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PRODUCT RADICAL CLASSES OF ℓ-GROUPS

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The main results of this paper concern product radical classes of ℓ-groups. We discuss the product radical mappings and the polar closure operator in the complete lattice $T_{1'23'}$, and generalize some results for torsion classes.

We use the standard terminology and notation of [1, 3, 5]. Throughout the paper G is an ℓ-group. We use the additive group notation. Let $\{G_\alpha \mid \alpha \in A\}$ be a family of ℓ-groups and let $\prod\limits_{\alpha\in A} G_\alpha$ be their direct product. For an element $g \in \prod\limits_{\alpha\in A} G_\alpha$, we denote the α-component of g by g_α. An ℓ-group G is said to be a subdirect product of ℓ-groups G_α, in symbols $G \subseteq' \prod\limits_{\alpha\in A} G_\alpha$, if G is an ℓ-subgroup of $\prod\limits_{\alpha\in A} G_\alpha$ such that for each $\alpha \in A$ and each $g' \in G_\alpha$ there exists $g \in G$ with the property $g_\alpha = g'$. We denote the ℓ-subgroup of $\prod\limits_{\alpha\in A} G_\alpha$ consisting of the elements with only finitely many non-zero components by $\sum\limits_{\alpha\in A} G_\alpha$. It is called the direct sum of G_α. An ℓ-group G is said to be a completely subdirect product of G_α, if G is an ℓ-subgroup of $\prod\limits_{\alpha\in A} G_\alpha$ and $\sum\limits_{\alpha\in A} G_\alpha \subseteq G$.

Let G be an ℓ-group. $\mathcal{C}(G)$ will denote the complete lattice of all convex ℓ-subgroups of G. For $g \in G$, let $G(g)$ be the convex ℓ-subgroup generated by g. If $X \subseteq G$, $X_G^\perp = \{f \in G \mid \text{ for all } x \in X, |f| \wedge |x| = 0\}$ is called the polar of X in G. If there is no danger of confusion, we simply write $X^\perp$.

1. CLASSES OF ℓ-GROUPS

We can form new ℓ-groups from some original ℓ-groups. These structure methods include:

1. taking ℓ-subgroups,
1′. taking convex ℓ-subgroups,
2. forming joins of convex ℓ-subgroups,
2′. forming finite joins and chain joins of convex ℓ-subgroups,

3. forming completely subdirect products,
3′. forming direct products,
3″. forming direct sums,
4. taking ℓ-homomorphic images,
4′. taking complete ℓ-homomorphic images,
4″. taking ℓ-isomorphic images,
5. forming extensions, that is, G is an extension of A with respect to B if A is an ℓ-ideal of G and $B = G/A$.

A family $\mathcal{U}$ of ℓ-groups is called a class, if it is closed under some of the above structures. If a class $\mathcal{U}$ is closed under the structures i_1, i_2, i_3, i_4, i_5, we call $\mathcal{U}$ an $i_1i_2i_3i_4i_5$-class, where $i_1 \in \{1, 1', 4''\}$, $i_2 \in \{2, 2', 4''\}$, $i_3 \in \{3, 3', 3'', 4''\}$, $i_4 \in \{4, 4', 4''\}$, $i_5 \in \{5, 4''\}$. All our classes are always assumed to contain along with a given ℓ-group all its ℓ-isomorphic copies, so we can omit the index 4″. For example, we simply write the $1'2'$-class for the $1'2'4''4''4''$-class.

Thus, a radical class [7] is a $1'2$-class, a torsion class [12] is a 24-class, a hereditary torsion class [11] ia a $1'24$-class, a torsion-free class [11] is a $1'3$-class, a quasi-torsion class [10] is a $1'24'$-class, a complete torsion class [12] is a 245-class, a variety is a 134-class.

Let $T_{i_1i_2i_3i_4i_5}$ be the collection of all $i_1i_2i_3i_4i_5$-classes, It is clear that an $i_1 \dots i_{k-1}i_k$-class is also an $i_1 \dots i_{k-1}$-class ($2 \leqslant k \leqslant 5$), that is, $T_{i_1 \dots i_{k-1} i_k} \subseteq T_{i_1 \dots i_{k-1}}$. We also have $T_{i_1 \dots i_k \dots i_5} \subseteq T_{i_1 \dots i'_k \dots i_5}$ ($1 \leqslant k \leqslant 5$).

We could have at most $3 \cdot 3 \cdot 4 \cdot 3 \cdot 2 = 216$ classes of ℓ-groups, but some of them coincide. For example, we will show that $T_{i_1i_2i_3i_4i_5} = T_{i_12'i_3i_4i_5}$ if $i_1 \neq 4''$. It is also clear that $T_{1i_23i_4i_5} = T_{1i_23'i_4i_5}$. In general, for $1'2i_3i_4i_5$-classes we have the following relations:

$$\begin{array}{ccccccccc} T_{1'23} & \subseteq & T_{1'23'} & \subseteq & T_{1'23''} & = & T_{1'2} & \supseteq & T_{1'24} \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\ T_{1'235} & \subseteq & T_{1'23'5} & \subseteq & T_{1'23''5} & = & T_{1'25} & \supseteq & T_{1'245} \end{array}$$

A $1'2'3'$-class is called a product radical class. A $1'2'3$-class is called a subproduct radical class. In this paper we mainly discuss the product radical classes. We will prove that most of the results similar to those from [11] are valid for product radical classes. First, we give some examples of product radical classes:

$\mathcal{H}$, the class of hyper-archimedean ℓ-groups. An ℓ-group belongs to $\mathcal{H}$ if and only if every ℓ-homomorphic image is archimedean.

$\mathcal{A}r$, the class of all archimedean ℓ-groups.

$\mathcal{CD}$, the class of completely distributive ℓ-groups.

$\mathcal{SP}$, the class of strongly projectable ℓ-groups, that is, ℓ-groups for which each polar is a cardinal summand.

$\mathcal{B}as$, the class of all ℓ-groups with a basis (an ℓ-group has a basis if it has a maximal pairwise disjoint set of basic element).

$\mathcal{C}$, the class of all complete ℓ-groups.

Since every variety of ℓ-groups is a torsion class [6] and a torsion-free class, so every variety is a product radical class. Let $\mathcal{L}$ be the variety of all ℓ-groups. A product radical class $\mathcal{R}$ is called proper if $\mathcal{R} \neq \mathcal{L}$.

2. The product radical mappings

Let $\mathcal{R}$ be a product radical class and G an ℓ-group. By Zorn's Lemma there exists a maximal convex ℓ-subgroups of G belonging to $\mathcal{R}$. We denote it by $\mathcal{R}(G)$. Since $\mathcal{R}$ is closed under finite joins, $\mathcal{R}(G)$ is the unique largest convex ℓ-subgroup of G belonging to $\mathcal{R}$. $\mathcal{R}(G)$ is called a product radical of G; it is invariant under all ℓ-automorphisms of G, and in particular it is an ℓ-ideal. Let $R(G) = \{\mathcal{R}(G) \mid \mathcal{R} \in T_{1'2'3'}\}$ [9].

We have the following elementary fact.

Theorem 2.1. *Suppose that $\mathcal{R}$ is a product radical class. Then*

(i) *if A is a convex ℓ-subgroup of G then $\mathcal{R}(A) = A \cap \mathcal{R}(G)$;*

(ii) *if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of ℓ-groups then $\mathcal{R}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big) = \prod\limits_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$.*

Conversely, if we associate with each ℓ-group G an ℓ-ideal $\mathcal{U}(G)$ subject to (i) *and* (ii) *above, and set $\mathcal{R} = \{G \mid \mathcal{U}(G) = G\}$, then $\mathcal{R}$ is a product radical class, and for each ℓ-group G, $\mathcal{R}(G) = \mathcal{U}(G)$.*

Proof. $A \cap \mathcal{R}(G)$ is a convex ℓ-subgroup of $\mathcal{R}(G)$ and belongs to $\mathcal{R}$, so $A \cap \mathcal{R}(G) \subseteq \mathcal{R}(A)$. $\mathcal{R}(A)$ is a convex ℓ-subgroup of G and belongs to $\mathcal{R}$, so $\mathcal{R}(A) \subseteq A \cap \mathcal{R}(G)$. Therefore $\mathcal{R}(A) = A \cap \mathcal{R}(G)$.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups. Then

$$\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) \supseteq \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda). \tag{1}$$

On the other hand, let $\bar{G}_\lambda = \Big\{g \in \prod\limits_{\lambda\in\Lambda} G_\lambda \mid \delta \neq \lambda \Rightarrow g_\delta = 0\Big\}$, then $\bar{G}_\lambda \cong G_\lambda$. We see that $\mathcal{R}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big) \cap \bar{G}_\lambda$ is a convex ℓ-subgroup of $\mathcal{R}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big)$ and $\bar{G}_\lambda$, so $\mathcal{R}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big) \cap \bar{G}_\lambda \subseteq \mathcal{R}(\bar{G}_\lambda)$. Since $\mathcal{R}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big)$ is a convex ℓ-subgroup of $\prod\limits_{\lambda\in\Lambda} G_\lambda$,

$$\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big)^+ \subseteq \prod_{\lambda\in\Lambda}\Big[\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big)^+ \cap \bar{G}_\lambda\Big] \subseteq \prod_{\lambda\in\Lambda}\Big[\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) \cap \bar{G}_\lambda\Big].$$

Hence

$$\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) \subseteq \prod_{\lambda\in\Lambda}\Big[\mathcal{R}\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) \cap \bar{G}_\lambda\Big] \subseteq \prod_{\lambda\in\Lambda} \mathcal{R}(\bar{G}_\lambda) = \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda). \tag{2}$$

Combining (1) and (2) we get (ii).

Conversely, suppose the function $\mathcal{U}$ satisfies (i) and (ii), and $\mathcal{R}\{G \mid \mathcal{U}(G) = G\}$. If $G \in \mathcal{R}$ and A is a convex ℓ-subgroup of G, then $\mathcal{U}(A) = A \cap \mathcal{U}(G) = A \cap G = A$, hence $A \in \mathcal{R}$. Next, suppose $\{C_\lambda \mid \lambda \in \Lambda\}$ is a family of convex ℓ-subgroups of an ℓ-group G, $C = \bigvee_{\lambda \in \Lambda} C_\lambda$, and $C_\lambda \in \mathcal{R}$ for each λ. Then $C_\lambda = \mathcal{U}(C_\lambda) = C_\lambda \cap \mathcal{U}(G) \in \mathcal{C}(\mathcal{U}(G))$, so $\bigvee_{\lambda \in \Lambda} C_\lambda \in \mathcal{C}(\mathcal{U}(G))$. But $\mathcal{U}(\mathcal{U}(G)) = \mathcal{U}(G)$ implies $\mathcal{U}(G) \in \mathcal{R}$. By the above we get $\bigvee_{\lambda \in \Lambda} C_\lambda \in \mathcal{R}$. This implies that $\mathcal{R}$ is closed under the structure 2, in particular, $\mathcal{R}$ is closed under the structure 2′.

Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of ℓ-groups, and $G_\lambda \in \mathcal{R}$ for each λ. Then $\mathcal{U}\Big(\prod_{\lambda \in \Lambda} G_\lambda\Big) = \prod_{\lambda \in \Lambda} \mathcal{U}(G_\lambda) = \prod_{\lambda \in \Lambda} G_\lambda$, hence $\prod_{\lambda \in \Lambda} G_\lambda \in \mathcal{R}$. Therefore $\mathcal{R}$ is a product radical class. $\mathcal{U}(G) \in \mathcal{R}$ implies $\mathcal{U}(G) \subseteq \mathcal{R}(G)$. On the other hand, $\mathcal{R}(G) = \mathcal{U}(\mathcal{R}(G)) = \mathcal{R}(G) \cap \mathcal{U}(G) \subseteq \mathcal{U}(G)$. Hence $\mathcal{R}(G) = \mathcal{U}(G)$. □

Any mapping $G \to \mathcal{U}(G)$ on the variety $\mathcal{L}$ of all ℓ-groups satisfying the above properties (i) and (ii) is called a product radical mapping. Thus there exists a 1-1 correspondence between the product radical classes and the product radical mappings. From the above proof we see that a product radical class is always closed under forming joins of convex ℓ-subgroups, by a product radical class we always mean a 1′23′-class. In the general we have

Corollary 2.2. *$T_{i_1 2 i_3 i_4 i_5} = T_{i_1 2' i_3 i_4 i_5}$ if $i_1 \neq 4''$. In particular, $T_{1'2'3'} = T_{1'23'}$.*

Proposition 2.3. *Suppose that $\mathcal{R}$ is a product radical class and $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of convex ℓ-subgroups of the ℓ-group G. Then*

(1) $\mathcal{R}\Big(\bigvee_{\lambda \in \Lambda} G_\lambda\Big) = \bigvee_{\lambda \in \Lambda} \mathcal{R}(G_\lambda)$,

(2) $\mathcal{R}\Big(\bigcap_{\lambda \in \Lambda} G_\lambda\Big) = \bigcap_{\lambda \in \Lambda} \mathcal{R}(G_\lambda)$.

The proof of this proposition is similar to the proof of Proposition 1.1 and Proposition 1.3 in [11].

3. The complete lattice $T_{1'23'}$

Suppose $\{\mathcal{U}_\lambda \mid \lambda \in \Lambda\} \subseteq T_{1'23'}$. Since the intersection of a family of product radical classes is also a product radical class, we can define

$$\bigwedge_{\lambda \in \Lambda} \mathcal{U}_\lambda = \bigcap_{\lambda \in \Lambda} \mathcal{U}_\lambda,$$

$$\bigvee_{\lambda \in \Lambda} \mathcal{U}_\lambda = \bigcap \{\mathcal{U} \in T_{1'23'} \mid \mathcal{U} \supseteq \mathcal{U}_\lambda \text{ for each } \lambda \in \Lambda\}.$$

Theorem 3.1. *$T_{1'23'}$ is a complete lattice. If $\{\mathcal{U}_\lambda \mid \lambda \in \Lambda\} \subseteq T_{1'23'}$, $\{\mathcal{U}_i \mid i = 1,\dots,n\} \subseteq T_{1'23'}$, then for each ℓ-group G,*

$$\left(\bigwedge_{\lambda\in\Lambda} \mathcal{U}_\lambda\right)(G) = \bigcap_{\lambda\in\Lambda} \mathcal{U}_\lambda(G) \tag{3}$$

and

$$\left(\bigvee_{i=1}^{n} \mathcal{U}_i\right)(G) = \bigvee_{i=1}^{n} \mathcal{U}_i(G), \tag{4}$$

where $\bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)$ is the convex ℓ-subgroup generated by $\mathcal{U}_i(G)$ $(i = 1,\dots,n)$. Hence $T_{1'23'}$ is a sublattice of $T_{1'2}$ and the meets of $T_{1'23'}$ agree which those of $T_{1'2}$.

P r o o f. The formula (3) is clear. We only prove (4). First, $G \to \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)$ is a product radical mapping. In fact, $\bigvee\limits_{i=1}^{n} \mathcal{U}_i(A) = \bigvee\limits_{i=1}^{n} (A \bigcap \mathcal{U}_i(G)) = A \bigcap (\bigvee\limits_{i=1}^{n} \mathcal{U}_i(G))$ for each $A \in \mathcal{C}(G)$. For any family $\{G_\delta \mid \delta \in \Delta\}$ of ℓ-groups, evidently $\prod\limits_{\delta\in\Delta} \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G_\delta) \supseteq \bigvee\limits_{i=1}^{n} \prod\limits_{\delta\in\Delta} \mathcal{U}_i(G_\delta)$. If $a = (\dots, a_\delta, \dots) \in \prod\limits_{\delta\in\Delta} \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G_\delta)$, then for each $\delta \in \Delta$ $a_\delta = a_{\delta_1} + \dots + a_{\delta_n}$, $a_{\delta_i} \in \mathcal{U}_i(G_\delta)$ $(1 \leqslant i \leqslant n)$. So $a = (\dots, a_{\delta_1}, \dots) + \dots + (\dots, a_{\delta_n}, \dots)$, where $(\dots, a_{\delta_i}, \dots) \in \prod\limits_{\delta\in\Delta} \mathcal{U}_i(G_\delta)$ $(1 \leqslant i \leqslant n)$. Hence $a \in \bigvee\limits_{i=1}^{n} \prod\limits_{\delta\in\Delta} \mathcal{U}_i(G_\delta)$. Therefore $\prod\limits_{\delta\in\Delta} \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G_\delta) = \bigvee\limits_{i=1}^{n} \prod\limits_{\delta\in\Delta} \mathcal{U}_i(G_\delta)$. Thus, (i) and (ii) of Theorem 2.1 are satisfied and $\bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)$ defines a product radical class $\mathcal{U} = \{G \mid G = \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)\}$. If $\mathcal{R}$ is a product radical class such so that $\mathcal{R} \supseteq \mathcal{U}_i$ $(1 \leqslant i \leqslant n)$ and $G \in \mathcal{U}$, then $\mathcal{R}(G) = \mathcal{R}(\bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)) = \bigvee\limits_{i=1}^{n} \mathcal{R}(\mathcal{U}_i(G)) = \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G) = G$ and $G \in \mathcal{R}$. It follows that $\mathcal{U} = \bigvee\limits_{i=1}^{n} \mathcal{U}_i$ and $\left(\bigvee\limits_{i=1}^{n} \mathcal{U}_i\right)(G) = \bigvee\limits_{i=1}^{n} \mathcal{U}_i(G)$. □

N o t e 1. From the formulas (3) and (4) we have $\mathcal{I} \wedge \left(\bigvee\limits_{i=1}^{n} \mathcal{U}_i\right) = \bigvee\limits_{i=1}^{n} (\mathcal{I} \wedge \mathcal{U}_i)$. Nonetheless, it is not generally true that $\mathcal{I}\left(\bigvee\limits_{\lambda\in\Lambda} \mathcal{U}_\lambda\right) = \bigvee\limits_{\lambda\in\Lambda} (\mathcal{I} \wedge \mathcal{U}_\lambda)$. So $T_{1'23'}$ is not a Brouwerian lattices. Nor is it generally true that $\mathcal{I} \vee \left(\bigwedge\limits_{\lambda\in\Lambda} \mathcal{U}_\lambda\right) = \bigwedge\limits_{\lambda\in\Lambda} (\mathcal{I} \vee \mathcal{U}_\lambda)$. In general, $\mathcal{I} \vee \left(\bigwedge\limits_{\lambda\in\Lambda} \mathcal{U}_\lambda\right) \subseteq \bigwedge\limits_{\lambda\in\Lambda} (\mathcal{I} \vee \mathcal{U}_\lambda)$.

Note 2. The general form of (4) is $\left(\bigvee_{\lambda\in\Lambda}\mathcal{U}_\lambda\right)(G)=\bigvee_{\lambda\in\Lambda}\mathcal{U}_\lambda(G)$. It is valid for radical classes and torsion classes. Let $\{\mathcal{U}_\lambda \mid \lambda\in\Lambda\}$ be a family of $1'2i_3i_4i_5$-classes. From the proof of Theorem 3.1 we know that $\left(\bigvee_{\lambda\in\Lambda}\mathcal{U}_\lambda\right)(G)=\bigvee_{\lambda\in\Lambda}\mathcal{U}_\lambda(G)$, if and only if $\mathcal{R}=\{G \mid G=\bigvee_{\lambda\in\Lambda}\mathcal{U}_\lambda(G)\}$ defines a $1'2i_3i_4i_5$-class if and only if $T_{1'2i_3i_4i_5}$ is a complete sublattice of $T_{1'2}$. $T_{1'23'}$ is not a complete sublatice of $T_{1'2}$.

Note 3. By Theorem 3.1 we see that $R(G)$ is a sublattice of $\mathcal{C}(G)$ for an ℓ-group G.

Since a product radical class is a radical class, for any two product radical classes $\mathcal{I}$ and $\mathcal{U}$ we also have their product $\mathcal{I}.\mathcal{U}=\{G \mid G/\mathcal{I}(G)\in\mathcal{U}\}$ [8].

Theorem 3.2. *$\mathcal{I}.\mathcal{U}$ is a product radical class whenever $\mathcal{I}$ and $\mathcal{U}$ are; if G is an ℓ-group, the product radical $\mathcal{I}.\mathcal{U}(G)$ is defined by the equation $\mathcal{I}.\mathcal{U}(G)/\mathcal{I}(G)=\mathcal{U}(G/\mathcal{I}(G))$. Consequently, $T_{1'23'}$ is a subsemigroup of $T_{1'2}$.*

Proof. We will prove that $\mathcal{I}.\mathcal{U}(G)$ satisfies (i) and (ii) of Theorem 2.1. Suppose that A is a convex ℓ-subgroup of G. To show that $\mathcal{I}.\mathcal{U}(A)=A\cap\mathcal{I}.\mathcal{U}(G)$ we prove that $[A\cap\mathcal{I}.\mathcal{U}(G)]/\mathcal{I}(A)=\mathcal{U}(A/\mathcal{I}(A))$.

$$\begin{aligned}
[A\cap\mathcal{I}.\mathcal{U}(G)]/\mathcal{I}(A) &= [A\cap\mathcal{I}.\mathcal{U}(G)]/[A\cap\mathcal{I}(G)]\\
&\cong [(A\cap\mathcal{I}.\mathcal{U}(G))\vee\mathcal{I}(G)]/\mathcal{I}(G)=[A\vee\mathcal{I}(G)]\cap\mathcal{I}.\mathcal{U}(G)/\mathcal{I}(G)\\
&= [A\vee\mathcal{I}(G)/\mathcal{I}(G)]\cap[\mathcal{I}.\mathcal{U}(G)/\mathcal{I}(G)]\\
&= [A\vee\mathcal{I}(G)/\mathcal{I}(G)]\cap\mathcal{U}(G/\mathcal{I}(G))=\mathcal{U}(A\vee\mathcal{I}(G))/\mathcal{I}(G)\\
&\cong \mathcal{U}(A/A\cap\mathcal{I}(G))=\mathcal{U}(A/\mathcal{I}(A)).
\end{aligned}$$

Next, let $\{G_\lambda \mid \lambda\in\Lambda\}$ be a family of ℓ-groups. Then

$$\begin{aligned}
&\Big[\prod_{\lambda\in\Lambda}\mathcal{I}.\mathcal{U}(G_\lambda)\Big]\Big/\mathcal{I}\Big(\prod_{\lambda\in\Lambda}G_\lambda\Big)=\Big[\prod_{\lambda\in\Lambda}\mathcal{I}.\mathcal{U}(G_\lambda)\Big]\Big/\Big[\prod_{\lambda\in\Lambda}\mathcal{I}(G_\lambda)\Big]\\
&=\prod_{\lambda\in\Lambda}\Big[\mathcal{I}.\mathcal{U}(G_\lambda)/\mathcal{I}(G_\lambda)\Big]=\prod_{\lambda\in\Lambda}\mathcal{U}(G_\lambda/\mathcal{I}(G_\lambda))\\
&=\mathcal{U}\Big(\prod_{\lambda\in\Lambda}(G_\lambda/\mathcal{I}(G_\lambda))\Big)=\mathcal{U}\Big(\prod_{\lambda\in\Lambda}G_\lambda\Big/\prod_{\lambda\in\Lambda}\mathcal{I}(G_\lambda)\Big)\\
&=\mathcal{U}\Big(\prod_{\lambda\in\Lambda}G_\lambda\Big/\mathcal{I}\Big(\prod_{\lambda\in\Lambda}G_\lambda\Big)\Big)=\mathcal{I}.\mathcal{U}\Big(\prod_{\lambda\in\Lambda}g_\lambda\Big)\Big/\mathcal{I}\Big(\prod_{\lambda\in\Lambda}G_\lambda\Big).
\end{aligned}$$

That is, $\mathcal{I}.\mathcal{U}\left(\prod_{\lambda\in\Lambda}G_\lambda\right)=\prod_{\lambda\in\Lambda}\mathcal{I}.\mathcal{U}(G_\lambda)$. Hence $\mathcal{I}.\mathcal{U}(G)$ is a product radical. It is clear that $G\in\mathcal{I}.\mathcal{U}$ if and only if $\mathcal{I}.\mathcal{U}(G)=G$. So $\mathcal{I}.\mathcal{U}$ is the product radical class defined by $\mathcal{I}.\mathcal{U}(G)$. □

Corollary 3.3. *Suppose $\mathcal{I}, \mathcal{U}, \mathcal{U}' \in T_{1'23'}$. If $\mathcal{U} \supseteq \mathcal{U}'$, then $\mathcal{I}.\mathcal{U} \supseteq \mathcal{I}.\mathcal{U}'$.*

Theorem 3.4. *Let $\mathcal{U}$, $\{\mathcal{I}_\lambda \mid \lambda \in \Lambda\}$, $\{\mathcal{I}_i \mid i = 1, \ldots, n\}$ be product radical classes. Then*
(1) $\mathcal{U}.\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda\Big) = \bigwedge_{\lambda\in\Lambda} \mathcal{U}.\mathcal{I}_\lambda$,
(2) $\bigvee_{i=1}^{n} \mathcal{U}.\mathcal{I}_i = \mathcal{U}.\Big(\bigvee_{i=1}^{n} \mathcal{I}_i\Big)$.

Proof. (1) By Theorem 3.1 we have

$$\begin{aligned}
\mathcal{U}.\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda\Big)(G)\Big/\mathcal{U}(G) &= \Big(\bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda\Big)(G/\mathcal{U}(G)) \\
&= \bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda(G/\mathcal{U}(G)) = \bigwedge_{\lambda\in\Lambda} [\mathcal{U}.\mathcal{I}_\lambda(G)/\mathcal{U}(G)] \\
&= \Big[\bigwedge_{\lambda\in\Lambda} \mathcal{U}.\mathcal{I}_\lambda(G)\Big]\Big/\mathcal{U}(G) = \Big(\bigwedge_{\lambda\in\Lambda} \mathcal{U}.\mathcal{I}_\lambda\Big)(G)\Big/\mathcal{U}(G).
\end{aligned}$$

Hence $\mathcal{U}.\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda\Big)(G) = \Big(\bigwedge_{\lambda\in\Lambda} \mathcal{U}.\mathcal{I}_\lambda\Big)(G)$ for any ℓ-group G, and so $\mathcal{U}.\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{I}_\lambda\Big) = \bigwedge_{\lambda\in\Lambda} \mathcal{U}.\mathcal{I}_\lambda$.

(2) It follows from Theorem 3.1 that

$$\begin{aligned}
\Big[\mathcal{U}.\Big(\bigvee_{i=1}^{n} \mathcal{I}_i\Big)\Big](G)\Big/\mathcal{U}(G) &= \Big(\bigvee_{i=1}^{n} \mathcal{I}_i\Big)(G/\mathcal{U}(G)) \\
&= \bigvee_{i=1}^{n} \mathcal{I}_i(G/\mathcal{U}(G)) = \bigvee_{i=1}^{n} [\mathcal{U}.\mathcal{I}_i(G)/\mathcal{U}(G)] \\
&= \Big[\bigvee_{i=1}^{n} \mathcal{U}.\mathcal{I}_i(G)\Big]\Big/\mathcal{U}(G) = \Big(\bigvee_{i=1}^{n} \mathcal{U}.\mathcal{I}_i\Big)(G)\Big/\mathcal{U}(G).
\end{aligned}$$

Therefore $\Big[\mathcal{U}.\Big(\bigvee_{i=1}^{n} \mathcal{I}_i\Big)\Big](G) = \Big(\bigvee_{i=1}^{n} \mathcal{U}.\mathcal{I}_i\Big)(G)$ for any ℓ-group G, and so $\mathcal{U}.\Big(\bigvee_{i=1}^{n} \mathcal{I}\Big) = \bigvee_{i=1}^{n} \mathcal{U}.\mathcal{I}_i$. □

In this section we define some new product radical classes from the old ones by taking closures of the product radicals. Suppose that $\mathcal{R}$ is a product radical class. Let $\mathcal{R}^{\perp} = \{G \mid \mathcal{R}(G) = 0\}$. Clearly $\mathcal{R}^{\perp}$ is also a product radical class. $\mathcal{R}^{\perp}$ is called the polar of $\mathcal{R}$.

Theorem 4.1. *For any product radical class $\mathcal{R}$, $\mathcal{R}^{\perp}(G) = \mathcal{R}(G)^{\perp}$.*

Proof. We show that $G \to \mathcal{R}(G)^{\perp}$ is a product radical mapping. Let $A \in \mathcal{C}(G)$. Then $\mathcal{R}(A)^{\perp} = (A \cap \mathcal{R}(G))_A^{\perp} = A \cap \mathcal{R}(G)^{\perp}$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups. Then $\left[\mathcal{R}\left(\prod\limits_{\lambda\in\Lambda} G_\lambda\right)\right]^{\perp}_{\prod\limits_{\lambda\in\Lambda} G_\lambda} = \left[\prod\limits_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)\right]^{\perp}_{\prod\limits_{\lambda\in\Lambda} G_\lambda} = \prod\limits_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)^{\perp}_{G_\lambda}$. Thus $\mathcal{R}(G)^{\perp}$ defines a product radical class $\mathcal{I}$. It is obvious that $\mathcal{I} = \mathcal{R}^{\perp}$. □

Let G be an ℓ-group. By Proposition 1.2.6 of [1], $\mathcal{R}(G)^{\perp}$ is the unique largest convex ℓ-subgroup for which $\mathcal{R}(G) \cap \mathcal{R}(G)^{\perp} = 0$.

This and Theorem 3.1 imply that $\mathcal{R}^{\perp}$ is the unique largest product radical class for which $\mathcal{R} \wedge \mathcal{R}^{\perp} = 0$. This complementation polar operator defines a Galois connection which has the following properties: Let $\mathcal{R}$ and $\mathcal{I}$ be product radical classes. Define $\mathcal{R}^{\perp\perp} = (\mathcal{R}^{\perp})^{\perp}$. Then

$$(5)\qquad \begin{aligned} &(1)\quad \mathcal{R} \subseteq \mathcal{R}^{\perp\perp};\\ &(2)\quad \text{if } \mathcal{R} \subseteq \mathcal{I}, \text{ then } \mathcal{R}^{\perp} \supseteq \mathcal{I}^{\perp};\\ &(3)\quad \mathcal{R}^{\perp} = \mathcal{R}^{\perp\perp\perp};\\ &(4)\quad (\mathcal{R} \vee \mathcal{I})^{\perp} = \mathcal{R}^{\perp} \wedge \mathcal{I}^{\perp}. \end{aligned}$$

From Theorem 4.1 in [4] we have

Corollary 4.2. *The polar operator in $T_{1'23'}$ agrees with that in $T_{1'2}$.*

From the formula (5) and Lemma 1 in [2] we get

Proposition 4.3. *The mapping $\mathcal{R} \to \mathcal{R}^{\perp\perp}$ is a closure operator in $T_{1'23'}$;*
(1) $\mathcal{R}^{\perp\perp} = (\mathcal{R}^{\perp\perp})^{\perp\perp}$;
(2) if $\mathcal{R} \subseteq \mathcal{I}$, then $\mathcal{R}^{\perp\perp} \subseteq \mathcal{I}^{\perp\perp}$;
(3) $(\mathcal{R} \cap \mathcal{I})^{\perp\perp} = \mathcal{R}^{\perp\perp} \cap \mathcal{I}^{\perp\perp}$.

A product radical class $\mathcal{R}$ is said to be a polar product radical class if $\mathcal{R} = \mathcal{R}^{\perp\perp}$. Let $T^p_{1'23'}$ be the set of all polar product radical classes. Then $T^p_{1'23'}$ is a complete Boolean algebra under inclusion, in which meets agree with those of $T_{1'23'}$ but joins need not.

A product radical class $\mathcal{I}$ is called complete (or idempotent), if $\mathcal{I} \in T_{1'23'5}$, that is $\mathcal{I}.\mathcal{I} = \mathcal{I}$. We now seek to give a more precise description of complete product radical classes. Let $\mathcal{I}$ be a product radical class and σ an ordinal number. We define an ascending sequence $\mathcal{I}, \mathcal{I}^2, \ldots, \mathcal{I}^\sigma, \ldots$ as follows:

$$\mathcal{I}^\sigma = \begin{cases} \mathcal{I}.\mathcal{I}^{\sigma-1} \text{ if } \sigma \text{ is not a limit ordinal,} \\ \{G \mid G = \bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G)\} \text{ if } \sigma \text{ is a limit ordinal.} \end{cases}$$

We will show that $\mathcal{I}^\sigma$ is a product radical class for each σ. In fact, using the transfinite inclusion we can show that

$$G \to \mathcal{I}^\sigma(G) = \begin{cases} \mathcal{I}.\mathcal{I}^{\sigma-1}(G) \text{ if } \sigma \text{ is not a limit ordinal,} \\ \bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G) \text{ if } \sigma \text{ is a limit ordinal} \end{cases}$$

are product radical mappings. It suffices to verify that $G \to \bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G)$ are product radical mappings for limit numbers σ. For any $A \in \mathcal{C}(G)$ we have $\mathcal{I}^\sigma(A) = \bigcup_{\alpha<\sigma} \mathcal{I}^\sigma(A) = \bigcup_{\alpha<\sigma} [A \cap \mathcal{I}^\alpha(G)] = A \cap [\bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G)] = A \cap \mathcal{I}^\sigma(G)$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups. Then

$$\mathcal{I}^\sigma\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) = \bigcup_{\alpha<\sigma} \mathcal{I}^\alpha\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) = \bigcup_{\alpha<\sigma} \prod_{\lambda\in\Lambda} \mathcal{I}^\alpha(G_\lambda)$$
$$\subseteq \prod_{\lambda\in\Lambda} \Big[\bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G_\lambda)\Big] = \prod_{\lambda\in\Lambda} \mathcal{I}^\sigma(G_\lambda).$$

On the other hand, let $a \in \prod_{\lambda\in\Lambda} \Big[\bigcup_{\alpha<\sigma} \mathcal{I}^\alpha(G_\lambda)\Big]$, $a = (\ldots, a_\lambda, \ldots)$, where $a_\lambda \in \mathcal{I}^{\alpha_\lambda}(G_\lambda)$ for $\lambda \in \Lambda$. Put $\overline{\mathcal{I}^{\alpha_\lambda}(G_\lambda)} = \{f \in \prod_{\lambda'\in\Lambda} \mathcal{I}^{\alpha_\lambda}(G_{\lambda'}) \mid \text{ if } \lambda' \neq \lambda,\ f_{\lambda'} = 0\}$. Then

$$\overline{\mathcal{I}^{\alpha_\lambda}(G_\lambda)} \subseteq \prod_{\lambda'\in\Lambda} \mathcal{I}^{\alpha_\lambda}(G_{\lambda'}) \subseteq \bigcup_{\alpha<\sigma} \prod_{\lambda\in\Lambda} \mathcal{I}^\alpha(G_\lambda).$$

So

$$a \in \prod_{\lambda\in\Lambda} \mathcal{I}^{\alpha_\lambda}(G_\lambda) = \prod_{\lambda\in\Lambda} \overline{\mathcal{I}^{\alpha_\lambda}(G_\lambda)} \subseteq \bigcup_{\alpha<\sigma} \prod_{\lambda\in\Lambda} \mathcal{I}^\alpha(G_\lambda).$$

Therefore

$$\mathcal{I}^\sigma\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) = \prod_{\lambda\in\Lambda} \mathcal{I}^\sigma(G_\lambda).$$

We define

$$\mathcal{I}^* = \bigcup_\sigma \mathcal{I}^\sigma.$$

Theorem 4.4. *Let $\mathcal{I}$ be a product radical class. Then $\mathcal{I}^*$ is a complete product radical class. It is the smallest complete product radical class containing $\mathcal{I}$. Hence $\mathcal{I}$ is complete if and only if $\mathcal{I} = \mathcal{I}^*$.*

Proof. Let G be an ℓ-group. For sufficiently large σ (depending on G), $\mathcal{I}^\sigma(G) = \mathcal{I}^{\sigma+1}(G) = \ldots$. For such σ, we define $\mathcal{I}^*(G) = \mathcal{I}^\sigma(G)$. Clearly $G \in \mathcal{I}^*$ if and only if $\mathcal{I}^*(G) = G$. We will show that $\mathcal{I}^*(G)$ satisfies (i) and (ii) of Theorem 2.1. For $A \in \mathcal{C}(G)$, $\mathcal{I}^\sigma(G) = \mathcal{I}^{\sigma+1}(G) = \ldots$ implies $\mathcal{I}^\sigma(A) = A \cap \mathcal{I}^\sigma(G) = A \cap \mathcal{I}^{\sigma+1}(G) = \mathcal{I}^{\sigma+1}(A) = \ldots$. So $\mathcal{I}^*(A) = \mathcal{I}^\sigma(A) = A \cap \mathcal{I}^\sigma(G) = A \cap \mathcal{I}^*(G)$.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups. For sufficiently large σ $\mathcal{I}^\sigma\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big) = \mathcal{I}^{\sigma+1}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big) = \ldots$. Hence $\prod\limits_{\lambda\in\Lambda} \mathcal{I}^\sigma(G_\lambda) = \prod\limits_{\lambda\in\Lambda} \mathcal{I}^{\sigma+1}(G_\lambda) = \ldots$, and so $\mathcal{I}^\sigma(G_\lambda) = \mathcal{I}^{\sigma+1}(G_\lambda) = \ldots$ for each $\lambda \in \Lambda$. It follows that

$$\mathcal{I}^*\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) = \mathcal{I}^\sigma\Big(\prod_{\lambda\in\Lambda} G_\lambda\Big) = \prod_{\lambda\in\Lambda} \mathcal{I}^\sigma(G_\lambda) = \prod_{\lambda\in\Lambda} \mathcal{I}^*(G_\lambda).$$

This proves that $\mathcal{I}^*$ is a product radical class.

By using the transfinite induction we can show that $\mathcal{I}^*.\mathcal{I}^\sigma = \mathcal{I}^*$, so $\mathcal{I}^*$ is complete. If $\mathcal{U}$ is a complete product radical class containing $\mathcal{I}$ then by another induction approach we have $\mathcal{I}^\sigma \subseteq \mathcal{U}$ for each ordinal σ. Thus $\mathcal{I}^* \subseteq \mathcal{U}$ as claimed. □

$\mathcal{I}^*$ is called the completion of $\mathcal{I}$.

Similarly to Theorem 1.7 in [11] we have

Proposition 4.5. *Let $\mathcal{I}$ be a product radical class, and let G be an ℓ-group. Then $\mathcal{I}^*(G) \subseteq \mathcal{I}(G)^{\perp\perp}$. That is, $\mathcal{I}^* \subseteq \mathcal{I}^{\perp\perp}$.*

Corollary 4.6. *A polar product radical class is complete, that is, $T^p_{1'23'} \subseteq T_{1'23'5} \subseteq T_{1'23'} \subseteq T_{1'2}$.*

From Proposition 4.3 and Proposition 4.5 we also get

Corollary 4.7. *For any product radical class $\mathcal{I}$, $(\mathcal{I}^*)^{\perp\perp} = \mathcal{I}^{\perp\perp}$.*

Now we give a more precise description of the polar product radical class.

Propositon 4.8. *Let $\mathcal{R}$ be a product radical class, then $\mathcal{R}^{\perp\perp} = \{G \mid \mathcal{R}(C) \neq 0$ for each convex ℓ-subgroup $C \neq 0$ of $G\}$.*

Proof. $\mathcal{R}^\perp(G)$ is the largest convex ℓ-subgrup C of G such that $\mathcal{R}(C) = 0$. So $\mathcal{R}^\perp(G) = 0$ if and only if $\mathcal{R}(C) \neq 0$ for each convex ℓ-subgroup $C \neq 0$ of G. It follows from Theorem 4.1 that $G \in \mathcal{R}^{\perp\perp}$ if and only if $\mathcal{R}^\perp(G) = 0$, if and only if $\mathcal{R}(C) \neq 0$ for each convex ℓ-subgroup $C \neq 0$ of G. □

The following theorem is a direct consequence of Proposition 4.8.

Theorem 4.9. *Let $\mathcal{R}$ be a product radical class. Then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is a polar product radical class.*

(2) *If* $\mathcal{R}(C) \neq 0$ *for each convex* ℓ*-subgroup* C *of* G*, then* $G \in \mathcal{R}$*.*

(3) *If for each* $0 < x \in G$ *there exists an element* $0 < y \leqslant nx$ *(with a suitable integer* n*) such that* $G(y) \in \mathcal{R}$*, then* $G \in \mathcal{R}$*.*

Corollary 4.10. *Let* $\mathcal{I}$ *and* $\mathcal{R}$ *be product radical classes and* $\mathcal{I} \cap \mathcal{R} = 0$*. Then* $\mathcal{I}^{\perp\perp} \cap \mathcal{R}^{\perp\perp} = 0$ *and* $\mathcal{I}^* \cap \mathcal{R}^* = 0$*.*

Proof. Suppose $\mathcal{I} \cap \mathcal{R} = 0$ and $0 \neq G \in \mathcal{I}^{\perp\perp} \cap \mathcal{R}^{\perp\perp}$. It follows from Proposition 4.8 that $\mathcal{I}(C) \neq 0$ for each convex ℓ-subgroup $C \neq 0$ of G. In particular, $\mathcal{I}(G) \neq 0$. $\mathcal{I}(G) \in \mathcal{R}^{\perp\perp}$ implies $\mathcal{R}(\mathcal{I}(G)) \neq 0$. Thus $o \neq \mathcal{R}(\mathcal{I}(G)) \in \mathcal{I} \cap \mathcal{R}$. This contradicts $\mathcal{I} \cap \mathcal{R} = 0$. Hence $\mathcal{I}^{\perp\perp} \cap \mathcal{R}^{\perp\perp} = 0$. It follows from Proposition 4.5 and $\mathcal{I}^{\perp\perp} \cap \mathcal{R}^{\perp\perp} = 0$ that $\mathcal{I}^* \cap \mathcal{R}^* = 0$. □

From Proposition 4.4 in [4] and the above Proposition 3.2 and Corollary 4.2 we get

Corollary 4.11. *For any product radical class* $\mathcal{R}$*,* $\mathcal{R}^{\perp\perp}$ *is complete.*

This corollary is also a consequence of Proposition 4.3(1) and Corollary 4.6.

Similarly to Theorem 4.8 in [4] we have

Propositon 4.12. *The mapping* $\mathcal{R} \to \mathcal{R}^{\perp\perp}$ *is a semigroup endomorphism in* $T_{1'23'}$*.*

Corollary 4.13. $T^p_{1'23'}$ *is a subsemigroup of* $T_{1'23'}$*.*

5. 1′23′-HOMOGENEOUS ℓ-GROUPS

For a family X of ℓ-groups we denote by $\mathcal{R}(X)$ the intersection of all $\mathcal{I} \in T_{1'23'}$ with $X \subseteq \mathcal{I}$. It is said to be the product radical class generated by X. The product radical class generated by an ℓ-group G is denoted by $\mathcal{R}_G$. For a family X of ℓ-groups let $J(X)$ be the joins $G = \bigvee_{\lambda \in \Lambda} G_\lambda$ with $G_\lambda \in X \cap \mathcal{C}(G)(\lambda \in \Lambda)$. Let $P(X)$ and $C(X)$ denote the classes of ℓ-groups which are products or convex ℓ-subgroups, respectively, of elements of X. Clearly $J(X)$, $P(X)$ and $C(X)$ are the classes containing X and belonging to T_2, $T_{3'}$ and $T_{1'}$, respectively.

Theorem 5.1. *Suppose that* X *is any family of* ℓ*-groups. Then* $\mathcal{R}(X) = JCP(X)$ *provided* $CP(X)$ *is closed under forming finite joins of convex* ℓ*-subgroups.*

Proof. It is clear that $JCP(X)$ is closed under taking convex ℓ-subgroups and forming joins of convex ℓ-subgroups. We proceed in the following two steps to show that $JCP(X)$ is closed under forming the direct products.

(a) $CP(X)$ is closed under forming the direct product. In fact, let $\{G_\alpha \mid \alpha \in A\} \subseteq CP(X)$, that is $G_\alpha \in \mathcal{C}\Big(\prod_{\alpha_\lambda \in \Lambda_\alpha} G_{\alpha_\lambda}\Big)$ $(\alpha \in A)$ where $G_{\alpha_\lambda} \in X$ for each α_λ. Then $\prod_{\alpha \in A} G_\alpha \in \mathcal{C}\Big(\prod_{\alpha \in A}\Big(\prod_{\alpha_\lambda \in \Lambda_\alpha} G_{\alpha_\lambda}\Big)\Big) = \mathcal{C}\Big(\prod_{\alpha_\lambda \in \Lambda_\alpha} G_{\alpha_\lambda}\Big)$.

(b) Let $\{G_{\alpha_\lambda} \mid \alpha_\lambda \in \Lambda_\alpha\} \subseteq CP(X)$ and $G^\alpha = \bigvee_{\alpha_\lambda \in \Lambda_\alpha} G_{\alpha_\lambda}$ where $G_{\alpha_\lambda} \in \mathcal{C}(G^\alpha)$ $(\alpha \in A)$. For each $\alpha \in A$ put $G_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}} = \bigvee_{i=1}^{n} G_{\alpha_{\lambda_i}}$ where $\alpha_{\lambda_i} \in \Lambda_\alpha$ $(i = 1, \dots, n)$. Let $\mathcal{H}_\alpha$ be the set of all ℓ-groups of the form $G_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}$ $(\alpha \in A)$. By the assumption $\mathcal{H}_\alpha \subseteq CP(X)$ and clearly $\mathcal{H}_\alpha \subseteq \mathcal{C}(G^\alpha)$. By (a), $\prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\} \in CP(X)$. It is also clear that each $\prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\}$ is a convex ℓ-subgroup of $\prod_{\alpha \in A} G^\alpha$. Then

$$\bigvee \prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\} \subseteq \prod_{\alpha \in A} G^\alpha. \tag{6}$$

For any $a = (\dots, a_\alpha, \dots) \in \prod_{\alpha \in A} G^\alpha$, a_α belongs to some $H_\alpha \in \mathcal{H}_\alpha$ $(\alpha \in A)$. Consequently, $a = (\dots, a_\alpha, \dots)$ belongs to some $\prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\}$. Hence

$$\prod_{\alpha \in A} G^\alpha \subseteq \bigvee \prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\}. \tag{7}$$

Combining (6) and (7) we get

$$\prod_{\alpha \in A} G^\alpha = \bigvee \prod\{H_\alpha \in \mathcal{H}_\alpha \mid \alpha \in A\}.$$

Therefore $\prod_{\alpha \in A} G^\alpha \in JCP(X)$.

Thus $JCP(X)$ is a product radical class containing X. It is obvious that $JCP(X)$ is the smallest product radical class containing X. □

In another paper we will determine the product radical classes generated by the integer group Z and by the real group R using the structure theory of a complete ℓ-group [13] and Theorem 5.1. The main results are:

The following assertions are equivalent:

(1) $G \in \mathcal{R}_Z$,

(2) G is an ideal subdirect product of Z,

(3) G is a complete ℓ-group which has no continuous convex ℓ-subgroup, and each convex ℓ-subgroup of G has a singular element.

The following assertions are equivalent:

(1) $G \in \mathcal{R}_R$,

(2) G is an ideal subdirect product of R,

(3) G is a complete ℓ-group which has no continuous convex ℓ-subgroup, and for each convex ℓ-subgroup of K of G we have $|K| > \aleph_0$.

Proposition 5.2. *Let G be an ℓ-group. Then there exists a unique largest product radical class $\mathcal{R}^G$ such that $\mathcal{R}^G(G) = 0$.*

Proof. $\mathcal{R}_G(G) = G$ implies $\mathcal{R}_G^{\perp}(G) = [\mathcal{R}_G(G)]^{\perp} = G^{\perp} = 0$ by Theorem 4.1. Suppose that $\mathcal{I}$ is a product radical class so that $\mathcal{I}(G) = 0$ and $\mathcal{I} \supseteq \mathcal{R}_G^{\perp}$. Then

$$\mathcal{I}^{\perp\perp} \supseteq \mathcal{I} \supseteq \mathcal{R}_G^{\perp} \tag{8}$$

and $\mathcal{I}^{\perp\perp}(G) = (\mathcal{I}(G)^{\perp})^{\perp} = 0$. On the other hand, $(\mathcal{I}^{\perp\perp})^{\perp}(G) = [\mathcal{I}^{\perp\perp}(G)]^{\perp} = G$, that is $G \in \mathcal{I}^{\perp\perp\perp}$ and $\mathcal{I}^{\perp\perp\perp} \supseteq \mathcal{R}_G$. It follows from the formula (5) that

$$\mathcal{I}^{\perp\perp} = (\mathcal{I}^{\perp\perp\perp})^{\perp} \subseteq \mathcal{R}_G^{\perp}. \tag{9}$$

(8) and (9) infer $\mathcal{I}^{\perp\perp} = \mathcal{R}_G^{\perp}$ and $\mathcal{I} = \mathcal{R}_G^{\perp}$. Thus $\mathcal{R}_G^{\perp}$ is the largest product radical class $\mathcal{R}^G$ such that $\mathcal{R}^G(G) = 0$. □

Corollary 5.3. *For any ℓ-group G, $\mathcal{R}^G \cap \mathcal{R}_G = 0$.*

Since $\mathcal{R}^G.\mathcal{R}^G(G)/\mathcal{R}^G(G) = \mathcal{R}^G(G/\mathcal{R}^G(G))$, that is $\mathcal{R}^G.\mathcal{R}^G(G) = \mathcal{R}^G(G) = 0$, so $\mathcal{R}^G$ is complete.

An ℓ-group G is called $1'23'$-homogeneous if for each product radical class $\mathcal{I}$, either $G \in \mathcal{I}$ or else $\mathcal{I}(G) = 0$. If G is $1'23'$-homogeneous, then $\mathcal{R}^G$ is meet irreducible. Conversely, let a proper product radical class $\mathcal{R}$ be meet irreducible. Let $\mathcal{Y}$ be its cover. Select $G \in \mathcal{Y} \setminus \mathcal{R}$. Put $G_0 = \mathcal{R}(G)^{\perp}$. Then $G_0 \neq 0$ and $\mathcal{R}(G_0) = 0$ by Theorem 4.1. If $\mathcal{I}$ is a product radical class with $\mathcal{I}(G_0) \neq 0$, then $\mathcal{R} \vee \mathcal{I} \neq \mathcal{R}$ by the formula (4). Thus $\mathcal{Y} \subseteq \mathcal{R} \vee \mathcal{I}$ and $G_0 = \mathcal{Y}(G_o) \subseteq \mathcal{R}(G_0) + \mathcal{I}(G_0) = \mathcal{I}(G_0)$, i.e. $G_0 \in \mathcal{I}$. Hence G_0 is $1'23'$-homogeneous. Clearly $\mathcal{R} \subseteq \mathcal{R}^{G_0}$. If $\mathcal{R} \neq \mathcal{R}^{G_0}$, then $\mathcal{R} \subseteq \mathcal{Y} \subseteq \mathcal{R}^{G_0}$. But $\mathcal{Y}(G_0) = G_0$, which contradicts $\mathcal{R}^{G_0}(G_0) = 0$. Therefore $\mathcal{R} = \mathcal{R}^{G_0}$.

The above discussion yields the following result:

Theorem 5.4. *A product radical class $\mathcal{R}$ is meet irreducible if and only if $\mathcal{R} = \mathcal{R}^G$ for some $1'23'$-homogeneous ℓ-group G.*

Corollary 5.5. *Any meet irreducible product radical class is complete.*

If G is $1'23'$-homogeneous, then $\mathcal{R}^G \vee \mathcal{R}_G$ is the cover of $\mathcal{R}^G$, and so $\mathcal{R}_G$ is the cover of $\mathcal{R}_G \wedge \mathcal{R}^G = 0$. Hence $\mathcal{R}_G$ is join irreducible. Conversely, if a product radical class $\mathcal{R}$ is join irreducible, then since $\mathcal{R} = \bigvee_{G \in \mathcal{R}} \mathcal{R}_G$, we have $\mathcal{R} = \mathcal{R}_G$ for some G in $\mathcal{R}$.

Finally, we give a sufficient and necessary condition under which an ℓ-group G is $1'23'$-homogeneous.

Proposition 5.6. *Let G be an ℓ-group. Then $R(G)$ is lattice isomorphic into the interval $[0, \mathcal{R}_G]$ of the lattice $T_{1'23'}$.*

Proof. For each $G_1 \in R(G)$, put $\varphi(G_1) = \mathcal{R}_G$. It is easy to show that φ is a lattice isomorphism from $R(G)$ into $[0, \mathcal{R}_G]$. □

Since G is $1'23'$-homogeneous if and only if $|R(G)| \leqslant 2$, we get

Theorem 5.7. *An ℓ-group G is $1'23'$-homogeneous if and only if $\mathcal{R}_G$ is an atom of $T_{1'23'}$.*

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