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ON DIRECT PRODUCT DECOMPOSITIONS OF DIRECTED SETS

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In this note it will be shown that a result of the paper [1] concerning direct product decompositions of a directed set into finitely many factors cannot be extended for direct product decompositions having an infinite number of factors. This solves a question proposed by M. Kolibiar. An analogous question dealing with a more general situation will also be investigated.

1. Preliminaries

Let $\mathscr{P} = (P; \leqq)$ be a partially ordered set.

1.1. Definition. (Cf. [1].) *An equivalence Θ on P will be said to be a congruence relation on $\mathscr{P}$ if the following conditions are satisfied:*

(i) *For each $a \in P$, $[a]\Theta (= \{x \in P | x\Theta a\})$ is a convex subset of $\mathscr{P}$.*

(ii) *If $a, b, c \in P$, $a \leqq c$, $b \leqq c$, and $a\Theta b$, then there is $d \in P$ such that $a \leqq d \leqq c$, $b \leqq d$ and $a\Theta d$.*

(iii) *If $a, b, u, v \in P$, $u \leqq a \leqq v$, $u \leqq b \leqq v$ and $u\Theta a$ $(a\Theta v)$, then there is $t \in P$ such that $b \leqq t \leqq v$, $a \leqq t$ $(u \leqq t \leqq b$, $t \leqq a)$ and $b\Theta t$.*

It is remarked in [1] that the conditions (ii) and (iii) of the above definition can be replaced by the following condition:

(iv) *If $a, b, c, d \in P$, $a \leqq c \leqq d$, $b \leqq d$ $(a \geqq c \geqq d$, $b \geqq d)$ and $a\Theta b$, then there is $e \in P$ such that $c \leqq e \leqq d$, $b \leqq e$ $(c \geqq e \geqq d$, $b \geqq e)$ and $c\Theta e$.*

Let Θ be a congruence relation on $\mathscr{P}$. We put $P/\Theta = \{[a]\Theta | a \in P\}$. For $a, b \in P$ we set $[a]\Theta \leqq [b]\Theta$ if there exist $a_1 \in [a]\Theta$ and $b_1 \in [b]\Theta$ such that $a_1 \leqq b_1$. As usual, we denote $\mathscr{P}/\Theta = (P/\Theta, \leqq)$.

Let $\{\Theta_i\}_{i \in I}$ be a set of congruence relations on $\mathscr{P}$. Let us consider the following conditions:

(1) $\bigwedge(\Theta_i | i \in I) = \mathrm{id}_p$.

(2) $\bigvee(\Theta_i | i \in I) = P \times P$.

(3) *For each family* $(x_i|i\in I)$ *of elements of* P *there exists an element* $x\in P$ such that $x\Theta_i x_i$ for all $i\in I$.

The direct product of partially ordered sets $\mathscr{P}_i$ $(i\in I)$ will be denoted by $\Pi(P_i|i\in I)$.

1.2. Lemma. *Let* $\mathscr{P}$ *and* $\mathscr{P}_i$ $(i\in I)$ *be directed sets. Let* φ *be an isomorphism of* $\mathscr{P}$ *onto* $\Pi(\mathscr{P}_i|i\in I)$. *For* $x, y\in\mathscr{P}$ *and* $i\in I$ *we put* $x\Theta_i y$ *if* $\varphi(x)(i)=\varphi(y)(i)$. *Then* (i) Θ_i *is a congruence relation on* $\mathscr{P}$, *and* (ii) *the indexed set* $\{\Theta_i|i\in I\}$ *fulfils the conditions* (1), (2), *and* (3).

Proof. The assertion (ii) is obvious; (i) follows from [1], Theorem 4.

Under these assumptions the isomorphism φ is said to be a direct product decomposition of $\mathscr{P}$.

If $\mathscr{P}$ is as in 1.2 and φ' is an isomorphism of $\mathscr{P}$ onto $\Pi(\mathscr{P}_i|i\in I)$ such that (under analogous denotations as above) we have $\Theta_i=\Theta_i'$ for each $i\in I$, then the direct decompositions φ and φ' will be considered to be equal. Let us put

$$\chi(\varphi)=\{\Theta_i|i\in I\}.$$

1.3. Theorem. (Cf. [1], Theorems 5 and 7.) *Let* $\mathscr{P}$ *be a directed set. Then* χ *is a one-to-one correspondence between direct product decompositions of* $\mathscr{P}$ *into finitely many (say* n*) factors and the families* $(\Theta_i|i\in I)$, $I=\{1, 2, \ldots, n\}$ *of congruence relations of* $\mathscr{P}$ *satisfying* (1), (2) *and* (3). *If* $(\Theta_i|i\in I)$ fulfils these conditions, then $\mathscr{P}\simeq\Pi(\mathscr{P}/\Theta_i|i\in I)$.

M. Kolibiar raised the question (oral communication) whether the above theorem can be extended to the case of a direct product decomposition with infinitely many factors. By a counter-example we shall show that the answer is "No".

2. An example

Let I be an infinite set, $\operatorname{card} I=\alpha$. Let P be the set of all functions $p: I\to\{0, 1\}$. For $p_1, p_2\in P$ we put $p_1\leqq p_2$ if $p_1(i)\leqq p_2(i)$ for each $i\in I$. Then $\mathscr{P}=(P; \leqq)$ is a Boolean algebra.

Let $i\in I$ and let p_1, p_2 be elements of P. We set $p_1\Theta_i p_2$ if $p_1(i)=p_2(i)$. Then we obviously have

2.1. Lemma. *For each* $i\in I$, Θ_i *is a congruence relation on* P *and the family* $\{\Theta_i|i\in I\}$ *fulfils the conditions* (1), (2) *and* (3). $\mathscr{P}$ *is isomorphic to* $\Pi(\mathscr{P}/\Theta_i|i\in I)$.

Let β be a cardinal, $\aleph_0\leqq\beta\leqq\alpha$. We define a binary relation $\leqq_\beta$ on P as follows. Let $p_1, p_2\in P$. We put $p_1\leqq_\beta p_2$ if some of the following conditions is valid:

($\beta(1)$) $\operatorname{card}\{i \in I | p_1(i) \neq 0\} < \beta$ and $p_1 \leqq p_2$.
($\beta(2)$) $\operatorname{card}\{i \in I | p_2(i) \neq 1\} < \beta$ and $p_1 \leqq p_2$.
($\beta(3)$) $\operatorname{card}\{i \in I | p_1(i) \neq p_2(i)\} < \beta$ and $p_1 \leqq p_2$.
The following two lemmas are easy to verify.

2.2. Lemma. *Under the above assumptions,* $\mathscr{P}_\beta = (P; \leqq_\beta)$ *is a directed set which is not isomorphic to* $\mathscr{P}$. *If* $\aleph_0 \leqq \gamma \leqq \alpha$, $\beta \neq \gamma$, *then* $\mathscr{P}_\beta \neq \mathscr{P}_\gamma$.

2.3. Lemma. *Let* $\aleph_0 \leqq \beta \leqq \alpha$, $p \in P$, $i \in I$. *Then* $[p]\Theta_i$ *is a convex subset in* $\mathscr{P}_\beta$.

2.4. Lemma. *Let* $\aleph_0 \leqq \beta \leqq \alpha$, $i \in I$. *Then* Θ_i *satisfies the condition* (iv) *for* $(P; \leqq_\beta)$.

Proof. Let $a, b, c, d \in P$, $a \leqq_\beta c \leqq_\beta d$, $b \leqq_\beta d$ and $a\Theta_i b$. We have to verify that there exists $e \in P$ such that $c \leqq_\beta e \leqq_\beta d$, $b \leqq_\beta e$ and $c\Theta_i e$.

First suppose that $a(i) = 0$. Then $b(i) = 0$ as well. If $c(i) = 0$, then let $e \in P$ such that $e(i) = 0$ and $e(j) = d(j)$ for each $j \in I \backslash \{i\}$. If $c(i) = 1$, then we put $e = d$.

Now let $a(i) = 1$. Then $c(i) = b(i) = 1$ and we put $e = d$.

In all the mentioned cases we have $c \leqq_\beta e \leqq_\beta d$, $b = e$ and $c\Theta_i e$. The case $a \geqq_\beta c \geqq_\beta d$, $b \geqq_\beta d$ can be treated analogously.

For each cardinal β with $\aleph_0 \leqq \beta \leqq \alpha$ and each $i \in I$, the partially ordered set $\mathscr{P}_\beta/\Theta_i$ is isomorphic to $\mathscr{P}/\Theta_i$. Because $\mathscr{P}$ is isomorphic to $\Pi(\mathscr{P}/\Theta_i | i \in I)$, in view of 2.2, 2.3 and 2.4 we infer:

2.5. Proposition. *Let* $\aleph_0 \leqq \beta \leqq \alpha$. *Then*

(i) $\{\Theta_i | i \in I\}$ *is a set of congruence relations on* $\mathscr{P}_\beta$ *satisfying the conditions* (1), (2) *and* (3);

(ii) *the partially ordered set* $\mathscr{P}_\beta$ *fails to be isomorphic to* $\Pi(\mathscr{P}_\beta/\Theta_i | i \in I)$.

2.6. Corollary. *The assertion of Theorem* 1.3 *cannot be extended for direct product decompositions having an infinite number of factors.*

Let us recall the following result.

2.7. Theorem. ([1], Theorems 6 and 8.) *Let* $\mathscr{P}$ *be a directed set such that*

(∗) *every closed interval of* $\mathscr{P}$ *satisfies the ascending chain condition.*

Then χ *is a one-to-one correspondence between direct product decompositions of* $\mathscr{P}$ *and families of congruence relations of* $\mathscr{P}$ *satisfying* (1), (2) *and* (3). *If* $(\Theta_i | i \in I)$ *fulfils these conditions, then* $\mathscr{P} \simeq \Pi(\mathscr{P}/\Theta_i | i \in I)$.

From 2.5 we obtain:

2.8. Corollary. *The assumption* (∗) *cannot be cancelled in Theorem* 2.7.

3. Congruence relations corresponding to direct factors

Let $\mathscr{P} = (P; \leqq)$ be a diredted set and let $\varphi: \mathscr{P} \to \Pi(\mathscr{P}_i | i \in I)$ be a direct product decomposition of $\mathscr{P}$. Let $i \in I$ and let Θ_i be as in 1.2. Then Θ_i is said to be a congruence relation corresponding to the direct factor $\mathscr{P}_i$. Such congruence relations will be said to be *d*-congruence relations. Let Con $\mathscr{P}$ be the system of all congruence relations on $\mathscr{P}$.

In view of the negative result established in Section 2 the natural question arises whether we can arrive at a positive result if we consider (instead of Con $\mathscr{P}$) an appropriate subset S of Con $\mathscr{P}$ containing all *d*-congruence relations. (If the answer was positive, then we could modify the notion of congruence relation on $\mathscr{P}$ by allowing only those congruence relations which belong to S).

More exactly, we can ask whether each directed set $\mathscr{P}$ satisfies the following condition:

(c) *There exists a system* $S \subseteq \mathrm{Eq}\, P$ *such that*

(i) *each d-congruence relation of* $\mathscr{P}$ *belongs to* S;

(ii) *if* $\{\Theta_i \mid i \in I\}$ *is a set of equivalence relations belonging to S such that* (1), (2) *and* (3) *are valid, then there exists a direct product decomposition* $\varphi: \mathscr{P} \to \Pi(\mathscr{P}_i \mid i \in I)$ *such that for each* $i \in I$, $\mathscr{P}_i$ *is isomorphic to* $\mathscr{P}/\Theta_i$.

By investigating this question let us first remark that by proving the assertion (i) of Propos. 2.5 we did not apply Lemma 1.2 (i.e., Theorem 4 of [1] was not used). If we apply 1.2, then we obtain an alternative proof of (an augmented version) the assertion (i) of Propos. 2.5. (Cf. 3.2 below; the validity of (1), (2) and (3) is obvious.) We can proceed as follows.

Let $\mathscr{P}$, α, β and $\mathscr{P}_\beta$ be as in Section 2. Let i be a fixed element of I. Denote

$$P_i = \{p \in P : p(j) = 0 \text{ for each } j \in I \setminus \{i\}\},$$
$$P_i' = \{p \in P : p(i) = 0\}.$$

P_i and P_i' are partially ordered under the partial order inherited from $\mathscr{P}_\beta$; then $\mathscr{P}_i = (P_i; \leqq_\beta)$ and $\mathscr{P}_i' = (P_i'; \leqq_\beta)$ are directed sets. Consider the mapping $\varphi: P \to P_i \times P_i'$ defined by $\varphi(p) = (q, r)$, where $p(i) = q(i)$ and $p(j) = r(j)$ for each $j \in I \setminus \{i\}$. Then we obviously have

3.1. Lemma. $\varphi: \mathscr{P}_\beta \to \mathscr{P}_i \times \mathscr{P}_i'$ *is a direct product decomposition of* $\mathscr{P}_\beta$.

Let p_1 and p_2 be elements of P. Let Θ_i be as in Section 2. Next we put $p_1 \Theta_i' p_2$ if $\varphi(p_i) = (q_i, r_i)$ for $i = 1, 2$ and $r_1 = r_2$. Then according to 1.2, we obtain:

3.2. Lemma. *For each* $i \in I$, Θ_i *and* Θ_i' *are d-congruence relations on* $\mathscr{P}_\beta$.

The following proposition shows that the answer to the question proposed above is "No"; hence there is no possibility of strengthening the notion of congruence relation for directed sets in order to obtain "nice" relations between congruences and direct product decompositions.

3.3. Proposition. *Let* $\mathscr{P}_\beta$ *be as in Section* 2. *Then* $\mathscr{P}_\beta$ *does not satisfy the condition* (c).

Proof. By way of contradiction, assume that there exists $S \subseteq \mathrm{Eq}\, P$ such that (i) and (ii) from the condition (c) are valid. According to (i) and Lemma 3.2, all Θ_i $(i \in I)$ belong to S. Then in view of 2.5 (ii), the condition (ii) fails to hold.

REFERENCES

[1] KOLIBIAR, M.: Congruence relations and direct decompositions of ordered sets. Acta scient. mathem. 51, 1987, 129—135.

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О ПРЯМЫХ РАЗЛОЖЕНИЯХ НАПРАВЛЕННЫХ МНОЖЕСТВ

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Резюме

Результаты етой заметки касаются соотношений между конгруэнциями направленного множества и его прямыми разложениями с бесконечным числом сомножителей.