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ON THE CONNECTEDNESS OF THE SET OF FIXED POINTS OF A COMPACT OPERATOR IN THE FRÉCHET SPACE $C^m(\langle b,\infty),\mathbf{R}^n)$

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INTRODUCTION

Several authors (e.g. N. Aronszajn in [2], M. Hukuhara in [7], M. A. Krasnosel'skij and A. I. Perov in [8], G. Stampacchia in [14], F. E. Browder and G. P. Gupta in [4], G. Vidossich in [19], S. Szufla in [15]–[18], R. R. Achmerov, M. I. Kamenskij, A. S. Potapov in [1], M. A. Krasnosel'skij, P. P. Zabrejko in [9] and B. N. Sadovskij in [13]) have investigated the compactness as well as the connectedness of the set of all fixed points of a compact operator or an operator of a more general type mostly in a Banach space. Only few of them have been interested in this problem in a more general space (P. Morales in [12], Š. Belohorec in [3], Z. Kubáček in [10] and K. Czarnowski, T. Pruszko in [5]). Here the results from a Banach space will be extended to a Fréchet space. Our considerations will be based on the following results which are given as Lemmas.

Lemma 1 ([10], p. 422). *Let X be a Hasdorff topological vector space, M a non-empty closed subset of X, $F\colon M \to X$ a compact mapping, and let B denote the neighborhood base of the point 0 consisting of balanced sets. Let the following conditions be satisfied:*

(i) *for each set $U \in B$ there exists a compact mapping $F_U\colon M \to X$ such that*

$$F(x) - F_U(x) \in U \quad \text{for each} \quad x \in M;$$

(ii) *for each $U \in B$ and for each $x \in U$ the equation*

$$y - F_U(y) = x$$

has a unique solution $y \in M$.

Then the set S of fixed points of the mapping F is nonempty, compact and connected.

Lemma 2 ([6], pp. 89–90, [20], pp. 55–56). *Let $(X, \|\cdot\|)$ be a real Banach space, Ω a non-empty open and bounded subset of X, $F\colon \bar{\Omega} \to X$ a compact mapping which satisfies the strengthened Leray – Schauder condition:*
there exists an $x_0 \in \Omega$ such that

$$F(x) - x_0 \neq \lambda(x - x_0) \quad \text{for each } x \in \partial\Omega \text{ and each } \lambda \geqslant 1.$$

Further, let there exist a sequence of compact mappings $F_p\colon \bar{\Omega} \to X$, $p = 1, 2, \dots$ with the properties

a) $\delta_p = \sup\{\|F_p(x) - F(x)\| : x \in \bar{\Omega}\} \to 0$ *for* $p \to \infty$;

b) *the equation (in y)*

$$y - F_p(y) = F(x) - F_p(x)$$

has at most one solution in $\bar{\Omega}$ for each $x \in \Omega$.

Then the set S of fixed points of the mapping F is non-empty, compact and connected.

The next Lemma is a consequence of the theorem in [11], p. 111.

Lemma 3. *Let (X, d) be a metric space and $\{S_m : m = 1, 2, \dots\}$ a sequence of non-empty compact and connected sets such that*

$$S_{m+1} \subset S_m \quad \text{for} \quad m = 1, 2, \dots.$$

Then $\bigcap\limits_{m=1}^{\infty} S_m$ is a non-empty compact and connected set.

We shall use the following notation.

Let $-\infty < b < \infty$ and let $n > 0$, $k \geqslant 0$ be integers, $I_b = \langle b, \infty)$, $|\cdot|$ a norm in $\mathbf{R}^n$. Let

$$X = C^k(I_b, \mathbf{R}^n), p_m(x) = \max\{|x(t)| + \ldots + |x^{(k)}(t)| : b \leqslant t \leqslant b + m\}$$

for each $x \in X$ and each $m = 1, 2, \dots$. The space $(X, \{p_m\})$ is a real Fréchet space and the convergence in this space means the uniform convergence of the functions and their first k derivatives on each interval $\langle b, b + m\rangle$, $m = 1, 2, \dots$.

Further, let

$$X_m = C^k(\langle b, b + m\rangle, \mathbf{R}^n) \text{ for each } m = 1, 2, \dots.$$

Then p_m is a norm in X_m and (X_m, p_m) is a real Banach space.

Let $h > 0$ and $\psi \in C^k(\langle -h, 0\rangle, \mathbf{R}^n)$. Let $\varphi, \varphi_p \in C(I_b, (0, \infty))$, $p = 1, 2, \ldots$ where the sequence $\{\varphi_p\}$ is nonincreasing in I_b and $\lim_{p\to\infty} \varphi_p(t) = 0$ for each $t \in I_b$.

Denote

$$
\begin{aligned}
M &= \{x \in X : |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leqslant \psi(t) \\
&\qquad \text{for each } t \in I_b \text{ and } x^{(j)}(b) = \psi^{(j)}(0),\ j = 0, 1, \ldots, k\}, \\
M_m &= \{x \in X_m : |x(t) - \psi(0)| + \ldots + |x^{(k)}(t) - \psi^{(k)}(0)| \leqslant \varphi(t), \\
&\qquad t \in \langle b, b+m\rangle \text{ and } x^{(j)}(b) = \psi^{(j)}(0), j = 0, 1, \ldots, k\}, \quad m = 1, 2, \ldots.
\end{aligned}
$$

$M(M_m)$ is a closed, convex and bounded set in X (in X_m, $m = 1, 2, \ldots$). Clearly, if $x \in M$ or $x \in M_{m+p}$, then $x\big|_{\langle b, b+m\rangle} \in M_m$ for each $m = 1, 2, \ldots, p = 1, 2, \ldots$. Here and in the sequel $f\big|_{\langle a, b\rangle}$ denotes the restriction of the function f to the interval $\langle a, b\rangle$.

Main results

The approximation Lemma which follows represents the main tool in obtaining the new results.

Lemma 4. *Let the spaces X, X_m, $m = 1, 2, \ldots$, the functions ψ, φ and the sets M, M_m, $m = 1, 2, \ldots$ be as above. Let there exist mappings $T \colon M \to X$, $T_m \colon M_m \to X_m$, $m = 1, 2, \ldots$ with the properties*

(1) $x\big|\langle b, b+m\rangle = y\big|\langle b, b+m\rangle \Rightarrow T(x)\big|\langle b, b+m\rangle = T(y)\big|\langle b, b+m\rangle$ *for each* $x, y \in M$, $m = 1, 2, \ldots$;

(2) $T_m(x\big|\langle b, b+m\rangle) = T(x)\big|\langle b, b+m\rangle$ *for each* $x \in M$, $M = 1, 2, \ldots$;

(3) $x\big|\langle b, b+m\rangle = y\big|\langle b, b+m\rangle \Rightarrow T_{m+p}(x)\big|\langle b, b+m\rangle = T_{m+p}(y)\big|\langle b, b+m\rangle$ *for each* $x, y \in M_{m+p}$, $m = 1, 2, \ldots, p = 1, 2, \ldots$;

(4) $T_m(x\big|\langle b, b+m\rangle) = T_{m+p}(x)\big|\langle b, b+m\rangle$ *for each* $x \in M_{m+p}$, $m = 1, 2, \ldots$, $p = 1, 2, \ldots$.

Further, let the set S_m^* of all fixed points of the operator T_m be nonempty, compact and connected in the space X_m. Then the set S of all fixed points of the operator T is nonempty, compact and connected in the space X.

Proof. Let $m_0 \geqslant 1$ be a fixed integer. Let

$$S_m = \{x\big|\langle b, b+m_0\rangle : x \in S_m^*\} \text{ for all } m \geqslant m_0.$$

Fix an arbitrary $m \geqslant m_0$. Clearly $S_m \neq \emptyset$. Since the mapping from X_m to X_{m_0} which to each function $x \in X_m$ assigns the restriction $x|\langle b, b+m_0\rangle$ is continuous, S_m is compact and connected. Since $m \geqslant m_0$ is arbitrary, by Lemma 3 we get that

$$P_{m_0} = \bigcap_{m=m_0}^{\infty} S_m \neq \emptyset, \tag{6}$$

and it is a compact and connected set.

Denote by S the set of all fixed points of the operator T. If $x \in S$, then in view of (6)

$$y_m = x|\langle b, b+m\rangle \in S_m^* \text{ for each } m \geqslant m_0$$

and hence

$$y = y_m|\langle b, b+m_0\rangle = x|\langle b, b+m_0\rangle \in P_{m_0}.$$

Conversely, let $y \in P_{m_0}$. Then for each $m \geqslant m_0$ there is a $y_m \in S_m^*$ such that $y_m|\langle b, b+m_0\rangle = y$. We shall show that there is an $x \in S$ such that $y = x|\langle b, b+m_0\rangle$. Consider the sequence $\{y_m\}_{m=m_0+1}^{\infty}$. As by (4) the sequence $\{y_m|\langle b, b+m_0+1\rangle\} \subset S_{m_0+1}^*$ and the last set is compact, there exists a subsequence $\{y_{m_1}\}$ of the sequence $\{y_m\}$ and a point $\tilde{y}_1 \in S_{m_0+1}^*$ such that the sequence $\{y_{m_1}^{(j)}|\langle b, b+m_0+1\rangle\}$ converges uniformly to $\tilde{y}_1^{(j)}$ on $\langle b, b+m_0+1\rangle$, $j = 0, \ldots, k$. By mathematical induction we get a sequence of sequences

$$\{y_{m_1}\}, \{y_{m_2}\}, \ldots, \{y_{m_r}\}, \ldots$$

such that

(i) the sequence $\{y_{m_1}\}$ is a subsequence of the sequence $\{y_m\}$;

(ii) $\{y_{m_{r+1}}\}$ is a subsequence of the sequence $\{y_{m_r}\}$ for $r = 1, 2, \ldots$;

(iii) the sequence $\{y_{m_r}^{(j)}|\langle b+m_0+r\rangle\}$ converges uniformly on $\langle b, b+m_0+r\rangle$ for $j = 0, \ldots, k$ and $\{y_{m_r}|\langle b, b+m_0+r\rangle\} \subset S_{m_0+r}^*$.

Then the diagonal sequence $\{y_{m_m}\}$ possesses the property that $\{y_{m_m}^{(j)}\}$ converges uniformly on each interval $\langle b, b+m_0+r\rangle$ to $x^{(j)}$ for $j = 0, \ldots, k$ where $x \in X$ is a certain function. As $y_{m_m}|\langle b, b+m_0+m\rangle \in S_{m_0+m}^*$, also $x|\langle b+m_0+m\rangle \in S_{m_0+m}^*$ and by (2), $x \in S$.

Hence $S \neq \emptyset$ and P_{m_0} is the set of restrictions to $\langle b, b+m_0\rangle$ of all functions belonging to S, for each $m_0 = 1, 2, \ldots$. Now we prove that S is a compact set in X.

Let $\{x_p\} \subset S$ be a sequence of points. Then by the compactness of the sets P_1, P_2, ... in the spaces X_1, X_2, ... respectively we get that there exist sequences

$$\{x_{p,1}\}, \{x_{p,2}\}, \ldots$$

such that

(i) $\{x_{p,1}\}$ is a subsequence of the sequence $\{x_p\}$;

(ii) $\{x_{p,r+1}\}$ is a subsequence of the sequence $\{x_{p,r}\}$ for $r = 1, 2, \ldots$;

(iii) the sequence $\{x_{p,r}\}$ together with its first k derivatives converges uniformly on $\langle b, b+r\rangle$.

Then the diagonal sequence $\{x_{p,p}\}$ converges in the space X to a point $x \in X$ with the property that $x\big|\langle b, b+m\rangle \in S_m^*$ and by (2), $x \in S$.

Finally, we prove that S is connected. If not, the set S can be decomposed into the union

$$S = \hat{K}_1 \cup \hat{K}_2$$

where $\hat{K}_1$, $\hat{K}_2$ are two non-empty, disjoint and compact sets. Let $m \geqslant 1$ be a natural number. Denote by $\hat{K}_{1m}$ and $\hat{K}_{2m}$ the sets of restrictions to $\langle b, b+m\rangle$ of the functions from $\hat{K}_1$ and $\hat{K}_2$, respectively. Hence we have

$$P_m = \hat{K}_{1m} \cup \hat{K}_{2m}.$$

The compactness of $\hat{K}_1, \hat{K}_2$ implies that $\hat{K}_{1m}, \hat{K}_{2m}$ are nonempty, compact sets in X_m. If they were disjoint, then P_m would not be connected in X_m. Hence there exist two elements $x_m \in \hat{K}_1$, $y_m \in \hat{K}_2$, $x_m \neq y_m$ such that their restrictions to $\langle b, b+m\rangle$ coincide. Thus

$$x_m\big|\langle b, b+m\rangle = y_m\big|\langle b, b+m\rangle\,. \tag{7}$$

Consider the sequences $\{x_m\}$, $\{y_m\}$. As $\{x_m\} \subset \hat{K}_1$, $\{y_m\} \subset \hat{K}_2$ and $\hat{K}_1$, $\hat{K}_2$ are compact in X, there exist two subsequences $\{x_{m_l}\}$, $\{y_{m_l}\}$ of the sequences $\{x_m\}$, $\{y_m\}$, respectively, and there exist two elements $x \in \hat{K}_1$, $y \in \hat{K}_2$ such that $\lim\limits_{l\to a} x_{m_l} = x$, $\lim\limits_{l\to\infty} y_{m_l} = y$ in X. Then with respect to (7) we have $x = y$. This contradicts the fact that $\hat{K}_1 \cap \hat{K}_2 = \emptyset$. Hence S is connected. □

Now by means of Lemmas 1 and 2 a sufficient condition for the sets S_m^* in Lemma 4 to be non-empty, compact and connected can be given. This is the content of the next theorem.

Theorem 1. *Suppose that all assumptions of Lemma 4 are satisfied except the assumption on the sets S_m^*, $m = 1, 2, \ldots$. Suppose, further, that for each $m = 1, 2, \ldots$*

$$T_m\colon M_m \subset X_m \to X_m \text{ is a compact mapping,} \tag{8}$$

and there exists a sequence $\{T_{mp}\}_{p=1}^{\infty}$ of mappings

$$T_{mp}\colon M_m \to X_m$$

with the following properties: For each $p = 1, 2, \ldots$

(9) $T_{mp}\colon M_m \subset X_m \to X_m$ *is a compact mapping;*

(10) $|T_m(x)(t) - T_{mp}(x)(t)| + \ldots + |(T_m(x))^{(k)}(t) - (T_{mp}(x))^{(k)}(t)| \leqslant \varphi_p(t)$ *for each $x \in M_m$ and each $t \in \langle b, b+m\rangle$,*

and either

(11) *there exists a function $\varphi_{*p} \in C(I_b, \langle 0,\infty))$ such that*

$$\varphi_{*p} + \varphi_p \leqslant \varphi \quad \text{in } I_b$$

and

$$|T_{mp}(x)(t) - \psi(0)| + \ldots + |(T_{mp}(x))^{(k)}(t) - \psi^{(k)}(0)| \leqslant \varphi_{*p}(t)$$

for all $x \in M_m$ and all $t \in \langle b, b+m\rangle$;

(12) *the operator $H_{mp}\colon M_m \to X_m$ which is defined by the relation*

$$H_{mp}(x) = x - T_{mp}(x) \quad \text{for all } x \in M_m$$

is injective on M_m,

or

(13) *there exists an $x_m \in \dot{M}_m$ (the interior of M_m) such that*

$$T_m(x) - x_m \neq \lambda(x - x_m)$$

for each $x \in \partial M$ and each $\lambda \geqslant 1$;

(14) *the equation*

$$H_{mp}(y) = x$$

has at most one solution in M_m for each $x \in X_m$ such that

$$|x(t)| + \ldots + |x^{(k)}(t)| \leqslant \varphi_p(t), \quad b \leqslant t \leqslant b+m.$$

(Here H_{mp} has the same meaning as in (12)).

Then the set S of all fixed points of the operator T is non-empty, compact and connected in the space X.

Proof. With respect to Lemma 4 it suffices to show that the set S_m^* of all fixed points of the operator T_m is non-empty, compact and connected for each $m = 1, 2, \ldots$. Hence, let $m \geqslant 1$ be an arbitrary but fixed number. Consider the case when the assumptions (11), (12) are satisfied. Then we apply Lemma 1 to the operator T_m in the space X_m. In this space we have two systems of balanced neighborhoods of 0:

$$U\Big(0,\frac{1}{j}\Big) = \Big\{x \in X_m : p_m(x) < \frac{1}{j}\Big\}, \quad j = 1,2,\ldots;$$
$$U_p = \{x \in X_m : |x(t)| + \ldots + |x^{(k)}(t)| \leqslant \varphi_p(t), b \leqslant t \leqslant b+m\},\ p = 1,2,\ldots.$$

By the Dini theorem the sequence $\{\varphi_p\}$ converges uniformly to 0 on $\langle b, b+m\rangle$ and both systems of neighborhoods determine the same topology in X_m. For each U_p there exists a compact mapping $T_{mp}\colon M_m \subset X_m \to X_m$ such that, in view of (10), $T_m(x) - T_{mp}(x) \in U_p$ for each $x \in M_m$.

As to the assumption (ii) in Lemma 1, by the assumption (12) it suffices to show that the equation

$$H_{mp}(y) = x \tag{15}$$

has at least one solution in M_m for each $x \in U_p$. So let us fix an arbitrary $x \in U_p$. Since M_m is a closed and convex set in X_m, the operator $T_{mp} + x\colon M_m \subset X_m \to X_m$ is compact and moreover

$$\begin{aligned}|T_{mp}(y)(t) - \psi(0)| + |x(t)| + \ldots + |(T_{mp}(y))^{(k)}(t) - \psi^{(k)}(0)| + |x^{(k)}(t)|\\ \leqslant \varphi_{*p}(t) + \varphi_p(t) \leqslant \varphi(t) \ \text{ for each } \ t \in \langle b, b+m\rangle,\end{aligned}$$

which means that $T_{mp} + x\colon M_m \to M_m$, by the Schauder fixed point theorem the equation (15) has a solution in M_m and the statement of the theorem follows.

When the assumptions (13) and (14) are fulfilled, then we use Lemma 2. We take (X_m, p_m) for the real Banach space, $\dot{M}_m$ for Ω and $T_m\colon M_m \subset X_m \to X_m$ for the compact mapping F. By (13) T_m satisfies the strengthned Leray-Schauder condition. When $\{T_{mp}\}_{p=1}^{\infty}$ is a sequence of compact mappings which approximates the mapping T_m, then by (10)

$$\begin{aligned}\delta_p &= \sup\{p_m(T_{mp}(x) - T_m(x))\colon x \in M_m\}\\ &= \max\{\varphi_p(t)\colon b \leqslant t \leqslant b+m\} \to 0 \ \text{ for } \ p \to \infty.\end{aligned}$$

Let $x \in \dot{M}_{\dot{m}}$. Then again by (10) $T_m(x) - T_{mp}(x) \in U_p$ and then (14) implies that the assumption b) of Lemma 2 is satisfied, too. By this Lemma the theorem is true. □

Theorem 1 will be applied to the initial value problem for a functional differential equation. First we consider a similar problem for an ordinary differential equation.

Let $\omega \in C(I_b, \langle 0, \infty))$, let $F \in C(\langle 0, \infty), (0, \infty))$ be a non-decreasing function and let $c \geqslant 0$. Then one can find that a necessary and sufficient condition for the problem

$$y'(t) = \omega(t)F(y + c), \quad y(b) = 0 \tag{16}$$

to have a unique solution on $\langle b, \infty)$ is that

$$\int_b^\infty \omega(s)\,\mathrm{d}s \leqslant \int_0^\infty \frac{\mathrm{d}v}{F(v + c)}.$$

Further, denote $H = C(\langle -h, 0\rangle, \mathbf{R}^n)$, $\|x\| = \max\{|x(s)| : -h \leqslant s \leqslant 0\}$ for each $x \in H$. Then $(H, \|\cdot\|)$ is a Banach space. If $x : \langle b - h, \infty) \to \mathbf{R}^n$ is a continuous function, then $x_t \in H$ is defined by $x_t(s) = x(t + s)$, $-h \leqslant s \leqslant 0$, for each $t \in I_b$. In the space $X^* = C(\langle b - h, \infty), \mathbf{R}^n)$ let the topology be defined by the seminorms $q_m(x) = \max\{|x(t)| : b - h \leqslant t \leqslant b + m\}$, $m = 1, 2, \ldots$, $x \in X^*$. Clearly $(X^*, \{q_m\}_{m=1}^\infty)$ is a Fréchet space.

Theorem 2. *Let $\psi \in H$, $f \in C(I_b \times H, \mathbf{R}^n)$. Let $\omega \in C(I_b, \langle 0, \infty))$, let $F \in C(\langle 0, \infty), (0, \infty))$ be a nondecreasing function and*

$$\int_b^\infty \omega(s)\,\mathrm{d}s \leqslant \int_0^\infty \frac{\mathrm{d}v}{F(v + |\psi(0)|)}. \tag{17}$$

Let

$$|f(t, \chi)| \leqslant \omega(t)F(\|\chi\|) \ \text{ for each } \ (t, \chi) \in I_b \times M^{**}, \tag{18}$$

where

$$M^{**} = \{x_t \in H : x \in C(\langle b - h, \infty), \mathbf{R}^n), |x(t) - \psi(0)| \leqslant \varphi(t) \text{ for each } t \in I_b, x_b = \psi\}$$

and φ is the solution of (16) on I_b with $c = |\psi(0)|$.

Then the problem

$$x'(t) = f(t, x_t), \quad b \leqslant t < \infty \tag{19}$$

$$x_b = \psi \tag{20}$$

has a solution satisfying the inequality

$$|x(t)-\psi(0)| \leqslant \varphi(t) \quad \text{for each } t \in I_b. \tag{21}$$

The set of all such solutions is compact and connected in the space X^*.

Proof. Consider the Fréchet space $(X, \{p_m\}_{m=1}^{\infty})$ where $X = C(I_b, \mathbf{R}^n)$, and the seminorms $p_m(x) = \max\{|x(t)|: b \leqslant t \leqslant b+m\}$, $m = 1, 2, \ldots$, $x \in X$. This space corresponds to the case $k = 0$ mentioned above. By virtue of (21) the problem (19), (20) is equivalent to the fixed point (f.p. for short) problem for the operator $T^*: M^* \to X^*$ which is defined by

$$T^*(x)(t) = \begin{cases} \psi(0) + \int_b^t f(s, x_s)\,\mathrm{d}s, & b \leqslant t < \infty, \\ \psi(t-b), & b-h \leqslant t \leqslant b \end{cases}$$

on the set $M^* = \{x \in X^*: x_b = \psi \text{ and } |x(t)-\psi(0)| \leqslant \varphi(t) \text{ for each } t \in I_b\}$.

Let

$$V = \{x \in X: x(b) = \psi(0)\},$$
$$V^* = \{x \in X^*: x_b = \psi\}.$$

Define the mapping $P: V \to V^*$ by

$$P(x)(t) = \begin{cases} x(t), & b \leqslant t < \infty, \\ \psi(t-b), & b-h \leqslant t \leqslant b, \end{cases} \quad \text{for each } x \in V.$$

Then P is a bijection of V onto V^* and since $x_p \to x$ in $V \subset X$ for $p \to \infty$ is equivalent to $P(x_p) \to P(x)$ in $V^* \subset X^*$ for $p \to \infty$, P is a homeomorphism of V onto V^*. Clearly the inverse mapping P^{-1} of P is defined by

$$P^{-1}(x) = x\big|\langle b, \infty) \quad \text{for each } x \in V^*.$$

Let $M = \{x \in X: |x(t)-\psi(0)| \leqslant \varphi(t) \text{ for each } t \in I_b \text{ and } x(b) = \psi(0)\}$. Consider now the mapping $T = P^{-1} \circ T^* \circ P\big|M$. Then $T: M \to X$ and

$$T(x)(t) = \psi(0) + \int_b^t f(s, x_s)\,\mathrm{d}s, \quad b \leqslant t < \infty,\ x \in M,\ x_b = \psi. \tag{22}$$

(In fact, the operator T should be defined by

$$T(x)(t) = \psi(0) + \int_b^t f(s, (P(x))_s)\,\mathrm{d}s, \quad b \leqslant t < \infty, x \in M,$$

but it is clear what (22) means. The same notation will be used for the operators T_p, T_m and T_{mp}, which will be defined on M in a similar way.)

Clearly $u \in M$ is a f.p. of T iff $P(u) \in M^*$ is a f.p. of T^*, and in view of the property of P, the set of all f.p. of T^* in M^* is non-empty, compact and connected in M^* iff the set of all f.p. of T in M has the same property. Thus we can apply Theorem 1 to the operator T.

The set M is closed in the Fréchet space X. Define operators $T_p\colon M \to X$ by

$$T_p(x)(t) = \begin{cases} \psi(0), & b \leqslant t \leqslant b + \frac{1}{p}, \\ \psi(0) + \int_b^{t-1/p} f(s, x_s)\,\mathrm{d}s, & b + \frac{1}{p} \leqslant t < \infty, \ x \in M, \ x_b = \psi. \end{cases}$$

Then (18) yields

$$|T(x)(t) - T_p(x)(t)| \leqslant \begin{cases} \int_b^t \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, \ b \leqslant t \leqslant b + \frac{1}{p}, \\ \int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, \ b + \frac{1}{p} \leqslant t < \infty, \\ \qquad x \in M, \ x_b = \psi. \end{cases}$$

Denote by $\varphi_p(t)$ the right-hand side of the last inequality. Hence

$$\varphi_p(t) = \begin{cases} \int_b^t \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, \ b \leqslant t \leqslant b + \frac{1}{p}, \\ \int_{t-1/p}^t \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, \ b + \frac{1}{p} \leqslant t < \infty, \ p = 1, 2, \ldots. \end{cases}$$

Clearly $\{\varphi_p\}$ is a nonincreasing sequence on I_b and $\lim_{p \to \infty} \varphi_p(t) = 0$ for each $t \in \langle b, \infty)$.

Further, when we define

$$\varphi_{*p}(t) = \begin{cases} 0, \ b \leqslant t \leqslant b + \frac{1}{p}, \\ \int_b^{t-1/p} \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, \ b + \frac{1}{p} \leqslant t < \infty, \ p = 1, 2, \ldots \end{cases}$$

then

$$|T_p(x)(t) - \psi(0)| \leqslant \varphi_{*p}(t), t \in I_b, \ p = 1, 2, \ldots, x \in M, \ x_b = \psi$$

and by (16)

$$\varphi_{*p}(t) + \varphi_p(t) = \int_b^t \omega(s) F(\varphi(s) + |\psi(0)|)\,\mathrm{d}s = \varphi(t),$$

for each $t \in I_b$.

Further, the operators $T_m, T_{mp}\colon M_m \subset X_m \to X_m$ defined by

$$T_m(x)(t) = \psi(0) + \int_b^t f(s, x_s)\,\mathrm{d}s, b \leqslant t \leqslant b + m, x_b = \psi,$$

$$T_{mp}(x)(t) = \begin{cases} \psi(0), \ b \leqslant t \leqslant b + \frac{1}{p}, \\ \psi(0) + \int_b^{t-1/p} f(s, x_s)\,\mathrm{d}s, \ b + \frac{1}{p} \leqslant t \leqslant b + m \\ \qquad \text{for } \ m = 1, 2, \ldots, \ p = 1, 2, \ldots \end{cases}$$

are compact. This can be shown in the usual way.

The last step in checking the assumptions of Theorem 1 consists of proving (12). Let the mapping $H_{mp}\colon M_m \to X_m$ be defined by

$$H_{mp}(x) = x - T_{mp}(x) \quad \text{for all} \quad x \in M_m,\ x_b = \psi,\ m = 1,2,\dots,\ p = 1,2,\dots .$$

Consider two elements $x, y \in M_m$, $x \neq y$. Then there exists a $t_0\colon b < t_0 \leqslant b + m$ such that $x(t_0) \neq y(t_0)$. Two cases may occur:

a) If $t_0 \in \langle b, b + \frac{1}{p}\rangle$, then $H_{mp}(x)(t_0) = x(t_0) - \psi(0) \neq y(t_0) - \psi(0) = H_{mp}(y)(t_0)$;

b) there is a $t_1 \geqslant b + \frac{1}{p}$ such that $T_1 = \sup\{\tau > b\colon x(t) = y(t) \text{ for } t \in \langle b, \tau)\}$. Then there exists a $t_0 \in (t_1, t_1 + \frac{1}{p})$ such that $x(t_0) \neq y(t_0)$. This implies that $T_{mp}(x)(t_0) = \psi(0) + \int_b^{t_0 - 1/p} f(s, x_s)\,\mathrm{d}s = \psi(0) + \int_b^{t_0 - 1/p} f(s, y_s)\,\mathrm{d}s = T_{mp}(y)(t_0)$ and hence $H_{mp}(x)(t_0) \neq H_{mp}(y)(t_0)$.

In both cases the operator H_{mp} is injective on M_m and all assumptions of Theorem 1 are satisfied. By this theorem the statement of Theorem 2 follows. □

References

[1] Ахмеров, Р. Р., Каменский, М. И., Потапов, А. С.: Меры некомпактности и уплотняющие операторы, Наука, Новосибирск, 1986.

[2] *Aroszajn, N.*: Le correspondant topologique de l'unicité dans la théorie des équation différentielles, Annals of Mathematics *43* (1942), 730–738.

[3] *Belohorec, Š.*: Generalization of a certain theorem of N. Aroszajn and its application in the theory of functional differential equations, Manuscript. (In Slovak.)

[4] *Browder, F. E., and Gupta, G. P.*: Topological degree and non-linear mappings of analytic type in Banach spaces, Math. Anal. Appl. *26* (1969), 390–402.

[5] *Czarnowski, K., and Pruszko, T.*: On the structure of fixed point sets of compact maps in B_0 spaces with applications to integral and differential equations in unbounded domain, J. Math. Anal. Appl. *54* (1991), 151–163.

[6] *Deimling, K.*: Nichtlineare Gleichungen und Abbildungsgrade, Springer–Verlag, Berlin Heidelberg New York, 1974.

[7] *Hukuhara, M.*: Sur une généralization d'un thèoreme de Kneser, Proc. Japan Acad. *29* (154–155).

[8] Красносельский, М. А., Перов, А. И.: О существовании решений у некоторых нелинейных операторных уравнений, Доклады АН СССР *126* (1959), 15–18.

[9] Красносельский, М. А., Забрейко, П. П.: Геометрические методы нелинейного анализа, Наука, Moskva, 1975.

[10] *Kubáček, Z.*: A generalization of N. Aronszajn's theorem on connectedness of the fixed point set of a compact mapping, Czechoslovak Math. J. *37 (112)* (1987), 415–423.

[11] *Kuratowski, C.*: Topologie, Volume II, Pol. Tow. Mat., Warszawa, 1952.

[12] *Morales, P.*: Topological properties of the set of global solutions for a class of semilinear evolution equations in a Banach space. Atti del Convegno celebrativo del 1^0 centenario del Circolo Matematico di Palermo, Suppl. di Rendiconti del Circolo Matematico di Palermo, Serio II *8* (1985), 379–397.

[13] Садовский, Б. Н.: Предельно компактные и уплотняющие операторы, Успехи мат. наук *27:1* (1972), 81–146.

[14] *Stampacchia, G.*: Le transformazioni che presentano il fenomeno di Peano, Rend. Accad. Naz Lincei *7* (1949), 80–84.
[15] *Szufla, S.*: On the structure of solution sets of differential and integral equations in Banach spaces, Ann. Polon. Math *XXXIV* (1977), 165–177.
[16] *Szufla, S.*: On Volterra integral equations in Banach spaces, Funkcialaj Ekvacioj *20* (1977), 247–258.
[17] *Szufla, S.*: Sets of fixed points of nonlinear mappings in function spaces, Funkcialaj Ekvacioj *22* (1979), 121–126.
[18] *Szufla, S.*: On the existence of L^p-solutions of Volterra integral equations in Banach spaces, Funkcialaj Ekvacioj *27* (1984), 157–172.
[19] *Vidossich, G.*: On the structure of the set of solutions of nonlinear equations, J. Math. Anal. Appl. *34* (1971), 602–617.
[20] *Zeidler, E.*: Vorlesungen über nichtlineare Funktionalanalysis I — Fixpunktsätze, Teubner Verlagsgesellschaft, Leipzig, 1976.

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