Anatolij Dvurečenskij; Sylvia Pulmannová

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Mathematica Slovaca, Vol. 32 (1982), No. 2, 155--166

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ON JOINT DISTRIBUTIONS OF OBSERVABLES

ANATOLIJ DVUREČENSKIJ—SYLVIA PULMANNOVÁ

In the paper the joint distributions of an infinite set of observables on a logic are studied. In the special case of the Hilbert — space logic, the conditions of the existence of joint distributions for finite and infinite sets of observables are formulated.

Throughout the paper, the word *logic* means a σ-orthomodular lattice and the word *state* means a σ-additive probability measure on a logic. Both definitions of the latter notions together with some basic facts and physical interpretation can be found in [2]. The reader may also consult reference [2] for the basics on real observables. We shall deal with generalized observables, $\mathscr{X}$-observables, defined as follows. Let $\mathscr{X}$ be a complete separable metric space and $B(\mathscr{X})$ the σ-algebra of Borel sets of $\mathscr{X}$. An *$\mathscr{X}$-observable* is a map x from $B(\mathscr{X})$ to a logic L such that

(*i*) $x(\mathscr{X})=1$,

(*ii*) $x(E)\perp x(F)$ if $E\cap F=\emptyset$, $E, F\in B(\mathscr{X})$,

(*iii*) $x(\bigcup E_i)=\bigvee x(E_i)$ if $E_i\cap E_j=\emptyset$, $i\neq j$, $i, j=1, 2, \ldots$.

We shall frequently use the following simple observation. If $f\colon \mathscr{X}\to\mathscr{X}_1$ is a Borel measurable mapping between two complete separable metric spaces, then $f\circ x\colon E\mapsto x(f^{-1}(E))$, $E\in B(\mathscr{X}_1)$ is an $\mathscr{X}_1$-observable.

Obviously, we obtain the "traditional" observable if we set $\mathscr{X}=R^1$ and L is the logic $L(H)$ of all closed subspaces of a separable Hilbert space (real or complex). As known, there is a one-to-one correspondence between R^1-observables and self-adjoint operators on H [3].

Since we assume the space $\mathscr{X}$ to be fixed throughout the paper, we shall write simply an observable instead of an $\mathscr{X}$-observable.

Suppose we are given observables $x_1, x_2, \ldots, x_n\colon B(\mathscr{X})\to L$ and a state m on L. We say that the collection $x_1, \ldots, x_n$ has a *joint distribution* in the state m if there is a probability measure p on $B(\mathscr{X}^n)$ such that

$$p(E_1\times\ldots\times E_n)=m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big) \tag{1}$$

for any $E_i\in B(\mathscr{X})$, $i=1, 2, \ldots, n$. Evidently, such a measure p is then unique and we may (and shall) denote by $p^m_{x_1}, \ldots, x_n$ the measure corresponding to the collection $x_1, \ldots, x_n$ and the state m. This type of joint distributions was introduced by S. Gudder [4]. It is called the *type*-1 *joint distribution.*

We denote by $Com(x_1, \ldots, x_n)$ for a collection $x_1, \ldots, x_n$ of observables the set of all states on L in which the joint distribution exists. Let us recall a useful criterion for a state to belong to $Com(x_1, \ldots, x_n)$ (c. f. [1]). In order to simplify the expressions, let us state first a few conventions. Put $D=\{0, 1\}$ and denote by d_i the i-th coordinate of a point $d \in D^n$, $n \in N$. Write $E^{d_i}=E$ if $d_i=1$, and $E^{d_i}=E^c=\mathscr{X}-E$ if $d_i=0$, for any $E \in B(\mathscr{X})$. The criterion reads as follows. The observables $x_1, \ldots, x_n$ have a joint distribution in a state m iff

$$\sum_{d \in D^n} m\Big(\bigwedge_{i=1}^n x_i(E_i^{d_i}\Big)=1 \tag{2}$$

for any $E_1, E_2, \ldots, E_n \in B(\mathscr{X})$.

Let us set

$$a(E_1, E_2, \ldots, E_n)=\bigvee_{d \in D^n} \bigwedge_{i=1}^n x_i(E_i^{d_i}), \tag{3}$$

$E_i \in B(\mathscr{X})$. One sees easily that criterion (2) can be reformulated as follows:

$$m \in Com(x_1, \ldots, x_n) \text{ if } m(a(E_1, \ldots, E_n))=1 \tag{4}$$

for any $E_1, \ldots, E_n \in B(\mathscr{X})$.

Proposition 1. *(i) Let $f_1, \ldots, f_n$ be Borel measurable functions from $\mathscr{X}$ to $\mathscr{X}_1$. If $p^m_{x_1, \ldots, x_n}$ exists, then $p^m_{f_1 \circ x_1, \ldots, f_n \circ x_n}$ exists as well and there holds*

$$p^m_{x_1, \ldots, x_n}(f_1^{-1}(E_1) \times \ldots \times f_n^{-1}(E_n))=p^m_{f_1 \circ x_1, \ldots, f_n \circ x_n}(E_1 \times \ldots \times E_n) \tag{5}$$

for any $E_1, \ldots, E_n \in B(\mathscr{X}_1)$.

(ii) $p^m_{x_1, \ldots, x_n}$ exists iff for any choice of real valued Borel measurable functions $f_1, \ldots, f_n$ the observables $f_1 \circ x_1, \ldots, f_n \circ x_n$ from $B(R^1)$ into L have the joint distribution in the state m.

Proof. Part (*i*) and the necessary condition of (*ii*) are evident. Now let for all $f_1, \ldots, f_n: \mathscr{X} \to R^1$ the observables $f_1 \circ x_1, \ldots, f_n \circ x_n$ have the joint distribution in m. Let us put $f_i=\chi_{E_i}$, $i=1, 2, \ldots, n$, and let p be the joint distribution of $f_1 \circ x_1, \ldots, f_n \circ x_n$. Then

$$1=p(\{0, 1\} \times \{0, 1\} \times \ldots \times \{0, 1\})=\sum_{d \in D^n} p(\{d_1\} \times \ldots \times \{d_n\})=$$

$$=\sum_{d \in D^n} m\Big(\bigwedge_{i=1}^n x_i(\chi_{E_i}^{-1}(\{d_i\}))\Big)=\sum_{d \in D^n} m\Big(\bigwedge_{i=1}^n x_i(E_i^{d_i})\Big),$$

which implies, by criterion (2), the the validity of (*ii*).

Corollary 2. *If $x_1, \ldots, x_n$ are mutually compatible observables, then they have a joint distribution in any state.*

Proof. Suppose that $f_1, \ldots, f_n$ are Borel functions. If $x_1, \ldots, x_n$ are mutually compatible observables, then $f_1 \circ x_1, \ldots, f_n \circ x_n$ are mutually compatible real observables. By a result of Varadarajan [2], they have a joint distribution in any state and the rest follows from Proposition 1 (*ii*).

Let us recall another standard notion. A sequence m_n of states *converges weakly* to a state m if $m_n(a) \to m(a)$ for any $a \in L$.

The *carrier* of a state m (if it exists) is an element $a \in L$ such that $m(b)=0$ iff $a \perp b$.

A state m_0 is a *superposition* of a collection M of states iff $m(a)=0$ for every $m \in M$ implies $m_0(a)=0$.

The following lemma is an easy consequence of criterion (4).

Lemma 3. *Let a be the carrier of the state m. Then $m \in Com(x_1, \ldots, x_n)$ iff $a \leqslant a(E_1, \ldots, E_n)$ for any $E_1, \ldots, E_n \in B(\mathscr{X})$.*

Proposition 4. *Let m_1 be a state with 1 for the carrier. Then $x_1, \ldots, x_n$ are mutually compatible observables iff $m_1 \in Com(x_1, \ldots, x_n)$.*

Proof. If $x_1, \ldots, x_n$ are mutually compatible, then clearly $m_1 \in Com(x_1, \ldots, x_n)$. Conversely, let $m_1 \in Com(x_1, \ldots, x_n)$. Then $m_1 \in Com(x_i, x_j)$ for any $1 \leqslant i, j \leqslant n$, and, according to Lemma 3, we have

$$1 = x_i(E) \wedge x_j(F) \vee x_i(E)^\perp \wedge x_j(F) \vee x_i(E) \wedge x_j(F)^\perp \vee x_i(E)^\perp \wedge x_j(F)^\perp$$

for any $E, F \in B(\mathscr{X})$, and this identity yelds that $x_i \leftrightarrow x_j$ for any $i, j = 1, 2, \ldots, n$ (c. f. [2]).

Proposition 5. Let $\emptyset \neq M \subset Com(x_1, \ldots, x_n)$. *If m_0 is a superposition of the states of M, then $m_0 \in Com(x_1, \ldots, x_n)$.*

Proof. Let $m \in M$. Then the equality $m(a(E_1, \ldots, E_n)^\perp)=0$ implies $m_0(a(E_1, \ldots, E_n)^\perp)=0$ and therefore $m_0 \in Com(x_1, \ldots, x_n)$.

The next two propositions result from Proposition 5 with regard to the following observations: (*i*) a state m_a with the carrier a is a superposition of a state m_b with the carrier b iff $a \leqslant b$, (*ii*) the state $m = \sum_{i=1}^{\infty} c_i m_i$, where $\sum_{i=1}^{\infty} c_i = 1$, $0 \leqslant c_i \leqslant 1$, is a superposition of the states $\{m_i\}_{i=1}^{\infty}$, and every m_i is a superposition of the state m.

Proposition 6. *Let m_{a_i}, $i=1, 2, \ldots$, m_c be states with the carriers a_i, $i=1, 2, \ldots$, and c, respectively, and let $c \leqslant \bigvee_i a_i$. If $m_{a_i} \in Com(x_1, \ldots, x_n)$ for $i=1, 2, \ldots$, then $m_c \in Com(x_1, \ldots, x_n)$.*

Proposition 7. *Let $m = \sum_{i=1}^{\infty} c_i m_i$, $c_i > 0$, $\sum_{i=1}^{\infty} c_i = 1$. Then $m \in Com(x_1, \ldots, x_n)$ iff $m_i \in Com(x_1, \ldots, x_n)$ for any i.*

Theorem 8. *$Com(x_1, \ldots, x_n)$ is a σ-convex sequentially weakly complete subspace in the space of all states of L.*

Proof. By Proposition 7, $Com(x_1, \ldots, x_n)$ is σ-convex. If $\{m_i\}_i$ is a Cauchy sequence in the weak topology, then due to [5], Th. 2.2, the formula $m(a) = \lim m_i(a)$ defines a state on L. Therefore there is $\lim p_i(E_1 \times \ldots \times E_n) = p(E_1 \times \ldots \times E_n) = m\left(\bigwedge_{j=1}^{n} x_j(E)\right)$ and, consequently, there is $\lim p_i(A) = p(A)$ for any A of the algebra generated by all the rectangle sets. As $m(a(E_1, \ldots, E_n)) = \lim m_i(a(E_1, \ldots, E_n)) = 1$, we obtain $m \in Com(x_1, \ldots, x_n)$ by criterion (4).

Now let $L(H)$ be the logic of all closed subspaces of a Hilbert space H (real or complex) with an inner product $(.,.)$ and let $x_1, \ldots, x_n : B(\mathscr{R}) \to L(H)$ be observables. For $M \in L(H)$, let us put P^M for the projector onto M. There is a one-to-one correspondence between the elements of $L(H)$ and their projections. If $\varphi \in H$ is a unit vector, then $m_\varphi : M \mapsto (P^M\varphi, \varphi)$, $M \in L(H)$, is a state on $L(H)$. Moreover, the Gleason theorem asserts that any state on $L(H)$, $3 \leq \dim H \leq \aleph_0$, is of the form $m(M) = tr(TP^M)$, $M \in L(H)$, where T is the density operator. In the sequel we suppose that $3 \leq \dim H \leq \aleph_0$.

Theorem 9. *The observables $x_1, \ldots, x_n$ on the logic $L(H)$ have a joint distribution in a state $m = m_\varphi$ iff*

$$P^{x_1(E_1)} P^{x_2(E_2)} \ldots P^{x_n(E_n)} \varphi = P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})} \varphi \tag{6}$$

for every $E_1, \ldots, E_n \in B(\mathscr{R})$ end every permutation $(i_1, \ldots, i_n)$ of the set $(1, 2, \ldots, n)$.

The proof of the latter theorem follows from the next lemmas.

Lemma 10. *Let φ be an arbitrary element of H. If (6) holds for $x_1, \ldots, x_n$ and for φ, then*

$$P^{x_1(E_1) \wedge x_2(E_2) \wedge \ldots \wedge x_n(E_n)} \varphi = P^{x_1(E_1)} P^{x_2(E_2)} \ldots P^{x_n(E_n)} \varphi \tag{7}$$

for every $E_1, \ldots, E_n \in B(\mathscr{R})$.

Proof. It is known that the equality $P^M P^N \varphi = P^N P^M \varphi$ implies $P^{M \wedge N} \varphi = P^M P^N \varphi$. The rest is an elementary induction.

Lemma 11. *If (6) holds for $x_1, \ldots, x_n$ and $\|\varphi\| = 1$, then $x_1, \ldots, x_n$ have a joint distribution in the state $m = m_\varphi$.*

Proof. By the property (7) we have

$$\sum_{d \in D^n} m\left(\bigwedge_{i=1}^{n} x_i(E_i^{d_i})\right) = \sum_{d \in D^n} \left(P^{\bigwedge_{i=1}^{n} x_i(E_i^{d_i})} \varphi, \varphi\right) =$$

$$\sum_{d \in D^n} (P^{x_1(E_1^{d_1})} \ldots P^{x_n(E_n^{d_n})} \varphi, \varphi) = (\varphi, \varphi) = 1.$$

The criterion (1) implies that $x_1, \ldots, x_n$ have a joint distribution in the state $m = m_\varphi$.

Lemma 12. ([4], *Lemma* 3.5). *Let* $\{\varphi_i\}_i$ *be an orthonormal set of vectors in* H. *If a vector* φ *satisfies* $\|\varphi\|^2 = \sum_i |(\varphi, \varphi_i)|^2$, then $\varphi = \sum_i (\varphi, \varphi_i)\varphi_i$.

Lemma 13. *Let* $M_1, \ldots, M_n \in L(H)$ *and let* $d \in D^n$, $0 \neq \varphi \in \bigwedge_{i=1}^n M_i^{d_i}$, *where* $M^{d_i} = M$ *if* $d_i = 1$ *and* $M^{d_i} = M^\perp$ *if* $d_i = 0$. *Then*

$$P^{M_{i_1}} \ldots P^{M_{i_n}}\varphi = P^{M_1} \ldots P^{M_n}\varphi \tag{8}$$

for every permutation $(i_1, \ldots, i_n)$ *of* $(1, 2, \ldots, n)$.

Proof. We have $P^{M_1} \ldots P^{M_n}\varphi = \varphi$ iff $d_i = 1$, $i = 1, \ldots, n$; otherwise $P^{M_1} \ldots P^{M_n}\varphi = 0$. The proof follows immediately.

Proof of Theorem 9. The sufficient condition was proved in Lemma 11.

The necessary condition. Let $E_1, \ldots, E_n \in B(\mathscr{X})$ be given. By criterion (1) we have

$$\sum_{d \in D^n} \left(\bigwedge_{i=1}^n P^{x_i(E_i^{d_i})}\varphi, \varphi \right) = 1. \tag{9}$$

Let us set $M(d) = \bigwedge_{i=1}^n x_i(E_i^{d_i})$, $d \in D^n$. The vectors $\{P^{M(d)}\varphi \colon d \in D^n\}$ are orthogonal and the equality (9) yields

$$\|\varphi\|^2 = 1 = \sum_{\{d \in D^n \colon M(d)\varphi \neq 0\}} \left| \left(\varphi, \frac{P^{M(d)}}{\|P^{M(d)}\varphi\|} \varphi \right) \right|^2.$$

We see by Lemma 12 that the vector φ is a linear combination of vectors $P^{M(d)}\varphi \in \bigwedge_{i=1}^n x_i(E_i^{d_i})$ and by Lemma 13 that the condition (6) is satisfied.

We recall that any density operator can be written in the form $T = \sum_{i=1}^\infty c_i P^{[\varphi_i]}$ for some partition of unity $\{c_i \colon i \in N\}$ and an orthonormal system $\{\varphi_i \colon i \in N\}$, where $P^{[\varphi_i]} \colon \varphi \mapsto (\varphi, \varphi_i)\varphi_i$ is the projector on the subspace generated by the vector φ_i.

Theorem 14. *Let* $x_1, \ldots, x_n$ *be observables on the logic* $L(H)$. *Let* $T \colon \varphi \mapsto \sum_i c_i(\varphi, \varphi_i)\varphi_i$ *be a density operator. Let* $m = m_T \colon M \mapsto tr(P^M T)$ *be the state induced by* T. *Then*

(i) $x_1, \ldots, x_n$ *have a joint distribution in the state* $m = m_T$ *iff the condition* (6) *holds for any* φ_i, (*ii*) *if* $x_1, \ldots, x_n$ *are bounded real observables and* $X_1, \ldots, X_n$ *are corresponding self-adjoint operators on H, then the following conditions are equivalent*

(*a*) $x_1, \ldots, x_n$ *have a joint distribution in the sate* $m = m_T$,

(*b*) $P^{x_1(E_1)} \ldots P^{x_n(E_n)}\varphi_i = P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})}\varphi_i$,

(*c*) $X_1 \ldots X_n\varphi_i = X_{i_1} \ldots X_{i_n}\varphi_i$,

(*d*) $X_1 \ldots X_n T = X_{i_1} \ldots X_{i_n} T$

for any $i = 1, 2, \ldots, E_1, \ldots, E_n \in B(R^1)$ *and any permutation* $(i_1, i_2, \ldots, i_n)$ *of* $(1, 2, \ldots, n)$.

Proof. The statement (*i*) follows from Theorem 9 and Proposition 7. Using the properties of spectral measures of $X_1, \ldots, X_n$ and Theorem 9 we obtain (*ii*).

Theorem 15. *Let* $x_1, \ldots, x_n$ *be observables on the logic* $L(H)$. *The set* H_0 *of all vectors* $\varphi \in H$ *for which* (6) *holds is a closed subspace of H which is compatible with any* $x_i(E)$, *that is,* $P^{x_i(E)}P^{H_0} = P^{H_0}P^{x_i(E)}$ *for any* $E \in B(\mathscr{X})$, $i = 1, 2, \ldots, n$. *Moreover,*

$$H_0 = \bigwedge_{(E_1, \ldots, E_n)} a(E_1, \ldots, E_n). \tag{10}$$

The state $m = m_T\colon M \mapsto tr(TP^M)$ *belongs to* $Com(x_1, \ldots, x_n)$ *iff the eigenvectors of T belong to* H_0.

Proof. It is easy to see that H_0 is a closed subspace of H. Now we show that

$$P^{x_j(E)}P^{H_0} = P^{H_0}P^{x_j(E)},$$

or, equivalently,

$$P^{x_j(E)}\varphi \in H_0, \quad \text{whenever} \quad \varphi \in H_0, \quad j = 1, 2, \ldots, n.$$

Let $(i_1, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$ and x_j be an observable. Let the integer j be at the s-th place, $1 \leq s \leq n$. We have

$$\begin{aligned} &P^{x_{i_1}(E_{i_1})} \ldots P^{x_{i_n}(E_{i_n})}P^{x_j(E)}\varphi = \\ &= P^{x_{i_1}(E_{i_1})} \ldots P^{x_j(E_j)}[P^{x_{i_{s+1}}(E_{i_{s+1}})} \ldots P^{x_{i_n}(E_{i_n})}P^{x_j(E)}\varphi] = \\ &= P^{x_{i_1}(E_{i_1})} \ldots P^{x_j(E_j)}P^{x_j(E)}P^{x_{i_{s+1}}(E_{i_{s+1}})} \ldots P^{x_{i_n}(E_{i_n})}\varphi = \\ &= P^{x_1(E_1)} \ldots P^{x_j(E_j \cap E)} \ldots P^{x_n(E_n)}\varphi. \end{aligned}$$

Similarly,

$$P^{x_1(E_1)} \ldots P^{x_n(E_n)}P^{x_j(E_j)}\varphi = P^{x_1(E_1)} \ldots P^{x_j(E_j \cap E)} \ldots P^{x_n(E_n)}\varphi,$$

and therefore $P^{x_j(E_j)}\varphi \in H_0$.

Now if $\varphi \in H_0$, then by Theorem 9 the joint distribution of $x_1, \ldots, x_n$ in the state m_φ exists, hence $m_\varphi(a(E_1, \ldots, E_n)) = 1$, $E_1, \ldots, E_n \in B(\mathscr{X})$. From this it follows that $\varphi \in a(E_1, \ldots, E_n)$ for any $E_1, \ldots, E_n \in B(\mathscr{X})$. On the other hand, if $\varphi \in \bigwedge_{E_1, \ldots, E_n} a(E_1, \ldots, E_n)$, then the joint distribution in the state m_φ exists by the criterion (4), therefore $\varphi \in H_0$. The last statement of the theorem follows from Theorem 14.

In the following theorem we shall derive a connection between the considered type of joint distribution and another type of joint distributions- the so-called type 2 joint distribution on the logic $L(H)$ (see [4]). For the sake of simplicity we shall consider only two observables. Before stating the theorem we have to recall some definitions and results. For details see [4] and [7].

If x is a real obsservable on $L(H)$, we write X for the corresponding self-adjoint operator on H.

If X and Y are linear operators on H with the domains $D(X)$, $D(Y)$, respectively, then their sum $X+Y$ is an operator defined on $D(X) \cap D(Y)$ such that $(X+Y)\varphi = X\varphi + Y\varphi$ for any $\varphi \in D(X) \cap D(Y)$.

We say that the real observables x and y on the logic $L(H)$ have *type* 2 *joint distribution* in a state m if the observables $\alpha x + \beta y$ exist (i.e. $\alpha X + \beta Y$ are self-adjoint operators) for any $\alpha, \beta \in R^1$ and if there is a measure μ on $B(R^2)$ such that

$$\mu\{(\omega_1, \omega_2) : \alpha\omega_1 + \beta\omega_2 \in E\} = m((\alpha x + \beta y)(E)) \tag{11}$$

for any $E \in B(R^1)$ (see [4]).

A subspace H' of H is *invariant* under an operator X (or H' *reduces* X) if

(1) $X(H' \cap D(X)) \subset H'$,

(2) $P^{H'} D(X) \subset D(X)$. (12)

The condition (1) is equivalent to

$$P^{H'} X P^{H'} = X P^{H'} \tag{13}$$

A subspace H' *orthogonally reduces* the operator X if both H' and $H'^{\perp}$ are invariant under X (see [7], p. 349).

Proposition 16. *A subspace H' of H orthogonally reduces X iff $XP^{H'} \supset P^{H'}X$ (i.e., iff X commutes with $P^{H'}$)* ([7], *p.* 352).

Proposition 17. *Any invariant subspace of a self-adjoint operator X orthogonally reduces X* ([7], *p.* 355).

Proposition 18. *The operator X' induced by a self-adjoint operator X in his invariant subspace H' is self-adjoint* ([7], *p.* 365).

Proposition 19. *A bounded operator Y commutes with a self-adjoint operator X (i.e. $XY \supset YX$) iff it commutes with the spectral measure of the operator X* ([7], *p.* 521).

Theorem 20. *Let x and y be real observables on the logic $L(H)$ and let X and Y be the corresponding self-adjoint operators on H. Let the domains $D(X)$ and $D(Y)$ be such that $D(X) \cap D(Y)$ is dense in H. Then for any $m \in Com(x, y)$ there exists the type 2 joint distribution of x, y in the state m and this type2 joint distribution is identical with the type 1 joint distribution.*

Proof. As $D(X) \cap D(Y)$ is dense in H, the linear combinations $\alpha X + \beta Y$ exist for any $\alpha, \beta \in R^1$. According to Proposition 7, it satisfies to consider a state $m = m_\varphi$, where φ is a unit vector in H. Let $H_0 \in L(H)$ be defined by (10). Then $\varphi \in H_0$ and the subspace H_0 is invariant under the operators X and Y (see Theorem 15 and Propositions 16, 17 and 19). The restrictions of X and Y to H_0, denoted by X_0 and Y_0, are self-adjoint operators on H_0 which may be treated as a Hilbert space in its own right (see Proposition 18). We show that the subspace H_0 is invariant under the operator $\alpha X + \beta Y (\alpha, \beta \in R^1)$. Indeed, we have

$$P^{H_0}(\alpha X + \beta Y)P^{H_0} = P^{H_0}\alpha X P^{H_0} + P^{H_0}\beta Y P^{H_0} =$$
$$= \alpha X P^{H_0} + \beta Y P^{H_0} = (\alpha X + \beta Y)P^{H_0}$$

and so the condition (1) in (12) is fulfilled. From the inclusions

$$P^{H_0}D(X) \subset D(X), \quad P^{H_0}D(Y) \subset D(Y) \tag{14}$$

we obtain

$$P^{H_0}D(X) \cap D(Y) \subset D(X) \cap D(Y)$$

and this means that (2) in (12) is fulfilled. Then restriction $(\alpha X) + \beta Y)_0$ of $\alpha X + \beta Y$ to H_0 is a self-adjoint operator and, clearly, $(\alpha X + \beta Y)_0 = \alpha X_0 + \beta Y_0$. As X_0 and Y_0 commute, there is the type 2 joint distribution (indentical with the type 1 joint distribution (see [4])) for the observables x_0 and y_0 in the state m_φ, in other words, there is a measure μ on $B(R^2)$ such that

$$\mu\{(\omega_1, \omega_2) : \alpha\omega_1 + \beta\omega_2 \in E\} = m_\varphi((\alpha x_0 + \beta y_0)(E)),$$

for any $\alpha, \beta \in R^1$ and any $E \in B(R^1)$. Hovewer,

$$m_\varphi((\alpha x_0 + \beta y_0)(E)) = (P^{(\alpha x_0 + \beta y_0)(E)}\varphi, \varphi) =$$
$$(P^{(\alpha x + \beta y)_0(E)}\varphi, \varphi) = (P^{(\alpha x + \beta y)(E)}\varphi, \varphi) = m_\varphi((\alpha x + \beta y)(E)).$$

From this we see that the type 2 joint distribution in the state m_φ exists. On the other hand,

$$\mu(E \times F) = m_\varphi(x_0(E) \wedge y_0(F)) = (P^{x_0(E) \wedge y_0(F)}\varphi, \varphi) =$$

$$= (P^{x_0(E)}P^{y_0(F)}\varphi, \varphi) = (P^{H_0}P^{x(E)}P^{H_0}P^{y(F)}P^{H_0}\varphi, \varphi) =$$

$$= (P^{x(E)}P^{y(F)}\varphi, \varphi) = (P^{x(E)\wedge y(F)}\varphi, \varphi) = m_\varphi(x(E)\wedge y(F)).$$

Hence, the two types of joint distributions, being identical for the observables x_0 and y_0 on $L(H_0)$, are identical also for x and y.

The notion of the joint distribution of observables can be generalized to an arbitrary system of observables $\{x_s: s\in S\}$. We say that a system of observables $\{x_s: s\in S\}$ has the joint distribution in a state m if any its finite subsystem has one.

Now let a system of observables $\{x_s: s\in S\}$ on a separable logic L be given. We recall that a logic L is *separable* if any subsystem of mutually orthogonal elements of L is at most countable. For any finite set $\emptyset\neq F=\{s_1, \dots, s_n\}\in S$ we put

$$a(F; E_1, \dots, E_n) = \bigvee_{d\in D^n}\bigwedge_{j=1}^{n} x_{s_j}(E_j^{d_j}), \tag{15}$$

where $D=\{0, 1\}$, $d=(d_1, \dots, d_n)\in D^n$, $E^{d_j}=E$ if $d_j=1$ and $E^{d_j}=E^c$ if $d_j=0$,

$$a_0(F) = \bigwedge_{E_1, \dots, E_n} a(F; E_1, \dots, E_n) \tag{16}$$

$$a_0 = \bigwedge_{F\subset S,\ F\ \text{finite}} a_0(F). \tag{17}$$

Lemma 21. *Let L be a separable logic and let $\{x_s: s\in S\}$ be a given system of observables. Then for finite sets F_1, $F_2\subset S$, $\emptyset\neq F_1\subset F_2$ we have*

$$a_0(F_2)\leq a_0(F_1). \tag{18}$$

Proof. Let $F_1=\{s_1, \dots, s_n\}$, $F_2=\{s_1, \dots, s_n, s_{n+1}, \dots, s_m)\}$. Then $a(F_2; E_1, \dots, E_n, 0, \dots, 0) = a(F_1; E_1, \dots, E_n)$ and therefore

$$a_0(F_2) = \bigwedge_{E_1, \dots, E_m} a(F_2; E_1, \dots, E_m) \leq \bigwedge_{E_1, \dots, E_n} a(F_2; E_1, \dots, E_n, 0, \dots, 0) =$$

$$= \bigwedge_{E_1, \dots, E_n} a(F_1; E_1, \dots, E_n) = a_0(F_1).$$

By Zierler [6] there is in a separable logic to any system of elements $\{a_\alpha\}_\alpha$ a countable subsystem $\{a_i\}_i$ such that

$$\bigvee_\alpha a_\alpha = \bigvee_{i=1}^{\infty} a_i \left(\bigwedge_\alpha a_\alpha = \bigwedge_{i=1}^{\infty} a_i\right).$$

Theorem 22. *A system of observables $\{x_s: s\in S\}$ on a separable logic has the joint distribution in a state m iff $m(a_0)=1$.*

Proof. If the given system of observables has the joint distribution in the state m, then for any finite subset $F=\{s_1, \dots, s_n\}\subset S$ we have $m\in Com(x_{s_1}, \dots, x_{s_n})$. Therefore Theorem 2.7 in [1] implies that $m(a_0(F))=1$. Due to the separability of

L there is a sequence of finite subsets $\{F_n\}_n$ of S such that $a_0 = \bigwedge_n a_0(F_n)$. Now we show that

$$m\Big(\bigwedge_{i=1}^{n} a_0(F_i)\Big) = 1$$

for any n. Indeed, if we put $F_n^* = \bigcup_{i=1}^{n} F_i$, then $a_0(F_n^*) \leqslant a_0(F_i)$, $i = 1, \ldots, n$; $n = 1, 2, \ldots$. As $m(a_0(F_n^*)) = 1$, we obtain $m(a_0) = \lim_n m\Big(\bigwedge_{i=1}^{n} a_0(F_i)\Big) = 1$. Since $a_0 \leqslant a_0(F)$, $F \subset S$, the converse implication follows from Theorem 2.7 in [1].

The following theorem is a generalization of Theorem 2.11 in [1] fot the case of an infinite set of observables.

Theorem 23. *Let L be a separable logic and let there be, for any $a \neq 0$, a state m_a with the carrier a. If the system of observables $\{x_s : s \in S\}$ has the joint distribution in the state m, then $a_0 \neq 0$ and $x_s(E) \leftrightarrow a_0$ for any $E \in B(\mathscr{X})$ and $s \in S$. Moreover, all the observables $x_{s0} = x_s \wedge a_0$ are compatible on the logic $L_{[0, a_0]} = \{b \in L : b \leqslant a_0\}$. For the element a_0 we have*

$$a_0 = \bigvee \{a \in L: \text{there is the joint distribution of } \{x_s : s \in S\} \text{ in } m_a\}. \tag{19}$$

Proof. Since $m(a_0) = 1$, it is clear that $a_0 \neq 0$. There is a sequence of finite subsets $\{F_n\}_n$ of S such that $a_0 = \bigwedge_n a_0(F_n)$. Suppose $s \in S$. We shall show that $x_s(E) \leftrightarrow a_0$ for any $E \in B(\mathscr{X})$. For the indexed set $F_n' = F_n \cup \{s\}$ we have $a_0(F_n') \leqslant a_0(F_n)$ (Lemma 16). Therefore $a_0 \geqslant \bigwedge_n a_0(F_n')$, but on the other hand $a_0 = \bigwedge_{F \subset S} a_0(F) \leqslant \bigwedge_n a_0(F_n')$. Hence, $a_0 = \bigwedge_n a_0(F_n')$. Theorem 2.11 in [1] implies that $x_s(E) \leftrightarrow a_0(F_n')$ for any $E \in B(\mathscr{X})$ and any n. Therefore $x_s(E) \leftrightarrow \bigwedge_n a_0(F_n') = a_0$ (see [2], Lemma 6.10).

Now it is easy to show that $x_{s0} : E \mapsto x_s(E) \bigwedge a_0$, $E \in B(\mathscr{X})$ is an observable on the logic $L_{[0, a_0]}$. We claim that $\{x_{s0} : s \in S\}$ are compatible observables on $L_{[0, a_0]}$. Since $m_{a_0}(a_0) = 1$, we have (Theorem 22) that $\{x_s : s \in S\}$ has the joint distribution in the state m_{a_0}. Therefore, for any $s, t \in S$

$$m_{a_0}((x_s(E) \wedge x_t(F) \vee x_s(E)^{\perp} \wedge x_t(F) \vee x_s(E) \wedge x_t(F)^{\perp} \vee x_s(E)^{\perp} \wedge x_t(F)^{\perp})) = 1 \tag{20}$$

for any $E, F \in B(\mathscr{X})$.

As $x_s(E)\wedge x_t(F)\leftrightarrow a_0$ for E, $F\in B(\mathscr{X})$, we have

$$m_{a_0}(x_s(E)\wedge x_t(F))=m_{a_0}(x_s(E)\wedge x_t(F)\wedge a_0)+m_{a_0}(x_s(E)\wedge x_t(F)\wedge a_0^{\perp})=$$
$$=m_{a_0}(x_s(E)\wedge x_t(F)\wedge a_0).$$

The latter fact and (20) imply that

$$\tilde{m}_{a_0}(x_{s0}(E)\wedge x_{t0}(F)\vee x_{s0}(E)^{\perp}\wedge x_{t0}(F)\vee x_{s0}(E)\wedge x_{t0}(F)^{\perp}\vee x_{s0}(E)^{\perp}\wedge x_{t0}(F)^{\perp})=1$$

for any E, $F\in B(\mathscr{X})$, where $\tilde{m}_{a_0}=m_{a_0}/L_{[0,\,a_0]}$. By the criterion (4) and Proposition 4, $x_{s0}\leftrightarrow x_{t0}$.

The equality (19) follows from the observation that the system $\{x_s\colon s\in S\}$ has the joint distribution in the state m_a iff $m_a(a_0)=1$ or, equivalently, iff $a\leqslant a_0$.

For the case of the logic $L(H)$ the following theorem holds.

Theorem 24. *Let $L(H)$ be the logic of all closed subspaces of a separable Hilbert space H and let $\{x_s\colon s\in S\}$ be a system of observables. Then it has the joint distribution in a state $m=\sum_j c_j m_{\Phi_j}$, $c_j\geqslant 0$, $\sum_j c_j=1$, $\{\Phi_j\}_j$ an orthogonal system of vectors, iff*

$$P^{x_{s_{i_1}}(E_{i_1})}\dots P^{x_{s_{i_n}}(E_{i_n})}\,\Phi_j=P^{x_{s_1}(E_1)}\dots P^{x_{s_n}(E_n)}\Phi_j \tag{21}$$

for any permutation $(i_1,\dots,i_n)$ of $(1,2,\dots,n)$; $E_1,\dots,E_n\in B(\mathscr{X})$ and any finite subsystem of observables and any vector Φ_j. If we put $H_0=\{\Phi\colon \Phi\in H,\ \Phi$ fulfils (21)$\}$, then $H_0\in L(H)$. Moreover, H_0 is the element defined by (17) and it reduces the observables $\{x_s\colon s\in S\}$.

Finally, let us note that an analogical division into three categories of compatibilities as in [1] may be done for the system of observables $\{x_s\colon s\in S\}$: Let for the system $\{x_s\colon s\in S\}$ the element a_0 be defined by (17) (L is a separable logic). We may say that the system $\{x_s\colon s\in S\}$ of observables is (*i*) *compatible* if $a_0=1$, (*ii*) *partially compatible* if $0\neq a_0\neq 1$, (*iii*) *totally incompatible if* $a_0=0$. Further investigation may then proceede similarly to that of [1].

REFERENCES

[1] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: Connection between joint distribution and compatibility. To be published in Rep. Math. Phys.
[2] VARADARAJAN, V. S.: Geometry of Quantum Theory, Vol. 1. Van Nostrand, New York 1968.
[3] PRUGOVEČKI, E.: Quantum Mechanics in Hilbert Space. Acad. Press, New York 1971.
[4] GUDDER, S. P.: Joint distributions of observables. J. Math. Mech. 18, 1968, 325—335.
[5] DVUREČENSKIJ, A.: On convergences of signed states. Math. Slovaca 3, 1978, 289—297.
[6] ZIERLER, N.: Axioms for nonrelativistic quantum mechanics. Pac. J. Math. 11, 1961, 1161—1169.
[7] ПЛЕСНЕР, А. И.: Спектральная теория линейных операторов. Наука, Москва 1965.

Received April 3, 1980

Ústav merania a meracej techniky SAV
Dúbravská cesta
842 19 Bratislava

Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

ЗАМЕЧАНИЕ О СОВМЕСТНОМ РАСПРЕДЕЛЕНИИ ВЕРОЯТНОСТИ НАБЛЮДАЕМЫХ

Анатолий Двуреченский—Цылвиа Пулманнова

Резюме

В этой статье исследуются критерии для существования совместного распределения вероятности наблюдаемых на логике и их следствия. Полученные результаты обобщаются для случая бесконечной системы наблюдаемых.