Mathematica Slovaca

Jarmila Hedlíková

Chains in modular ternary latticoids

Mathematica Slovaca, Vol. 27 (1977), No. 3, 249--256

Persistent URL: `http://dml.cz/dmlcz/128621`

Math. Slovaca 27, 1977, No. 3, 249—256

CHAINS IN MODULAR TERNARY LATTICOIDS

JARMILA HEDLÍKOVÁ

In this paper we consider a set M closed under a ternary operation (abc) satisfying the identities

(1) $(abb)=b$,

(2) $((abc)dc)=(ac(dcb))$.

We call M a *modular ternary latticoid* (it is a generalization of the median semilattice from [4]):

Note that in any modular lattice the ternary operation (abc) defined by

(3) $(abc)=((b\vee c)\wedge a)\vee(b\wedge c)=(b\vee c)\wedge(a\vee(b\wedge c))$

satisfies the identities (1) and (2) (see the introduction in [3]). Thus every modular lattice is a modular ternary latticoid.

[3, Theorem 1] gives a characterization of modular lattices with a least element by means of the ternary operation (3).

In a modular ternary latticoid we introduce the relation between, the notion of the segment (compare [4]), and the notion of the chain (the corresponding notion is the line in lattice, see [2]). We give some results which characterize chains. Moreover, we prove the Jordan-Hölder theorem for chains.

Throughout the paper, M will denote a modular ternary latticoid.

1. Basic concepts and properties

In [3, Lemma] for a modular ternary latticoid the following is shown

(4) $(bab)=b,\ (aab)=a$.

(5) $((abc)bc)=(acb)$.

(6) $(abc)=(acb)$.

(7) $((abc)ac)=(ac(abc))=(abc)$.

(8) $(ab(cab))=(abc)$.

(9) $(bac)=(cab)\to(abc)=(bac)$.

(10) $(abc)=c\to(bca)=c=(cab)$.

(11) $(a(ade)(bde))=(ade)$.

We say that x is *between* a and b and write axb if and only if $x=(axb)$. The *segment* (a, b) is defined as the set of all elements between a and b, i.e. $(a, b)=\{x \in M: axb\}$. From (6) and (10) it follows

(12) $axb \rightarrow x=(bxa)=(xab)$.

We get $(a, b)=\{(axb): x \in M\}$ from (6) and (7), $(a, a)=\{a\}$ from (4), and a, $b \in (a, b)=(b, a)$ from (1) and (2).

We will show that a modular ternary latticoid satisfies the following relations

(13) $(a, b) \subseteq (a, c) \rightarrow b \in (a, c)$.
(14) $(a, b)=(a, c) \rightarrow b=c$.
(15) $aba \rightarrow a=b$.
(16) aab, baa.
(17) $abc \rightarrow cba$.
(18) $abc \cdot bac \rightarrow a=b$.
(19) $abc \cdot acb \rightarrow b=c$.
(20) $abc \cdot acd \rightarrow bcd \cdot abd$.
(21) $abc \cdot acd \cdot ade \rightarrow bde$.

Let $b \in (a, c)$ and $x \in (a, b)$, these mean abc and axb. Applying (12) twice, (2), and again (12) we get $x=(bxa)=((cba)xa)=(ca(xab))=cax$, which gives $x \in (a, c)$ by (10). Thus (13) is proved.

From (6) we have (14): $b=(abc)=(acb)=c$.

(15) follows immediately from (4), (16) from (1) and (4), (17) and (18) from (12), and (19) from (6).

Now let abc, acd. Applying (6), (12), (2), (12), and (1) we have $(bcd)=(bdc)=((bac)dc)=(bc(dca))=(bcc)=c$, which means bcd. Further abd follows from $c \in (a, d)$ and $b \in (a, c)$ by (13), and (20) is proved.

(21) follows immediately from (20).

The notation of betweenness can be extended as follows: $abcd$ denotes $abc \cdot abd \cdot acd \cdot bcd$. Similarly for more than four terms. Thus the implication in (20) can be replaced by the other one $abc \cdot acd \rightarrow abcd$.

The segment (a, b) is called a *simple segment* if and only if it contains only the elements a, b. Clearly the segment (a, b) is simple if and only if $(axb) \in \{a, b\}$ for all $x \in M$ (or $(bxa) \in \{a, b\}$ for all $x \in M$).

Two segments (a, b), (c, d) are called *transposed segments* (or shortly *transposes*), when a, $c \in (b, d)$ and b, $d \in (a, c)$ or a, $d \in (b, c)$ and b, $c \in (a, d)$. The relation of transposition is reflexive and symmetric but need not be transitive. This shows the five-element modular ternary latticoid $\{O, I, a, b, c\}$ corresponding to the known five-element modular nondistributive lattice (O, I denote the least and the greatest element, respectively): $(abc)=(OaI)=a$, $(bac)=(ObI)=b$, $(cab)=(OcI)=c$, $(aOb)=(aOc)=(bOc)=O$, $(aIb)=(aIc)=(bIc)=I$ (the number of defining identities is reduced with regard to (1), (6), and (10)). The segments (b, I), (a, O) and (a, O), (c, I) are transposes but the segments (b, I), (c, I) are not transposed. Therefore we introduce the following definition.

Two segments (a, b), (c, d) are *projective* if and only if there exist segments $(x_0, y_0), \ldots, (x_n, y_n)$, $x_0 = a$, $y_0 = b$, $x_n = c$, $y_n = d$ such that the segments (x_{i-1}, y_{i-1}), (x_i, y_i) are transposes for $i = 1, \ldots, n$. We call the segments (x_i, y_i), $0 < i < n$, the *middle members* of that projectivity.

Now we prove the following: If (a, b), (c, d) are transposed segments and (a, b) is simple, then (c, d) must be also simple. It is sufficient to consider the case *bad*, *bcd*, *abc*, *adc*. Let *cxd*. Then by (20) $cxd \cdot cda \rightarrow cxda$ and $dxc \cdot dcb \rightarrow dxcb$. Since (a, b) is simple, $(axb) \in \{a, b\}$. If $(axb) = a$, then $x = (axx) = (ax(dxb)) = ((abx)dx) = (adx) = d$. If $(axb) = b$, this means *abx*, then by (20) $abx \cdot axc \rightarrow bxc$, which with *bcx* gives $x = c$. The segment (c, d) is simple.

The following notions will be needed. The elements $a, b, c, d \in M$ form a *cyclic quadruple* (a, b, c, d) when they are pairwise different and satisfy *abc*, *bcd*, *cda*, *dab*. A nonempty subset $R \subseteq M$ is a *chain* if and only if it satisfies the following two conditions

(a) For every three elements $a, b, c \in R$ one (at least) of the relations *abc*, *bca*, *cab*, holds.

(b) R does not contain a cyclic quadruple.

It is clear that a nonempty subset of a chain is a chain. An element $a \in R$ is an *end element* of a chain R if and only if for all $x, y \in R$ *axy* or *ayx* holds. The *length* of a finite chain R is the number of its elements minus 1.

2. Chains

In a chain there holds: $abc \cdot bcd \cdot b \neq c \rightarrow abd$. To prove it assume *abc*, *bcd* and $b \neq c$. By (20) we have $adb \cdot abc \rightarrow dbc$, which together with *dcb* gives $b = c$, further $dac \cdot dcb \rightarrow acb$, which with *abc* also gives $b = c$. Thus neither *adb* nor *dac* is possible. If *acd*, then by (20) $abc \cdot acd \rightarrow abd$. Let *adc* and *dab*. The elements a, b, c, d cannot be different (because otherwise they would form a cyclic quadruple). Because of $b \neq c$ there must be $a \neq c$, $a \neq d$, and $b \neq d$. If $a = b$ or $c = d$, then *abd* holds trivially.

Note that from the preceding statement there follows: $abc \cdot bcd \cdot b \neq c \rightarrow abcd$ in a chain.

Proposition 1. *Every chain R has at most two end elements a, b, which are characterized by the following property: for all $x \in R$ axb.*

Proof. Let a, b, c be end elements of a chain R and *acb*. Therefore *cab* or *cba* must hold. Then $c = a$ or $c = b$.

Let $a \neq b$ be end elements of a chain R, $x \in R$. There are two possibilities: axb and abx. Let there be abx. One of the relations bxa or bax must hold. If bax, then $a = b$, which is impossible. Then bxa, hence axb.

Let a, $b \in R$, $a \neq b$, and azb for all $z \in R$. We shall show that a, b are end elements of R. Take x, $y \in R$. The elements a, b, x, y can be assumed to be pairwise different. Now the case xay (xby by symmetry) can be eliminated as follows. Let xay. From yax, axb, $a \neq x$ there follows that yab, which with ayb gives $y = a$, a contradiction. Therefore axy or ayx must hold and analogously bxy or byx.

Proposition 2. *Let $R \subseteq M$ have more than four elements and let R satisfy condition (a). Then R is a chain.*

Proof. It is enough to show that no four elements of R form a cyclic quadruple. Assume that there exist pairwise different elements $x, y, z, t \in R$ for which xyz, yzt, ztx, and txy. Let $a \in R - \{x, y, z, t\}$. There are three possibilities: 1. xay, 2. axy, 3. ayx. The last two relations are symmetric.

In the first case using (20) we obtain $xay \cdot xyz \to xayz$ and $yax \cdot yxt \to yaxt$. If atz, then $zta \cdot zax \to tax$, which contradicts axt. The relation azt does not hold by symmetry. There remains zat. But then $taz \cdot tzy \to azy$, which contradicts ayz. Therefore the first relation does not hold.

In the second case there are three possibilities: tya, tay, yta. Let tya, then $txy \cdot tya \to xya$, which contradicts axy. From the relation tay it follows that $a = (tay) = (t(tay)(xay)) = (tax)$, which cannot hold for the same reasons as xay. Then yta must hold. By (20) $yxt \cdot yta \to xta$ and $yzt \cdot yta \to yzta$. Now we show that all three possibilities axz, azx, and xaz lead to a contradiction. Let axz, then $axz \cdot azy \to xzy$, but it does not hold. The possibility azx is symmetric. Finally, let xaz. But then $t = (xat) = (x(xaz)(taz)) = (xaz) = a$, which is a contradiction. From the preceding it follows that the second relation does not hold and also the third one.

Therefore the assumption was incorrect and the proposition is proved.

Proposition 3. *Every finite chain R with at least two elements has two end elements.*

Proof. Let $R = \{x_0, \ldots, x_n\}$ contain $n+1$ elements. The proposition will be proved by induction on the number of elements of the chain R.

1. If $R = \{x_0, x_1\}$, then x_0, x_1 are the end elements, because $x_0x_0x_1$ and $x_0x_1x_1$.

2. Let $n > 1$. Assume the proposition to be true for all $k < n$. Let a, b be end elements of a chain $\{x_0, \ldots, x_{n-1}\}$. There are three possibilities: ax_nb, abx_n, bax_n. The last two are symmetric. If ax_nb, then R has the end elements a, b. If abx_n, then for all $k < n$ by (20) $ax_kb \cdot abx_n \to ax_kx_n$. Clearly ax_nx_n. Then the chain R has the end elements a, x_n.

Proposition 4. *Let $n > 1$. $R = \{y_0, \ldots, y_n\}$ is a chain if and only if $R = \{x_0, \ldots, x_n\}$, where $x_0x_1 \ldots x_n$ (this means $x_ix_jx_k$ for all $i, j, k \in \{0, \ldots, n\}$, $i \leq j \leq k$).*

Proof. Let $R=\{y_0, \ldots, y_n\}$ be a chain of a length n. The first implication will be proved by induction on n.

1. The proof is clear for $n=2$.

2. Let $n>2$ and let the proposition be true for all $k<n$. Let us denote the end elements of the chain R by x_0, x_n. From the induction assumption it follows that $R-\{x_n\}=\{x_0, \ldots, x_{n-1}\}$, where $x_0x_1\ldots x_{n-1}$. It is sufficient to show $x_ix_jx_n$ for all i, $j\in\{0, \ldots, n-1\}$, $i\leqslant j$. Indeed by (20) $x_0x_ix_j\cdot x_0x_jx_n\rightarrow x_ix_jx_n$.

It is easy to see that $R=\{x_O, \ldots, x_n\}$, where $x_0x_1\ldots x_n$ does not contain a cyclic quadruple, which proves the second implication.

The chain R will be denoted by $R=x_0x_1\ldots x_n$.

Proposition 5. *Let* $x_0x_1\ldots x_n$ *and* $x_{i-1}xx_i$ *for some* $i\in\{1, \ldots, n\}$. *Then* $x_0x_1\ldots x_{i-1}xx_i\ldots x_n$.

Proof. It is sufficient to show that x_kxx_m and x_jx_kx for all j, k, $m\in\{0, \ldots, n\}$, $j\leqslant k<i\leqslant m$. Clearly $x_jx_{i-1}x_k$, $x_kx_ix_m$, and $x_jx_kx_j$. Using (20) we obtain $x_jxx_{i-1}\cdot x_jx_{i-1}x_k\rightarrow x_kxx_i$, further $x_kxx_i\cdot x_kx_ix_m\rightarrow x_kxx_m$, and finally $x_ixx_k\cdot x_ix_kx_j\rightarrow x_jx_kx$.

Corollary. *If* $x_0x_1\ldots x_n$ *is a maximal chain between the elements* x_0, x_n, *then* (x_{i-1}, x_i) $(i=1, \ldots, n)$ *are simple segments.*

Remark 1. A chain R is *maximal* if and only if there exists no chain $S\supseteq R$, $S\neq R$.

Using Zorn's lemma we obtain the proposition: Every chain is contained in a maximal chain.

Similarly: Every chain between the elements a, b is contained in a maximal chain between the elements a, b.

Proposition 6. *Let* R *be a chain,* $a\in R$. *Then* $R=S\cup T$, *where* S, T *are chains with the end element* a, $S\cap T=\{a\}$, *and* sat *for all* $s\in S$, $t\in T$.

Conversely: Let S, T *be chains with the end element* a, $S\cap T=\{a\}$, *and* sat *for all* $s\in S$, $t\in T$. *Then* $R=S\cup T$ *is a chain.*

Proof. If a is an end element of R, it is sufficient to put $S=R$ and $T=\{a\}$. If a is not an end element of R, then there exist x, $y\in R$ such that xay and x, a, y are pairwise different. Put $S=\{s\in R\colon\ axs \text{ or } asx\}$ and $T=\{t\in R\colon\ ayt \text{ or } aty\}$. Evidently $x\in S$, $y\in T$, and $a\in S\cap T$. If $v\in S\cap T$, then avx and avy, which with xay gives $v=a$, hence $S\cap T=\{a\}$. Let $v\in R$. Then in each of the possibilities $vxay$, $xvay$, $xavy$, and $xayv$ it follows that $v\in S\cup T$. Hence $R=S\cup T$. Now let z, $v\in S-\{a\}$ and zav. Each of the possibilities $xzav$, $zxav$, $zaxv$, and $zavx$ leads to a contradiction. Thus azv or avz must hold and a is the end element of the chain S and similarly of the chain T. Let $s\in S$, $t\in T$. Then we get sat for all four possibilities $xsaty$, $xsayt$, $sxaty$, and $sxayt$. Thus the first part of the proposition is proved.

To prove that R satisfy the condition (a) it is sufficient to consider the case x, $y \in S$, $z \in T$, and axy. Then $yxa \cdot yaz \rightarrow yxaz$. With respect to this fact and Proposition 2 R is a chain.

Remark 2. The chain as in Proposition 6 will be denoted by $R = SaT$. Evidently the length of the chain R (R finite) is the sum of the lengths of the chains S, T.

Corollary. *If R is a chain between the elements a, b and abc holds, then $Rb\{b, c\}$.*

It follows from the fact that for all $t \in R$ $atb \cdot abc \rightarrow tbc$ holds.

Proposition 7. *A nonempty subset $R \subseteq M$ is a chain if and only if $R = \{x_i\}_{i \in I}$, where I is an ordered set so that $x_i x_j x_k$ for all i, j, $k \in I$, $i \leqslant j \leqslant k$.*

Proof. Let R be a chain, $a \in R$. Let S, T be chains as in Proposition 6. Now the ordering on the set R will be given. For x, $y \in R$ let $x \leqslant y$ hold if and only if one of the following conditions holds

(i) $x, y \in S$ and xya,

(ii) $x, y \in T - \{a\}$ and axy,

(iii) $x \in S$ and $y \in T - \{a\}$.

We immediately obtain that $x \leqslant x$. If $x \leqslant y$ and $y \leqslant x$, then one of the following possibilities is true: x, $y \in S$, xya, yxa or x, $y \in T - \{a\}$, axy, ayx. In both cases $x = y$ holds. Let $x \leqslant y$ and $y \leqslant z$. If $x \in S$ and $z \in T - \{a\}$, then $x \leqslant z$. Let $x \in T - \{a\}$. Then y, $z \in T - \{a\}$ and axy, ayz, hence axz, which means $x \leqslant z$. If $z \in S$, then x, $y \in S$ and xya, yza, hence xza, and hence $x \leqslant z$. Note that in all three cases xyz holds. It is easy to see that $x \leqslant y$ or $y \leqslant x$ for the arbitrary elements x, $y \in R$. From these considerations it follows that R can be written in a desirable form.

Clearly $R = \{x_i\}_{i \in I}$ (where I has the meaning as above) is a chain, which proves the second implication.

Proposition 8. *Let R be a maximal chain between the elements a, b and x, $y \in R$. Then $S = R \cap (x, y) = \{z \in R : xzy\}$ is a maximal chain between the elements x, y.*

Proof. With respect to the symmetry we may assume the case $axyb$. Let $S_0 = S \cup \{t\}$ be a chain between the elements x, y, hence xty and further $axtyb$. The chains S_0 and $R_1 = R \cap (a, x)$ fulfil the assumptions of the second part of Proposition 6, hence $S_0 \cup R_1$ is a chain. $R \cup \{t\} = (S_0 \cup R_1) \cup R_2$, where $R_2 = R \cap (y, b)$ is a chain for the same reasons as $S_0 \cup R_1$. Hence $t \in R$, which with xty gives $t \in S$ and thus S is maximal.

Remark 3. Proposition 8 is true for an arbitrary maximal chain R. It can be proved similarly.

3. The Jordan—Hölder theorem for chains

Now we can prove the basic result.

Proposition 9. *(Jordan—Hölder theorem for chains in modular ternary latticoid-*

s.) Let R, S be maximal chains with end elements a, b in a modular ternary latticoid. Let the chain R be finite. Then there holds:

1. The chain S is finite and of the same length as R.

2. There exists a bijective mapping of the set of all simple segments of the chain R to the set of all simple segments of the chain S such that the corresponding simple segments are projective and for the middle members (p, q) of that projectivity apb, aqb holds.

Proof. Let R be of the length n. The proof will be given by induction on n.

For $n=0,1$ the proposition is clear.

Let $n>1$, $R=x_0x_1\ldots x_n$, $a=x_0$, $b=x_n$, and let the proposition be true for all $k<n$. From this it follows that $S-\{a, b\}\neq\emptyset$. Denote $R_0=\{a, x_1\}$ and $R_1=R\cap(x_1, b)$. If x, $y\in(a, b)$, then $(xya)=((bxa)ya)=(ba(yax))=(yax)$ and similarly $(xyb)=(ybx)$. There are two possibilities (with respect to the fact that the segment (a, x_1) is simple): 1. ax_1y for all $y\in S-\{a, b\}$, 2. x_1ay for some $y\in S-\{a, b\}$.

In the first case $x_1\in S$, because $S\cup\{x_1\}$ is a chain and S is maximal (if y_1, $y_2\in S-\{a\}$ and ay_1y_2, then $ax_1y_1\cdot ay_1y_2\rightarrow ax_1y_1y_2$). The chain R_1 has the length $n-1$. From the induction assumption there follows the validity of the proposition for the chains R_1 and $S_1=S\cap(x_1, b)$. Since $S=R_0x_1S_1$, the proposition is true for the chains R, S.

In the second case denote $z=(x_1yb)=(ybx_1)$, hence x_1zy, ax_1zb, and $ayzb$. Therefore the segments (a, x_1) and (y, z) are transposes. Since (a, x_1) is simple, (y, z) is simple. First of all, assume that $z=b$. If (a, y) is not a simple segment, the case is symmetric to $z\neq b$ (there exists $y'\in S-\{a, y, b\}$ such that $ay'yb$, hence x_1ay' and $z'\neq b$, where $z'=(x_1y'b)=(y'bx_1)$). Let (a, y) be simple (the chain S is of the length 2). The segments (a, y) and (x_1, b) are transposes, hence (x_1, b) is simple, $n=2$, and the proposition is true. Now let the elements z, b be different. The proposition is true for the chains R_1 and a maximal chain $R_2\supseteq x_1zb$ between the elements x_1, b. $R_2\cap(z, b)$ has the length $k\geqslant 1$ $(z\neq b)$, the length of $R_2\cap(x_1, z)$ is $n-1-k$. Denote $S_0=\{y, z\}$. The proposition is true for the chains $S_0z(R_2\cap(z, b))$, $S\cap(y, b)$ (they have the length $k+1<n$, because $z\neq x_1$; in the case $x_1=z$ there holds ayx_1, which with x_1ay gives $a=y$, a contradiction) and for the chains $R_0x_1(R_2\cap(x_1, z))$, $(S\cap(a, y))yS_0$ (they have the length $n-k<n$). We may summarize: the chain S is finite and has the length $(n-k-1)+(k+1)=n$.

The second part of the proposition follows from the induction assumption and from the fact that the segments (a, x_1), (y, z) are transposes.

REFERENCES

[1] BIRKHOFF, G.: Lattice theory. 3rd. Providence 1967.

[2] KOLIBIAR, M.: Linien in Verbänden. Anal. Ştiinţ. Univ. Iaşi, 1, 1965, 90—98.

[3] KOLIBIAR, M. and MARCISOVÁ, T.: On a question of J. Hashimoto. Mat. Čas. 24, 1974, 179—185.
[4] SHOLANDER, M.: Medians and betweenness. Proc. Amer. Math. Soc. 5, 1954, 801—807.

Received December 3, 1975

Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

ЦЕПИ В МОДУЛЯРНЫХ ТЕРНАРНЫХ СТРУКТУРОИДАХ

Ярмила Хедликова

Резюме

В статье рассматривается множество M с тернарной операцией (abc) удовлетворяющей тождествам $(abb)=b$ и $((abc)dc)=(ac(dcb))$. M называется модулярный тернарный структуроид. Всякая модулярная структура с подходящей тернарной операцией $(abc)=$ $=((b\vee c)\wedge a)\vee(b\wedge c)=(b\vee c)\wedge(a\vee(b\wedge c))$ есть модулярный тернарный структуроид. В M вво дятся – тернарное отношение между, понятие интервала и понятие цепи (соответствующее понятие в структуре – линия). В работе приведено несколько результатов характеризующих цепи в M и доказана теорема Жордана–Гельдера для цепей в M.