, 2, \ldots$). Hence, we see that $v_n \to v^*$, $T_c v^* = v^*$ and $v^*(t) > 0$, $t \in (0,1)$. Therefore, v^* is a positive, nondecreasing solution of (1), (BCc). □

Remark 5.1. We can easily get that w_* and v^* are the maximal and the minimal solution of (1), (BCc) in $P_c[b,a]$.

Corollary 5.1. *Assume that* (H1), (H2) *and* (H3b) *hold. If there exists a* $\delta \in (0,1)$ *such that*

(C5′) $f\colon [0,+\infty) \to [0,+\infty)$ *is nondecreasing;*

(C6′) $\overline{\lim\limits_{l\to 0}} f(l)/l^{p-1} > (B/\delta)^{p-1}$ *and* $\underline{\lim\limits_{l\to+\infty}} f(l)/l^{p-1} < (aA)^{p-1}$ *(in particular,* $\lim\limits_{l\to 0} f(l)/l^{p-1} = +\infty$ *and* $\lim\limits_{l\to+\infty} f(l)/l^{p-1} = 0$*), where* A*,* B *are defined as in Theorem* 5.1*,*

then there exist two constants $a > 0$ *and* $b > 0$ *such that* (1) *and* (BCc) *has two positive, nondecreasing solutions* $w^*, v^* \in P_c$ *with*

$$b \leqslant \|w^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_c)^n w_0 = w^*, \quad \text{where } w_0(t) = a,\ t \in [0,1],$$
$$b \leqslant \|v^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_c)^n v_0 = v^*, \quad \text{where } v_0(t) = bt,\ t \in [0,1].$$

As for (1), (BCd), we have the following results.

Theorem 5.2. *Assume that* (H1), (H2) *and* (H3b) *hold. If there exists a* $\delta \in (0,1)$ *and two positive numbers* $b < a$ *such that*

(C7) $f\colon [0,a] \to [0,+\infty)$ *is nondecreasing;*

(C8) $f((1-\delta)b) \geqslant (bB)^{p-1}$, $f(a) \leqslant (aA)^{p-1}$,
where $B = 1/\int_0^\delta \varphi_p^{-1}(\int_0^s q(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s$, $A = 1/(m+1)\varphi_p^{-1}(\int_0^1 q(r)\,\mathrm{d}r)$,

then (1), (BCd) *has at least two positive and nonincreasing solutions* $w^*, v^* \in P_d$ *such that*

$$b \leqslant \|w^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_d)^n w_0 = w^*, \quad \text{where } w_0(t) = a,\ t \in [0,1],$$
$$b \leqslant \|v^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} (T_d)^n v_0 = v^*, \quad \text{where } v_0(t) = b(1-t),\ t \in [0,1],$$

where

$$(T_d u)(t) = B\biggl(\varphi_p^{-1}\biggl(\int_0^1 q(s) f(u(s))\,\mathrm{d}s\biggr)\biggr) + \int_t^1 \varphi_p^{-1}\biggl(\int_0^s q(\tau) f(u(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \quad t \in [0,1].$$

P r o o f. The proof of Theorem 5.2 is similar to Theorem 5.1, and we omit it. $\square$

Corollary 5.2. *Assume that* (H1), (H2) *and* (H3b) *hold. If*

(C7′) $f\colon [0,+\infty)\to[0,+\infty)$ *is nondecreasing;*

(C8′) $\overline{\lim\limits_{l\to 0}} f(l)/l^{p-1} > (B/(1-\delta))^{p-1}$ *and* $\underline{\lim\limits_{l\to+\infty}} f(l)/l^{p-1} < (aA)^{p-1}$ *(in particular,* $\lim\limits_{l\to 0} f(l)/l^{p-1}=+\infty$ *and* $\lim\limits_{l\to+\infty} f(l)/l^{p-1}=0$*), where* A*,* B *are defined as in Theorem* 5.2*,*

then there exist two constants $a>0$ *and* $b>0$*, such that* (1), (BCd) *has two positive, concave and nonincreasing solutions* w^**,* v^* *with*

$$b\leqslant \|w^*\|\leqslant a \quad \text{and} \quad \lim_{n\to+\infty}(T_d)^n w_0 = w^*, \quad \text{where } w_0(t)=a,\ t\in[0,1],$$
$$b\leqslant \|v^*\|\leqslant a \quad \text{and} \quad \lim_{n\to+\infty}(T_d)^n v_0 = v^*, \quad \text{where } v_0(t)=b(1-t),\ t\in[0,1].$$

Example 5.1. Suppose $0<n$, $m<3$, and consider

$$(|u'|^2u')'(t)+\frac{1}{t(1-t)}[(u(t))^m+\ln((u(t))^n+1)]=0,\quad t\in(0,1), \tag{24}$$

subject to

$$u(0)-\frac{1}{3}u'(0)=0,\quad u(1)+(u'(1))^3=0, \tag{Ea}$$
$$u(0)-\frac{1}{3}u'(0)=0,\quad u(1)+\frac{1}{3}u'(1)=0, \tag{Eb}$$
$$u(0)-\frac{1}{3}u'(0)=0,\quad u'(1)=0, \tag{Ec}$$
$$u'(0)=0,\quad u(1)+\frac{1}{3}u'(1)=0. \tag{Ed}$$

By Corollary 3.1, Corollary 4.1, Corollary 5.1, and Corollary 5.2, we can get not only the existence but also the iteration of positive solutions for the BVP (24) subject to (Ea), (Eb), (Ec) and (Ed). But the results in [1] can only guarantee the existence of their positive solutions.

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