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ON LIPSCHITZ MIDCONVEX MULTIFUNCTIONS

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ABSTRACT. We prove that a midconvex multifunction with non-empty closed and bounded values is locally Lipschitzean provided that it is weakly bounded on a ball. Similar result has been obtained by B. Pshenichnyi in 1974 for convex multifunctions with closed graph when the space of values is of a finite dimension.

Let X and Y be real normed vector spaces and let $F\colon X \to 2^Y$ be a multifunction. The *domain* of F is defined as

$$\operatorname{dom} F = \{x \in X : \ F(x) \neq \emptyset\}$$

and its *graph* by

$$\operatorname{Gr} F = \{(x,y) \in X \times Y : \ y \in F(x)\}.$$

A multifunction F is said to be *midconvex* (*convex*) if

$$\frac{1}{2}[F(x)+F(y)] \subset F\Big(\frac{x+y}{2}\Big) \qquad \big(\lambda F(x) + (1-\lambda)F(y) \subset F\big(\lambda x + (1-\lambda)y\big)\big)$$

for every $x, y \in X$ (and $\lambda \in [0,1]$).

A multifunction F is said to be *closed* if its graph is closed in $X \times Y$. Pshenichnyi [2] showed that a convex closed multifunction is locally Lipschitzean on the interior of its domain provided that a set $F(\hat{x})$ is bounded for some $\hat{x} \in \operatorname{dom} F$ and $\dim Y < \infty$.

Let D be a non-empty open and convex subset of X. Throughout this note we shall assume that $\operatorname{dom} F = D$. We say that a multifunction F is *weakly upper bounded* on a set $U \subset D$ if there exists a bounded set $A \subset Y$ such that $U \subset F^{-}(A)$, i.e., $F(x) \cap A \neq \emptyset$ for all $x \in U$. Nikodem [1; p. 35, Corollary 3.3] proved that a midconvex and weakly upper bounded on a non-empty open subset of D multifunction with bounded values is continuous. The main goal of this note is to prove that such multifunctions actually are locally Lipschitzean.

We shall need the following lemma.

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Lemma 1. ([1; pp. 29–30, Lemma 3.1, Remark 3.1]) *Let X be a real vector space, $D \subset X$ be convex and Y be a topological vector space. If a multifunction $F\colon D \to 2^Y \setminus \{\emptyset\}$ (with closed bounded values) is midconvex, then*

$$\lambda F(x) + (1-\lambda)F(y) \subset F\big(\lambda x + (1-\lambda)y\big)$$

for all $x, y \in D$ and all diadic (rational) numbers $\lambda \in [0,1]$.

It is easy to see that if $\hat{x} \in \operatorname{Int}(\operatorname{dom} F)$ and $F(\hat{x})$ is bounded, then all values of the convex multifunction F are also bounded. A slightly stronger result may also be obtained for midconvex multifunctions.

Let D be a subset of a real vector space X. We say that $x_0 \in D$ belongs to the *algebraic relative interior* of D if for every $x \in D$ there exists an $\varepsilon > 0$ such that

$$tx + (1-t)x_0 \in D \qquad \text{for every} \quad |t| < \varepsilon .$$

Lemma 2. *Let X be a real vector space, $D \subset X$ be convex and Y be a topological vector space. If a multifunction $F\colon D \to 2^Y \setminus \{\emptyset\}$ is midconvex and $F(\hat{x})$ is bounded for some $\hat{x}$ belonging to the algebraic relative interior of D, then all values of F are also bounded.*

P r o o f. Let us fix an x belonging to D. Since $\hat{x}$ is an algebraic relative interior point od D, the point $-\frac{1}{2^n-1}x + \big(1 + \frac{1}{2^n-1}\big)\hat{x} =: y$ belongs to D for large enough n. For $\lambda = 1/2^n$ we obtain

$$\hat{x} = \lambda x + (1-\lambda)y ,$$

and

$$\lambda F(x) + (1-\lambda)F(y) \subset F(\hat{x}) .$$

Let us choose any point $z \in F(y)$. We have

$$\lambda F(x) \subset F(\hat{x}) + (\lambda - 1)z$$

and

$$F(x) \subset \frac{1}{\lambda}F(\hat{x}) + \left(1 - \frac{1}{\lambda}\right) z .$$

So the set $F(x)$ is bounded. □

In the sequel $B(x_0, r)$ denotes an open ball in X centered at x_0 and with the radius $r > 0$. S stands for the closed unit ball in Y, i.e.,

$$S = \big\{y \in Y : \ \|y\| \le 1\big\} .$$

For a bounded subset A of Y, $\|A\|$ denotes the supremum of the set $\big\{\|y\| : y \in A\big\}$.

In what follows we shall assume that all values of F are bounded although it is enough to assume that $F(\hat{x})$ is bounded for some $\hat{x} \in D$.

THEOREM 1. *Let X and Y be real linear normed spaces and let D be a non-empty open convex subset of X. Assume that $F\colon D \to 2^Y \setminus \{\emptyset\}$ with closed and bounded values is a midconvex multifunction. If F is weakly upper bounded on a ball in D, then there exists a positive number c such that*

$$\|F(x)\| \le c\big(1 + \|x\|\big) \tag{1}$$

for every $x \in D$.

Proof. By the hypothesis we can find a ball $B(x_0, r)$ contained in D and a bounded set $A \subset Y$ such that $F(y) \cap A \neq \emptyset$ for all $y \in B(x_0, r)$. We observe that there is a bounded set $C \subset Y$ such that $F(x) \subset C$ for $x \in B(x_0, r)$. In fact, let us fix an $x \in B(x_0, r)$. Of course, $0 \in F(x) - A$. We have $x_0 = \frac{x+y}{2}$ for some $y \in B(x_0, r)$. Since F is midconvex,

$$F(x) + F(y) \subset 2F(x_0)\,,$$

whence

$$F(x) \subset F(x) + F(y) - A \subset 2F(x_0) - A\,.$$

Setting $C := 2F(x_0) - A$ we obtain the observation, i.e.,

$$F(x) \subset C \qquad \text{for all} \quad x \in B(x_0, r)\,. \tag{2}$$

Now let y be an arbitrary point from $B(x_0, r)$. Suppose that $y \neq x_0$. Take a rational number $\lambda \in (0, 1)$ such that

$$\frac{\|y - x_0\|}{r} < \lambda < 2\frac{\|y - x_0\|}{r}\,. \tag{3}$$

Write

$$z = x_0 - \frac{1}{\lambda}(y - x_0)\,.$$

We see that $z \in B(x_0, r)$ and

$$x_0 = \frac{\lambda}{1+\lambda} z + \frac{1}{1+\lambda} y\,.$$

By Lemma 1

$$\frac{\lambda}{1+\lambda} F(z) + \frac{1}{1+\lambda} F(y) \subset F(x_0)\,,$$

whence

$$F(y) \subset (1+\lambda)F(x_0) - \lambda F(z)\,.$$

Putting $M := \|C\|$ we obtain by (2)

$$\|F(y)\| \le (1 + 2\lambda)M\,.$$

Denoting by c_0 the expression $M \max\left\{\frac{4}{r}, 1+\frac{4\|x_0\|}{r}, \frac{1}{1+\|x_0\|}\right\}$ we get by (3)

$$\|F(y)\| \le c_0(1+\|y\|)\,. \tag{4}$$

Now let $y \in D\setminus B(x_0,r)$. We can find a rational number λ from the interval $(0,1)$ such that

$$z := \lambda x_0 + (1-\lambda)y \in B(x_0,r) \qquad \text{and} \qquad z \notin B\Big(x_0, \frac{r}{2}\Big)\,.$$

So we have

$$\frac{r}{2} \le \|\lambda x_0 + (1-\lambda)y - x_0\| < r\,.$$

Consequently

$$\frac{r}{2(1-\lambda)} \le \|y - x_0\| < \frac{r}{1-\lambda}\,. \tag{5}$$

By the midconvexity of F and by (2) we obtain

$$\lambda F(x_0) + (1-\lambda)F(y) \subset F(z) \subset C\,.$$

Hence we have

$$F(y) \subset \frac{1}{1-\lambda}\big(C - \lambda F(x_0)\big)\,.$$

Thus by (5)

$$\|F(y)\| \le \frac{2M}{1-\lambda} \le \frac{4M}{r}\|y - x_0\| \le \frac{4M}{r}\big(\|y\| + \|x_0\|\big)\,.$$

Define c by $\max\left\{c_0, \frac{4M}{r}, \frac{4M\|x_0\|}{r}\right\}$. Then we have

$$\|F(y)\| \le c(1+\|y\|)\,.$$

This inequality and (4) completes the proof. □

Let K be a convex cone in X, i.e., $x+y \in K$ and $\lambda x \in K$ for every $x, y \in K$ and $\lambda > 0$. Consider a multifunction $F\colon K \to 2^Y$. We say that F is *superadditive* if $F(x) + F(y) \subset F(x+y)$. Also we say that F is *positively homogeneous* if $F(\lambda x) = \lambda F(x)$ for $\lambda > 0$ and $x \in K$. If F is both positively homogeneous and superadditive, then F is said to be *superlinear.*

As a consequence of Theorem 1 we can obtain the following corollary.

COROLLARY 1. *Let X and Y be real normed spaces and let K be a non-empty open convex cone in X. Assume that $F\colon K \to 2^Y \setminus \{\emptyset\}$ is a superlinear multifunction with bounded closed values in Y. If F is weakly upper bounded on an open non-empty subset of K, then there exists $c > 0$ such that*

$$\|F(x)\| \le c\|x\| \qquad \textit{for} \quad x \in K\,. \tag{6}$$

P r o o f . From Theorem 1 we have

$$\|F(x)\| \leq c\big(1+\|x\|\big)$$

for every $x \in K$ and some $c > 0$. The homogeneity of F yields

$$\|\lambda F(x)\| \leq c\big(1+\lambda\|x\|\big) \qquad \text{for all} \quad \lambda > 0\,,$$

whence

$$\|F(x)\| \leq \frac{c}{\lambda} + c\|x\|\,.$$

Letting $\lambda \to \infty$ we obtain (6). □

It is known that a midconvex and a bounded above at a neighbourhood of a point of D single-valued function defined on an open convex set $D \subset Y$ is locally Lipschitz on D (see e.g. [3]). An analogous result is valid for midconvex weakly upper bounded multifunctions.

THEOREM 2. *If X, Y, D, F are as in Theorem 1, then F is locally Lipschitz on D. Moreover, if $K \subset D$ is a non-empty bounded set such that*

$$\bigcup\big\{B(x,r):\ x \in K\big\} \subset D \tag{7}$$

for some $r > 0$, then F is Lipschitz on K.

P r o o f . Take an arbitrary $x \in D$. There exists a positive number r such that the ball $B(x,2r)$ is contained in D. Choose $y, z \in B(x,r)$ such that $y \neq z$. We can find a positive rational number λ for which

$$\frac{\|y-z\|}{r} < \lambda < \frac{2\|y-z\|}{r}\,. \tag{8}$$

Further put

$$y_1 = y + \frac{1}{\lambda}(y-z)\,. \tag{9}$$

Hence, in view of (8), $y_1 \in B(x,2r)$ and

$$y = \frac{1}{1+\lambda} z + \frac{\lambda}{1+\lambda} y_1\,.$$

Since F is midconvex, we get by Lemma 1

$$\frac{1}{1+\lambda}F(z) + \frac{\lambda}{1+\lambda}F(y_1) \subset F(y)\,.$$

Let $u \in F(z)$ and $v \in F(y_1)$. We have

$$u + \frac{\lambda}{1+\lambda}(v-u) \in F(y)\,,$$

whence

$$u \in F(y) + \frac{\lambda}{1+\lambda}(\|u\| + \|v\|)S \subset F(y) + \lambda(\|u\| + \|v\|)S. \tag{10}$$

In virtue of Theorem 1 we infer

$$\begin{aligned}\|u\| &\leq \|F(z)\| \leq c(1 + \|z\|) \\ &\leq c(1 + \|z - x\| + \|x\|) \leq c(1 + r + \|x\|)\end{aligned}$$

and

$$\begin{aligned}\|v\| &\leq \|F(y_1)\| \leq c(1 + \|y_1\|) \\ &\leq c(1 + \|y_1 - x\| + \|x\|) \leq c(1 + 2r + \|x\|).\end{aligned}$$

Consequently, by (10) and (8),

$$u \in F(y) + \frac{2c}{r}(2 + 3r + 2\|x\|)\|y - z\|S,$$

whence

$$F(z) \subset F(y) + \frac{2c}{r}(2 + 3r + 2\|x\|)\|y - z\|S$$

since u was chosen arbitrarily in $F(z)$. By reversing the roles of y and z in the above argument we obtain the inclusion

$$F(y) \subset F(z) + \frac{2c}{r}(2 + 3r + 2\|x\|)\|y - z\|S.$$

The two last relations imply that

$$h(F(z), F(y)) \leq \frac{2c}{r}(2 + 3r + 2\|x\|)\|y - z\|, \tag{11}$$

where h denotes the Hausdorff metric derived from the norm in Y. This means that F fulfils in $B(x, r)$ the Lipschitz condition with the constant $\frac{2c}{r}(2 + 3r + 2\|x\|)$.

Now, let $K \subset D$ be a bounded set and let (7) hold for some $r > 0$. Take $y, z \in K$ such that $y \neq z$. We can find a constant d such that $\|x\| \leq d$ for all $x \in K$. Choose also a positive rational number λ such that inequality (8) is fulfilled. Further define y_1 by (9). Making use of (8) we get

$$\|y_1 - y\| = \frac{\|y - z\|}{\lambda} < r,$$

which means that $y_1 \in B(y, r)$. Moreover, relation (10) holds true for arbitrary $u \in F(z)$ and $v \in F(y_1)$. By Theorem 1

$$\|u\| \leq c(1 + \|z\|) \leq c(1 + d)$$

and

$$\|v\| \le c\big(1 + \|y_1\|\big) \le c(1 + r + d)\,.$$

Hence, by (8) and (10),

$$F(z) \subset F(y) + \frac{2c}{r}(2 + r + 2d)\|y - z\|S\,.$$

Since the roles y and z are symmetric, we may interchange them in the above relation, whence the inclusion

$$F(y) \subset F(z) + \frac{2c}{r}(2 + 2d + r)\|y - z\|S$$

results. Thus we have

$$h\big(F(z), F(y)\big) \le \frac{2c}{r}(2 + 2d + r)\|y - z\|\,.$$

The last inequality, valid for arbitrary $y, z \in K$, says that F is Lipschitz on K. □

COROLLARY 2. *Let X, Y, D be as in Theorem 1 and let $F\colon X \to 2^Y \setminus \{\emptyset\}$ be a midconvex multifunction with bounded closed values. If F is weakly upper bounded on an open non-empty subset of X, then F fulfils on X the Lipschitz condition.*

Proof. Take arbitrary $y, z \in X$. We can find $r_0 > 0$ such that $y, z \in B(0, r)$ for all $r \ge r_0$. From the proof of Theorem 2 (see relation (11)) we have

$$h\big(F(y), F(z)\big) \le \frac{2c}{r}(2 + 3r)\|y - z\|\,.$$

Letting $r \to \infty$, we get

$$h\big(F(y), F(z)\big) \le 6c\|y - z\|\,,$$

i.e., f fulfils on X the Lipschitz condition with the constant $6c$. □

REFERENCES

[1] NIKODEM, K.: *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politech. Łódz. Mat. **559** (1989), 1–75.

[2] PSHENICHNYI, B.: *Convex multivalued mappings and their conjugates*, Cybernetics **10** (1974), 453–464.

[3] ROBERTS, A. W.—VARBERG, D. E.: *Convex functions*, New York-London, Academic Press, 1973.

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