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SUBALGEBRA MODULAR, DISTRIBUTIVE AND BOOLEAN VARIETIES OF SEMIGROUPS

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Let A be an algebra. By Sub(A) we denote the lattice of all subalgebras of A, including the empty set, under inclusion. A variety $\mathscr{V}$ is said to be ***subalgebra modular*** (***distributive***) if every algebra A from $\mathscr{V}$ has a modular (distributive) lattice Sub(A) (see [1] and [2]). Characterizations of semigroups S having the modular (distributive or boolean) lattices Sub(S) are well known (see [3]).

The aim of this paper is to describe all varieties of semigroups S whose subsemigroup lattices Sub(S) are modular, distributive or boolean. We shall use the results on tolerance modular (distributive, boolean) semigroup varieties. Recall that a *tolerance* on a semigroup S is a reflexive and symmetric subsemigroup of the direct product $S \times S$. By Tol(S) we denote the lattice of all tolerances on S with respect to set inclusion (see [4] and [5]). A variety $\mathscr{V}$ of semigroups is called *tolerance modular* (*distributive, boolean*) if every semigroup S from $\mathscr{V}$ has a modular (distributive, boolean) lattice Tol(S).

By Ref(S) (Sym(S)) we denote the lattice of all reflexive (symmetric, respectively) subsemigroups of $S \times S$ for arbitrary semigroup S. See [6].

By $\mathscr{W}(i = j)$ we denote the variety of all semigroups satisfying the identity $i = j$. Terminology and notation not defined here may be found in [7] and [8].

It is easy to show the following:

Lemma 1. *Let S be a semigroup. Then the lattices* Tol(S), Ref(S) *and* Sym(S) *are sublattices of lattice* Sub($S \times S$) *and* Tol(S) = Ref(S) $\cap$ Sym(S).

Lemma 2. *Let S be a semigroup. Then the lattice* Sub(S) *is embedded into the lattice* Sym(S).

Proof. For each $A \in \mathrm{Sub}(S)$ we put $\varphi(A) = \{(a,a); a \in A\}$. Clearly $\varphi(A) = \mathrm{Sym}(S)$ and so $\varphi\colon \mathrm{Sub}(S) \to \mathrm{Sym}(S)$. It is easy to show that φ is a lattice isomorphism. □

Lemma 3. *Let G be a semigroup which is a periodic group. If the lattice* $\mathrm{Sub}(G \times G)$ *is modular, then G is commutative.*

Proof. This follows from Theorem of [9] and from the well known fact that every subsemigroup of a periodic group is a subgroup. □

Let S be a semigroup. By $E(S)$ we denote the set of all idempotents of S. For any element x of S, by $\langle x \rangle$ we denote the subsemigroup of S generated by x. Denote by $\vee$ or $\wedge$ the join or the meet, respectively, in the lattice $\mathrm{Sub}(S)$.

Lemma 4. *Let S be a semigroup. If the lattice* $\mathrm{Sub}(S)$ *is modular, $x \in S$ and $e \in E(S)$, then $\langle x \rangle \vee \langle e \rangle = \langle x \rangle \cup \langle e \rangle$.*

See Lemma V.2.8 of [10].

For every semigroup S we put $S^2 = \{ab; a, b \in S\}$.

Lemma 5. *Let S be a semigroup. If S^2 is a commutative periodic subgroup of S, then the lattice* $\mathrm{Sub}(S)$ *is modular.*

Proof. It is clear that a semigroup S is an ideal extension of a commutative periodic group S^2, for which $\mathrm{Sub}(S^2)$ is modular, by a nilsemigroup S/S^2, in which every subsemigroup generated by any two subsemigroups of S/S^2 coincides with their set theoretic union. It follows from Lemma V.2.15 of [10] that $\mathrm{Sub}(S)$ is modular. □

Lemma 6. *Let S be a semigroup from $\mathscr{W}(xy = x^2) \cup \mathscr{W}(yx = x^2) \cup \mathscr{W}(xy = uv)$. Then the lattice* $\mathrm{Sub}(S)$ *is distributive.*

Proof. Let $S \in \mathscr{W}(xy = x^2) \cup \mathscr{W}(yx = x^2) \cup \mathscr{W}(xy = uv)$. It is easy to show that for $A, B \in \mathrm{Sub}(S)$ we have $A \wedge B = A \cap B$ and $A \vee B = A \cup B$. □

Theorem 1. *For a variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Ref}(S)$ *is modular for each $S \in \mathscr{V}$;*
2. $\mathrm{Tol}(S)$ *is modular for each $S \in \mathscr{V}$;*
3. $\mathscr{V} \subseteqq \mathscr{W}((xy)^{n+1} = xy) \cap \mathscr{W}((xyx)^n = x^n)$ *for a positive integer n.*

Proof. $1 \Rightarrow 2$. This follows from Lemma 1.

$2 \Rightarrow 3$. See Theorem 3 of [11].

$3 \Rightarrow 1$. This follows from Part II of the proof of Theorem 3 in [11] if we replace $\mathrm{Tol}(S)$ by $\mathrm{Ref}(S)$. □

Theorem 2. *For a variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Sym}(S)$ *is modular for each* $S \in \mathscr{V}$;
2. $\mathrm{Sub}(S)$ *is modular for each* $S \in \mathscr{V}$;
3. $\mathscr{V} \subseteqq \mathscr{W}(xy = x^2)$ *or* $\mathscr{V} \subseteqq \mathscr{W}(yx = x^2)$ *or* $\mathscr{V} \subseteqq \mathscr{W}(xy = yx) \cap \mathscr{W}(xy = xy(uv)^n)$ *for a positive integer* n.

Proof. $1 \Leftrightarrow 2$. This follows from Lemma 2 and Lemma 1.

$2 \Rightarrow 3$. Let $\mathscr{V}$ be a subalgebra modular variety of semigroups. It follows from Lemma 1 that $\mathscr{V}$ is tolerance modular and so according to Theorem 1 we have

$$\mathscr{V} \subseteqq \mathscr{W}\big((xy)^{n+1} = xy\big) \cap \mathscr{W}\big((xyx)^n = x^n\big) \tag{1}$$

for a positive integer n. It is clear that $E(S) \neq \emptyset$ for every semigroup S from $\mathscr{V}$.

Case 1. $\operatorname{card} E(S) = 1$ for every $S \in \mathscr{V}$.

It follows from (1) that S^2 is a periodic subgroup of a semigroup S from $\mathscr{V}$. Evidently $S^2 \times S^2 \in \mathscr{V}$ and so according to Lemma 3, S^2 is commutative. We have

$$\mathscr{V} \subseteqq \mathscr{W}(xyuv = uvxy) \cap \mathscr{W}\big(xy = xy(uv)^n\big)$$

for a positive integer n. Then we obtain

$$xy = xy(xy)^n = xy(xy)^{2n} = (xyx)(yx)^{2n-1}y = (yx)^{2n-1}y(xyx) = yx(yx)^{2n} = yx.$$

Consequently, we have

$$\mathscr{V} \subseteqq \mathscr{W}(xy = yx) \cap \mathscr{W}\big(xy = xy(uv)^n\big) \tag{2}$$

for a positive integer n.

Case 2. In $\mathscr{V}$ there is a semigroup T such that $\operatorname{card} E(T) \geqslant 2$.

Let $e, f \in E(T)$, $e \neq f$. It follows from Lemma 4 that $ef \in \{e, f\} = F$.

Case 2a. $ef = e$

According to Lemma 4, we have $fe \in F$. If $fe = e$, then by (1) we obtain that $f = f^n = (fef)^n = (ef)^n = e$, which is a contradiction. Therefore $fe = f$. Consequently, $F \in \mathscr{V}$.

We shall show that

$$\mathscr{V} \subseteqq \mathscr{W}(x^2 = x^3). \tag{3}$$

Let S be a semigroup from $\mathscr{V}$ and let $a \in S$. By virtue of (1), we have $h = a^{2n} \in E(S)$ and $ha^2 = a^2$. Evidently $S \times F \in \mathscr{V}$ and so, by Lemma 4, we obtain

$$\langle (a, e) \rangle \vee \langle (h, f) \rangle = \langle (a, e) \rangle \cup \langle (h, f) \rangle .$$

Hence we have $(h, f)(a, e) = (ha, f) = (h, f)$. Therefore $ha = h$ and so $a^2 = ha^2 = ha = h$. Consequently, $a^3 = ha = h = a^2$.

Now, we shall prove that

$$(4) \qquad \mathscr{V} \subseteqq \mathscr{W}(x^2y^2 = x^2).$$

Let S be a semigroup from $\mathscr{V}$ and $a, b \in S$. It follows from (3) that $a^2, b^2 \in E(S)$ and so from Lemma 4 and (3) we have $a^2b^2 \in \{a^2, b^2\}$. On the contrary, suppose that $a^2b^2 \neq a^2$. Then $a^2b^2 = b^2$ and $a^2 \neq b^2$. Evidently $S \times F \in \mathscr{V}$ and so, by Lemma 4, we have

$$\langle (a^2, e) \rangle \vee \langle (b^2, f) \rangle = \langle (a^2, e) \rangle \cup \langle (b^2, f) \rangle .$$

Hence $(a^2b^2, e) = (a^2, e)(b^2, f) = (a^2, e)$, a contradiction. Thus we obtain $a^2b^2 = a^2$.

It follows from (4) and (3) that $x^2y = (x^2y^2)y = x^2y^3 = x^2y^2 = x^2$. By virtue of (1) and (3), we have $x^2 = x^{2n} = (xyx)^{2n} = (xyx)^2 = xyx^2yx = xyx^3 = xyx^2$ and so, by (4), $x^2 = xyx^2 = (xy)^2x^2 = (xy)^2$. Using (1) we can get $xy = (xy)^{n+1} = (xy)^2$ and so $xy = x^2$. Thus we have

$$(5) \qquad \mathscr{V} \subseteqq \mathscr{W}(xy = x^2).$$

Case 2b. $ef = f$.

This is dual to Case 2a and so we obtain that

$$(6) \qquad \mathscr{V} \subseteqq \mathscr{W}(yx = x^2).$$

$3 \Rightarrow 2$. Let $\mathscr{V}$ be a variety of semigroups satisfying (2). According to Lemma 5, $\mathscr{V}$ is subalgebra modular. Let $\mathscr{V}$ be a variety of semigroups satisfying (5) or (6). Then, by Lemma 6, $\mathscr{V}$ is subalgebra modular. □

Theorem 3. *For a variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Ref}(S)$ *is distributive for each* $S \in \mathscr{V}$*;*
2. $\mathrm{Tol}(S)$ *is distributive for each* $S \in \mathscr{V}$*;*
3. $\mathscr{V} \subseteqq \mathscr{W}(xyz = xz)$.

Proof. $1 \Rightarrow 2$. This follows from Lemma 1.

$2 \Rightarrow 3$. See Theorem 1 of [12].

$3 \Rightarrow 1$. This follows from Part II of the proof of Theorem 1 in [12] if we replace $\mathrm{Tol}(S)$ by $\mathrm{Ref}(S)$. □

Theorem 4. *For a variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Sym}(S)$ *is distributive for each* $S \in \mathscr{V}$;
2. $\mathrm{Sub}(S)$ *is distributive for each* $S \in \mathscr{V}$;
3. $\mathscr{V} \subseteqq \mathscr{W}(xy = x^2)$ *or* $\mathscr{V} \subseteqq \mathscr{W}(yx = x^2)$ *or* $\mathscr{V} \subseteqq \mathscr{W}(xy = uv)$.

Proof. $1 \Leftrightarrow 2$. This follows from Lemma 2 and Lemma 1.

$2 \Rightarrow 3$. Let $\mathscr{V}$ be a subalgebra distributive variety of semigroups. According to Lemma 1, $\mathscr{V}$ is tolerance distributive and so, by Theorem 3, we have

$$\mathscr{V} \subseteqq \mathscr{W}(xyz = xz). \tag{7}$$

Using Theorem 2 we can suppose that $\mathscr{V} \subseteqq \mathscr{W}(xy = yx) \cap \mathscr{W}\big(xy = xy(uv)^n\big)$ for a positive integer n. It follows from (7) that $xy = xy(uv)^n = (uv)^n xy(uv)^n = uv$.

$3 \Rightarrow 2$. Let $\mathscr{V}$ be a variety of semigroups satisfying (5) or (6) or

$$\mathscr{V} \subseteqq \mathscr{W}(xy = uv). \tag{8}$$

It follows from Lemma 6 that $\mathscr{V}$ is subalgebra distributive. □

We shall say that a variety $\mathscr{V}$ of semigroups is *subalgebra boolean* if every semigroup S from $\mathscr{V}$ is *subalgebra boolean*, i.e. the lattice $\mathrm{Sub}(S)$ is boolean.

Theorem 5. *For a variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Ref}(S)$ *is boolean for each* $S \in \mathscr{V}$;
2. $\mathrm{Tol}(S)$ *is boolean for each* $S \in \mathscr{V}$;
3. $\mathscr{V} \subseteqq \mathscr{W}(xyx = x)$ *or* $\mathscr{V} \subseteqq \mathscr{W}(xy = uv)$.

Proof. $1 \Rightarrow 2$. Suppose that $\mathrm{Ref}(S)$ is a boolean lattice. If follows from Lemma 1 that $\mathrm{Tol}(S)$ is a distributive lattice. For each $A \in \mathrm{Ref}(S)$ we put $\psi(A) = \{(a,b);\ (b,a) \in A\}$. It is easy to show that ψ is a lattice automorphism on $\mathrm{Ref}(S)$.

Now, we shall prove that $\mathrm{Tol}(S)$ is boolean. Let $A \in \mathrm{Tol}(S) \subseteqq \mathrm{Ref}(S)$. Clearly $\psi(A) = A$. There is $B \in \mathrm{Ref}(S)$ such that $A \wedge B = \mathrm{id}_S$ and $A \vee B = S \times S$. Hence we have $A \wedge \psi(B) = \psi(\mathrm{id}_S) = \mathrm{id}_S$ and $A \vee \psi(B) = \psi(S \times S) = S \times S$. Therefore $B = \psi(B)$ and so $B \in \mathrm{Tol}(S)$.

$2 \Rightarrow 3$. This follows from Theorem 2 of [12].

$3 \Rightarrow 1$. First, we shall show that the variety of all rectangular bands $\mathscr{RB} = \mathscr{W}(xyx = x)$ satisfies

$$\mathscr{RB} \subseteqq \mathscr{W}(xyz = xz) \tag{9}$$

and

$$\mathscr{RB} \subseteqq \mathscr{W}(x^2 = x). \tag{10}$$

Indeed, we have $xyz = xy(zxz) = x(yz)xz = xz$ and $x^2 = x^3 = x$. It follows from (9) and Theorem 3 that $\mathrm{Ref}(S)$ is distributive for each $S \in \mathscr{RB}$.

Now, we shall prove that $\mathrm{Ref}(S)$ is boolean for each $S \in \mathscr{RB}$. Let $A \in \mathrm{Ref}(S)$. Put $B = \big((S\times S)\setminus A\big)\cup \mathrm{id}_S$ and $C = \{(a,b);\ a,b \in S$, where $(a,ba) \in B$ and $(a,ab) \in B\}$. Evidently $\mathrm{id}_S \leqslant C$. Let $(a,b),(c,d) \in C$. Suppose that $(a,b)(c,d) = (ac,bd) \notin C$. Then, by (9), we have $(ac,bc) = (ac,bdac) \notin B$ or $(ac,ad) = (ac,acbd) \notin B$. If $(ac,bc) \notin B$, then $(ac,bc) \in A$ and $ac \neq bc$. It follows from (9) and (10) that $(a,ba) = (ac,bc)(a,a) \in A$ and $a \neq ba$. Thus we get $(a,ba) \notin B$ and so $(a,b) \notin C$, which is a contradiction. Analogously we can show that $(ac,ad) \notin B$ implies $(c,d) \notin C$, a contradiction. Therefore we have $(ac,bd) \in C$. Hence we obtain $C^2 \subseteqq C$ and so $C \in \mathrm{Ref}(S)$.

Let $a,b \in S$. By virtue of (9) and (10) we have $(a,b) = (a,ba)(a,ab)$. We shall show that $(a,b) \in A \vee C$. We have the following possibilities:

Case 1. $(a,ba) \notin B$ and $(a,ab) \notin B$. Then we get $(a,ba) \in A$ and $(a,ab) \in A$.

Case 2. $(a,ba) \notin B$ and $(a,ab) \in B$. Then we have $(a,ba) \in A$. By virtue of (9) and (10), we obtain $\big(a,a(ab)\big) = (a,ab) \in B$ and $\big(a,(ab)a\big) = (a,a) \in B$. Therefore $(a,ab) \in C$.

Case 3. $(a,ba) \in B$ and $(a,ab) \notin B$. This is dual to Case 2.

Case 4. $(a,ba) \in B$ and $(a,ab) \in B$. Then $(a,b) \in C$.

Consequently, $A \vee C = S \times S$.

Suppose that $(a,b) \in A \wedge C = A \cap C$. Then $(a,ba),(a,ab) \in B$. By virtue of (9) and (10), we have $(a,ba) = (a,b)(a,a) \in A$ and so $a = ba$. Analogously we have $a = ab$ and so $a = a^2 = (ba)(ab) = b$. Therefore $A \wedge C = \mathrm{id}_S$.

Consequently, the lattice $\mathrm{Ref}(S)$ is boolean for every rectangular band S.

It follows from Theorem 3 that $\mathrm{Ref}(S)$ is distributive for each $S \in \mathscr{Z} = \mathscr{W}(xy = uv)$. Let $S \in \mathscr{Z}$. Evidently, S is a zero-semigroup. Let $A \in \mathrm{Ref}(S)$. Put $B = (S \times S \setminus A)\cup \mathrm{id}_S$. Clearly $B \in \mathrm{Ref}(S)$. We have $A \wedge B = \mathrm{id}_S$ and $A \vee B = S \times S$. Therefore $\mathrm{Ref}(S)$ is boolean. □

Theorem 6. *For a nontrivial variety $\mathscr{V}$ of semigroups the following conditions are equivalent:*

1. $\mathrm{Sym}(S)$ *is boolean for each* $S \in \mathscr{V}$*;*

2. Sub(S) *is boolean for each* $S \in \mathscr{V}$;
3. $\mathscr{V} = \mathscr{W}(xy = x)$ *or* $\mathscr{V} = \mathscr{W}(yx = x)$.

Proof. $1 \Rightarrow 3$ and $2 \Rightarrow 3$. According to Theorem 4, we have (5) or (6) or (8). Therefore $\mathscr{V}$ satisfies (3). We shall show that

$$\mathscr{V} \subseteqq \mathscr{W}(x^2 = x). \tag{11}$$

On the contrary, suppose that a is an element of a semigroup S from $\mathscr{V}$ such that $a^2 \neq a$.

Case 1. Suppose that Sym(S) is boolean. It follows from (3) that $A = \{(a^2, a^2)\} \in \text{Sym}(S)$. According to one of (5), (6) and (8), there exists $B \in \text{Sym}(S)$ such that $A \cup B = A \vee B = S \times S$ and $A \cap B = A \wedge B = \emptyset$. Therefore $(a, a) \in B$ and so $(a^2, a^2) \in B$, a contradiction.

Case 2. Assume that Sub(S) is boolean. Then (putting $A = \{a^2\}$) we analogously obtain a contradiction.

It is easy to show that from (11) we have $\mathscr{V} \subseteqq \mathscr{W}(xy = x) = \mathscr{L}$ or $\mathscr{V} \subseteqq \mathscr{W}(yx = x) = \mathscr{R}$. It is well known (see [13]) that $\mathscr{L}$ and $\mathscr{R}$ are minimal varieties.

$3 \Rightarrow 1$ and 2. Let $\mathscr{V} \in \{\mathscr{L}, \mathscr{R}\}$. It is easy to show that for every semigroup S from $\mathscr{V}$ the lattice Sub(S) is the lattice of all subsets of S. Therefore $\mathscr{V}$ is subalgebra boolean. Analogously we can show that the lattice Sym(S) is the lattice of all symmetric subsets of $S \times S$ and so it is boolean. □

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