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CONVERGENCE OF WEIGHTED SUMS OF RANDOM VARIABLES IN VECTOR LATTICES

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In the present paper the order-convergence of weighted sums $S_n = \sum_{k=1}^{n} a_{nk} f_k$ is obtained under various conditions on the weights $\{a_{nk}\}$ and random variables f_k. I recall that in many spaces (e.g. L^p-spaces, $1 \leq p < \infty$) this convergence is stronger than convergence in the norm. The first theorem extends a result of Rohatgi [6] for weighted sums of random variables to vector lattices. The other main result is an order-version of a theorem of Padgett and Taylor [4]. My notation will follow [1] and [2]. (See also [3] and [4].)

In what follows I shall consider functions with values in an Archimedean vector lattice E.

Definition 1. *Let (Z, S, P) be a probability space. A sequence $\{f_n\}$ of functions from Z to E converges to a function f almost uniformly if for every $\varepsilon > 0$ there exists a set $A \in S$ such that $P\{A\} < \varepsilon$ and $\{f_n\}$ converges relatively uniformly to f uniformly on $Z - A$ (i.e. there exists a sequence $\{a_n\}$ of real numbers converging to 0 and an element $r \in E$ such that $|f_n(z) - f(z)| \leq a_n r$ for each $z \in Z - A$).*

Definition 2. *A function $f\colon Z \to E$ is called a random variable if there exists a sequence $\{f_n\}$ of countably valued random variables such that $\{f_n\}$ converges to f almost uniformly.*

Proposition 1. *Let E be a vector lattice equipped with a locally solid complete metrizable linear topology, P be a complete probability measure. Then each random variable is a random element (i.e. a measurable map from Z to E).*

Proof. There exists a sequence $\{A_k\}$, $A_k \in S$ such that $P\{A_k^C\} < k^{-1}$ and $|f_n(z) - f(z)| \leq a_n^k b_k$, $b_k \in E$ for all $z \in A_k$, $k = 1, 2, \ldots$ For each neighbourhood U of zero there exists a continuous monotonous Riesz pseudo-norm r such that $\{x \in E;\ r(x) < 1\} \subset U$ (see [2]). Because of this and the above inequalities we have that $f_n(z) \to f(z)$ in the topology for each $z \in Z$ except possibly a set of probability 0, since $P\{\cup A_k\}^C = 0$. Denote $\cup A_k$ by Z_0. An application of [4], prop. 2.1.3 implies that the restriction of f to Z_0 is a measurable function from Z_0 into E. Let B

be any Borel subset of E. We have $f^{-1}(B)=\{z\in Z_0;\ f(z)\in B\}\cup\{z\in Z-Z_0;\ f(z)\in B\}\in S$.

From now on P means a complete probability measure. Proposition 1 makes it possible to define independent, identically distributed and symmetric random variables in the usual manner, i.e. these definitions are straightforward extensions of the real case (see e.g. [4]).

Theorem 1. *Let E be a σ-complete vector lattice with the σ-property equipped with a locally solid complete metrizable linear topology. Let $\{a_{nk}\}$ be a double array of real numbers satisfying the conditions*

a) $\lim\limits_{n}\max\limits_{k}|a_{nk}|=0$

b) $\sup\limits_{n}\sum\limits_{k=1}^{\infty}|a_{nk}|^{r}<\infty$ *for some* $0<r<1$.

If f_n are pairwise independent, symmetric random variables in E such that $P\{|f_n|\leq a\}\geq P\{|f_1|\leq a\}$ for all positive elements $a\in E^{+}$ and all n and moreover $\sum\limits_{n=1}^{\infty}P\{|f_1|\leq na\}^{C}<\infty$ for some positive $a\in E$, then

$$\lim_{n} P\left\{z;\ \left|\sum_{k=1}^{\infty}a_{nk}f_k\right|\leq \varepsilon u\right\}=1$$

for each $\varepsilon>0$ and some positive $u\in E$.

Proof. For each n let $\{f_n^k\}$ be a sequence of countably valued random variables converging almost uniformy to f_n. The set of all values of f_n^k will be denoted by $\{y_n\}_{n=1}^{\infty}$. Since E has the σ-property, this set is included in a principal ideal of E (i.e. the ideal generated by a single element, say u, $u\in E^{+}$, $a\leq u)I_u$. Put $y_0=u$ and consider the countable set $A=\left\{\sum\limits_{i=0}^{n}a_iy_i;\ n=0,1,\dots\right\}$ of all linear combinations of y_i with the rational coefficients a_i. The set

$$B=\bigcap_{r\in Q}\bigcup_{a\in A}\{x\in I_u;\ |x-a|\leq ru\}$$

where Q stands for the set of all rational numbers is a linear subspace of I_u.

By definition 1 there exists a set Z_0 of probability 1 such that $f_n^k(z)\to f_n(z)$ relatively uniformly for all n and all $z\in Z_0$ with at most countably many different regulators of the convergence. Because of this, the inequality $|f_n|\leq|f_n-f_n^k|+|f_n^k|$ which holds for each natural n and k and the assumption that E has the σ-property we obtain that all the values of f_n belong to B.

It is a well-known fact that I_u equipped with the o-unit norm is a Banach space. So is B as a closed subset of I_u. Moreover B is separable. Indeed for each $x\in B$ and each $\varepsilon > 0$ there exists an element $a \in A$ such that $\|x - a\|_u < \varepsilon$; $\| \ \|_u$ means the norm induced by u. This space will be denoted by $(B, \| \ \|_u)$.

I shall prove that f_n are pairwise independent, symmetric random variables from Z_0 to B. Since B is separable, its Borel sets are generated by open balls. Denote these Borel sets by W_S and denote by W_T the σ-algebra generated by subsets of B open with respect to the original topology. It suffices to show that $W_S \subset W_T$. We have the following equality for an open ball

$$\{x \in B; \|x - x_i\|_u < \varepsilon\} = \bigcup_n \{x \in B; \|x - x_i\|_u \leqslant \varepsilon(1 - n^{-1})\} =$$

$$\bigcup_n B \cap \{x \in I_u; \|x - x_i\|_u \leqslant \varepsilon(1 - n^{-1})\} = B \cap \bigcup_n \{x \in I_u; |x - x_i| \leqslant \varepsilon(1 - n^{-1})u\}.$$

It means that f_n are pairvise independent and symmetric random variables in $(B, \| \ \|_u)$. By hypothesis we have $P\{\|f_n\|_u \geqslant b\} \leqslant P\{\|f_1\|_u \geqslant b\}$ for all $b > 0$ and

$$E\|f_1\|_u^r < 1 + \sum_{n=1} P\{\|f_1\|_u > n\} = 1 + \sum_{n=1} P\{|f_1| \leqslant nu\}^C < \infty$$

for $0 < r < 1$. Now apply [5], th. 1 which says that $\sum_{k=1}^{\infty} a_{nk} f_k$ norm-converges to 0 in probability for each sequence $\{f_n\}$ of pairwise independent random variables in a Banach space such that $P\{\|f_n\| \geqslant b\} \leqslant P\{\|f_1\| \geqslant b\}$ for all $b > 0$, $n \geqslant 1$, $E\|f_1\|^r < \infty$ for some $0 < r < 1$ and the weights $\{a_{nk}\}$ satisfying the conditions a) and b) of our theorem. We obtain that

$$\lim_n P\left\{z \in Z_0; \left\|\sum_{k=1}^{\infty} a_{nk} f_k\right\|_u \leqslant \varepsilon\right\} = 1$$

for each $\varepsilon > 0$. Since $P\{Z_0\} = 1$ and because of the definition of the order-norm this result is equivalent to the following

$$\lim_n P\left\{z; \left|\sum_{k=1} a_{nk} f_k\right| \leqslant \varepsilon u\right\} = 1$$

for each $\varepsilon > 0$.

Theorem 2. *Let E be a σ-complete vector lattice with the σ-property equipped with a locally solid complete metrizable linear topology. Let f_n be independent identically distributed symmetric random variables such that*

$$\sum_{n=1}^{\infty} P\{z; |f_1(z)| \leqslant na\}^C < \infty$$

for some $a \in E^+$. Let $\{d_{nk}\}$ be an array of real numbers satisfying

$$\limsup_n \sum_{k=1}^{n} d_{nk}^2 < \infty$$

and define

$$a_{nk} = \begin{cases} \dfrac{d_{nk}}{n} & k=1, \dots, n \\ 0 & k>n. \end{cases}$$

Then $\sum_{k=1}^{n} a_{nk} f_k \to 0$ *relatively uniformly with probability 1.*

Proof. One can prove the theorem repeating step by step the argument given in the proof of theorem 1. Having proved that f_n are independent, identically distributed and symmetric random variables in a separable Banach space B we complete the proof as follows. Denoting the norm in B by $\| \ \|_u$ we have

$$E\|f_1\|_u \leq 1 + \sum_{n=1}^{\infty} P\{\|f_1\|_u > n\} = 1 + \sum_{n=1}^{\infty} P\{|f_1| \leq nu\}^C < \infty$$

(C stands for the set complement). It follows, by [4] th. 5.1.5 that $\left\|\sum_{k=1}^{n} a_{nk} f_k\right\|_u \to 0$ a.s. in the norm and consequently $\sum_{k=1}^{n} a_{nk} f_k \to 0$ relatively uniformly with probability 1.

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СХОДИМОСТЬ СУММ СЛУЧАЙНЫХ ВЕЛИЧИН СО ЗНАЧЕНИЯМИ В ВЕКТОРНОЙ РЕШЕТКЕ

Rastislav Potocký

Резюме

В работе доказываются теоремы о сходимости по упорядочению сумм случайных величин.