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Mathematica Slovaca, Vol. 43 (1993), No. 3, 327--336

Persistent URL: http://dml.cz/dmlcz/129587

Math. Slovaca, **43** (1993), No. 3, 327–336

Mathematica Slovaca

FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

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(*Communicated by Michal Zajac*)

ABSTRACT. In this paper we study in Banach spaces the existence of fixed points of asymptotically regular mappings. Specifically, we establish for these mappings some fixed point theorems in a Hilbert space, in L^p spaces, in Hardy spaces H^p and in Sobolev spaces $H^{p,k}$ for $1 < p < +\infty$ and $k \ge 0$. We extended results from the paper [6].

1. Introduction

Throughout this paper, E will always stand for a real Banach space with norm $\|\cdot\|$.

The concept of asymptotic regularity is due to F. Browder and V. Petryshyn (see [1]).

A mapping $T\colon E \to E$ into itself is said to be *asymptotically regular* if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all x in E.

It is known [5] that if T is nonexpansive, then $T_t := t\cdot I + (1-t)\cdot T$ is asymptotically regular for all $0 < t < 1$.

Recently, P. K. Lin in [10] has constructed a uniformly asymptotically regular Lipschitzian mapping acting on a weakly compact subset of ℓ^2 which has no fixed points.

Let $p > 1$ and denote by λ the number in $[0,1]$ and by $W_p(\lambda)$ the function $\lambda\cdot(1-\lambda)^p + \lambda^p\cdot(1-\lambda)$.

AMS Subject Classification (1991): Primary 47H10. Secondary 54H25.
Key words: Asymptotically regular mapping, Asymptotic center, Fixed point.

The functional $\|\cdot\|^p$ is said to be *uniformly convex* [20] on the Banach space E if

(*) there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x, y \in E$ the following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p.$$

H. K. Xu [19] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e. there exists a constant $c>0$ such that the moduli of convexity (see [5]), $\delta_E(\varepsilon) \ge c\cdot\varepsilon^p$ for all $0 \le \varepsilon \le 2$.

In this note we show some theorems on fixed points of asymptotically regular mappings in p-uniformly convex Banach spaces. The main result generalizes fixed point theorems proved in [6].

2. Preliminaries

Let A and B be a nonempty closed convex bounded subsets of E. Assume that Φ is a real-valued lower semicontinuous functional defined on

$$A \ominus B = \bigcup_{a\in A} a \ominus B = \bigcup_{a\in A} \{a-b:\ b\in B\}$$

and bounded on $a \ominus B$ for each $a \in A$. We note that all functionals Φ which will occur in the applications of the theorems, presented in this paper, have these properties.

An element z in A is said to be an *asymptotic center of the bounded sequence* $B = \{b_n\} \subset E$ *with respect to* Φ *and* A if

$$\Psi(z) = \inf_{a\in A} \Psi(a),$$

where

$$\Psi(a) = \limsup_{n\to\infty} \Phi(a-b_n).$$

R. Smarzewski in [15] (see [17]) has established the following:

THEOREM 1. *Let* $\Phi(a-b_n) := \Phi(a)$, $a \in A$, *such that for all* $b_n \in B$:

1) $\bigvee_{c>0} \bigwedge_{a\in A} \Phi(a) \ge c$,
2) $\bigwedge_{0<t<1} \bigwedge_{h,a\in A} \Phi\big(a+t(h-a)\big) - \Phi(a) \le t\big[\Phi(h)-\Phi(a)\big] - K\big(t,\|h-a\|\big)$,
where

$$K(t,s) = t\cdot\varphi\big((1-t)s\big) + (1-t)\cdot\varphi(ts), \qquad t,s\ge 0,$$

$\varphi\colon [0,+\infty) \to [0,+\infty)$ *is a continuous strictly increasing function with* $\varphi(0)=0$, *and c is constant. Then there exists a unique asymptotic center $z \in A$ of the sequence $B = \{b_n\}$ with respect to Φ and A. Moreover, we have*

$$\Psi(z) \le \Psi(a) - \varphi(\|z-a\|) \tag{1}$$

for all a in A. □

From $(*)$, it follows that the functional $\Phi\colon E \to \mathbb{R}$ defined by $\Phi(y) = \|y\|^p$, satisfies the assumptions of Theorem 1 with $\varphi(s) = c \cdot s^p$. Thus we have the following:

COROLLARY 1. *Let $p > 1$ and let E be a p-uniformly convex Banach space, C a nonempty closed convex subset of E and let $\{x_n\} \subset E$ be a bounded sequence. Then there exists a unique point z in C such that*

$$\limsup_{n\to\infty} \|x_k - z\|^p \le \limsup_{n\to\infty} \|x_k - x\|^p - c_p \cdot \|x-z\|^p$$

for every x in C, where $c_p > 0$ is the constant given in $(*)$. □

The following lemma is crucial in the proof of the main result.

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space E and let $\{n_i\}$ be an increasing sequence of natural numbers. Assume that $T\colon C \to C$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}$, T^m is continuous. If*

$$\tilde{\Psi}(x) := \limsup_{n\to\infty} \|x - T^{n_i}u\| = 0$$

for some $u \in C$ and $x \in C$, then $Tx = x$.

Proof. If $\tilde{\Psi}(x) = 0$, then $T^{n_i}u \to x$ for $i \to +\infty$. So

$$\begin{aligned}\|T^{n_i+m}u - x\| &\le \|T^{n_i+m}u - T^{n_i}u\| + \|T^{n_i}u - x\| \\ &\le \sum_{j=0}^{m-1} \|T^{n_i+j+1}u - T^{n_i+j}u\| + \|T^{n_i}u - x\|\end{aligned}$$

and from the asymptotic regularity of T, $T^{n_i+m}u \to x$ as $i \to +\infty$.

Since T^m is continuous, we have

$$T^m x = T^m\big(\lim_{i\to\infty} T^{n_i}u\big) = \lim_{i\to\infty} T^{n_i+m}u = x.$$

It is easily verified (by induction) that $T^{ms}x = x$ for $s = 1, 2, \ldots$.

Then

$$\|Tx - x\| = \|T^{ms+1}x - T^{ms}x\| \to 0$$

as $s \to +\infty$, so $Tx = x$. □

3. Main result

In this section, we prove a fixed point theorem for asymptotically regular mappings in p-uniformly convex Banach spaces by making use of the method of asymptotic centre.

To prove it, we recall that the *normal structure coefficient* $N(E)$ *of* E is defined (cf. [2]) by

$$N(E) = \inf\left\{ \frac{\operatorname{diam} C}{r_C(C)} : \begin{array}{c} C \text{ a bounded convex subset of } E \\ \text{consisting of more than one point} \end{array} \right\},$$

where

$$\operatorname{diam} C = \sup\{ \|x-y\| : x, y \in C \}$$

is the diameter of C and

$$r_C(C) = \inf_{x \in C} \Big(\sup_{y \in C} \|x-y\| \Big)$$

is the Chebyshev radius of C relative to itself.

E is said to have *uniformly normal structure* if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure (cf. [4]) and for a Hilbert space $\mathcal{H}$, $N(\mathcal{H}) = \sqrt{2}$. Recently, S. Pichugov [11] (cf. [13]) calculated that

$$N(L^p) = \min\{2^{1/p}, 2^{(p-1)/p}\}, \qquad 1 < p < +\infty .$$

Some estimates for the normal structure coefficient in other Banach spaces may be found in [14].

THEOREM 2. *Let $p > 1$ and let E be a p-uniformly convex Banach space, C a nonempty closed convex and bounded subset of E, $T\colon C \to C$ an asymptotically regular mapping. If*

$$\liminf_{n\to\infty} \|T^n\| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right)\right]^{1/p}$$

(where $\|T^n\|$ is the Lipschitz constant (norm) of T^n, N is the normal structure coefficient of E and c_p is the constant given in $(*)$), then T has a fixed point in C.

P r o o f. If $k < 1$, then T has a fixed point by Banach's theorem. Hence assume that $k \geq 1$. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\liminf_{n\to\infty} \|T^n\| = \lim_{i\to\infty} \|T^{n_i}\| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right)\right]^{1/p}.$$

Given an element $z_0 \in C$ and by Lemma 1, we can inductively construct a sequence $\{z_m\}$ such that z_m is the unique asymptotic center of the sequence $\{T^{n_i} z_{m-1}\}_{i\geq 1}$ with respect to the functional

$$\limsup_{i\to\infty} \|x - T^{n_i} z_{m-1}\|^p$$

over x in C.

Now for each $m \geq 1$, we set

$$\begin{aligned} D_m &= \limsup_{i\to\infty} \|z_m - T^{n_i} z_m\|\,, \\ r_m &= \limsup_{i\to\infty} \|z_{m+1} - T^{n_i} z_m\|\,. \end{aligned}$$

By the result of Casini-Maluta [3] and the asymptotical regularity of T, we have

$$\begin{aligned} r_m &\leq \frac{1}{N} \cdot \lim_{n\to\infty} \Big(\sup \big(\|T^{n_i} z_m - T^{n_j} z_m\| :\ i, j \geq n\big)\Big) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \Big(\limsup_{j\to\infty} \|T^{n_i} z_m - T^{n_j} z_m\|\Big) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \Big(\limsup_{j\to\infty} \big(\|T^{n_i} z_m - T^{n_i+n_j} z_m\| + \|T^{n_i+n_j} z_m - T^{n_j} z_m\|\big)\Big) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \Big(\limsup_{j\to\infty} \Big(\|T^{n_i}\| \cdot \|z_m - T^{n_j} z_m\| \\ &\qquad\qquad + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1} z_m - T^{n_j+v} z_m\|\Big)\Big) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \|T^{n_i}\| \cdot \limsup_{j\to\infty} \|z_m - T^{n_j} z_m\| \\ &= \frac{k}{N} \cdot \limsup_{j\to\infty} \|z_m - T^{n_j} z_m\|\,, \end{aligned}$$

i.e.

$$r_m \leq \frac{k}{N} \cdot D_m\,, \qquad m = 0, 1, 2, \dots\,,$$

where N is the normal structure coefficient of E.

For each fixed $m \geq 1$ and all n_i, n_j, we have from $(*)$:

$$\begin{aligned}
&\|\lambda z_{m+1} + (1-\lambda)T^{n_j} z_{m+1} - T^{n_i} z_m\|^p + c_p \cdot W_p(\lambda) \cdot \|z_{m+1} - T^{n_j} z_{m+1}\|^p \\
&\leq \lambda \cdot \|z_{m+1} - T^{n_i} z_m\|^p + (1-\lambda) \cdot \|T^{n_j} z_{m+1} - T^{n_i} z_m\|^p \\
&\leq \lambda \cdot \|z_{m+1} - T^{n_i} z_m\|^p + (1-\lambda) \cdot \big[\|T^{n_j} z_{m+1} - T^{n_i+n_j} z_m\| \\
&\qquad + \|T^{n_i+n_j} z_m - T^{n_i} z_m\|\big]^p \\
&\leq \lambda \cdot \|z_{m+1} - T^{n_i} z_m\|^p + (1-\lambda) \cdot \Big[\|T^{n_j}\| \cdot \|z_{m+1} - T^{n_i} z_m\| \\
&\qquad + \sum_{v=0}^{n_j-1} \|T^{n_i+v+1} z_m - T^{n_i+v} z_m\|\Big]^p .
\end{aligned}$$

Taking the limit superior as $i \to +\infty$ on each side, by definition of z_m and by the asymptotical regularity of T, we get

$$r_m^p + c_p \cdot W_p(\lambda) \cdot \|z_{m+1} - T^{n_j} z_{m+1}\|^p \leq \big(\lambda + (1-\lambda)k^p\big) r_m^p .$$

It then follows that

$$D_{m+1}^p \leq \frac{(1-\lambda)(k^p-1)}{c_p \cdot W_p(\lambda)} \cdot r_m^p \leq \frac{(1-\lambda)(k^p-1)}{c_p \cdot W_p(\lambda)} \cdot \frac{k^p}{N^p} \cdot D_m^p .$$

Letting $\lambda \uparrow 1$, we conclude that

$$D_{m+1} \leq \left[\frac{k^p(k^p-1)}{c_p^p \cdot N^p}\right]^{1/p} \cdot D_m := A \cdot D_m , \qquad m = 1, 2, \dots ,$$

where

$$A = \left[\frac{k^p(k^p-1)}{c_p^p \cdot N^p}\right]^{1/p} < 1$$

by assumption of the theorem.

Since

$$\|z_{m+1} - z_m\| \leq r_m + D_m \leq 2D_m \leq \cdots \leq 2 \cdot A^m \cdot D_0 \to 0$$

as $m \to +\infty$, it follows that $\{z_m\}$ is a Cauchy sequence. Let $z = \lim_{m\to\infty} z_m$. Then we have

$$\begin{aligned}
\|z - T^{n_i} z\| &\leq \|z - z_m\| + \|z_m - T^{n_i} z_m\| + \|T^{n_i} z_m - T^{n_i} z\| \\
&\leq \big(1 + \|T^{n_i}\|\big) \cdot \|z - z_m\| + \|z_m - T^{n_i} z_m\| .
\end{aligned}$$

Taking the limit superior as $i \to +\infty$ on each side, we get

$$\begin{aligned}
\limsup_{i\to\infty} \|z - T^{n_i} z\| &\leq (1+k) \cdot \|z - z_m\| + D_m \\
&\leq (1+k) \cdot \|z - z_m\| + A^m \cdot D_0 \to 0
\end{aligned}$$

as $m \to +\infty$. Therefore $Tz = z$ by Lemma 1. The proof is complete. □

4. The corollaries in Hilbert and L^p-spaces

In this section we give applications of the established inequalities analogous to (*) in some Banach spaces. Let us first begin with the following:

LEMMA 2.

(a) *In a Hilbert space* H, *this equality holds:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all x, y *in* H *and* $\lambda \in [0,1]$.

(b) *If* $1 < p \le 2$, *then we have for all* x, y *in* L^p *and* $\lambda \in [0,1]$,

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2 .$$

(The inequality (b) is contained in [18], [9].)

(c) *Assume* $2 < p < +\infty$ *and* t_p *is the unique zero of the function* $g(x) = -x^{p-1} + (p-1)x + p - 2$ *in the interval* $(1, +\infty)$. *Let*

$$c_p = (p-1)(1+t_p)^{2-p} = (1+t_p^{p-1})/\big((1+t_p)^{p-1}\big)$$

and we have the following inequality

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p$$

for all x, y *in* L^p *and* $\lambda \in [0,1]$.

(The inequality (c) is due essentially to Lim, see [8], [9] and [19].) □

Remark 1. All constants appearing in the inequalities of Lemma 2 (e.g. the $(p-1)$ and c_p) are the best possible, [9], [8].

By Lemma 2 we immediately obtain from Theorem 2 the following results:

COROLLARY 2. ([7]) *Let* C *be a nonempty bounded closed convex subset of a Hilbert space* H. *If* $T\colon C \to C$ *is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \sqrt{2},$$

then T *has a fixed point in* C. □

COROLLARY 3. *Let* C *be a nonempty bounded closed convex subset of* L^p $(1 < p \le 2)$. *If* $T\colon C \to C$ *is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4(p-1)\cdot 2^{(p-1)/p}}\right)\right]^{1/2},$$

then T *has a fixed point in* C. □

COROLLARY 4. *Let C be a nonempty bounded closed convex subset of L^p $(2 < p < +\infty)$. If $T\colon C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \left[\frac{1}{2}\left(1+\sqrt{1+8\cdot c_p}\right)\right]^{1/p},$$

then T has a fixed point in C. □

Remark 2. A simple calculation shows that this result is essentially more general than that given in [6] for L^p spaces, $2 < p < +\infty$.

5. The corollaries in other Banach spaces

Using the results of Prus, Smarzewski [12], [16] and Xu [19] we can obtain from Theorem 2 the fixed point theorem for asymptotically regular mapping for Hardy and Sobolev spaces.

Let H^p, $1 < p < +\infty$, denote the Hardy space of all functions x analytic in the unit disc $|z| < 1$ of the complex plane, such that

$$\|x\| = \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|x\left(r\,\mathrm{e}^{\mathrm{i}\Theta}\right)\right|^p \,\mathrm{d}\Theta\right)^{1/p} < +\infty .$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{r,p}(\Omega)$, $r \geq 0$, $1 < p < +\infty$, the Sobolev space of distributions x such that $\mathrm{D}^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha|\leq k} \int_\Omega |\,\mathrm{D}^\alpha x(\omega)|^p \,\mathrm{d}\omega\right)^{1/p} .$$

Let $(\Omega_\alpha, \Sigma_a, \mu_\alpha)$, $\alpha \in \Lambda$ be a sequence of positive measure spaces, where the index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < +\infty$ and $q = \max(2,p)$, the linear space of all sequences

$$x = \{x_\alpha \in X_\alpha :\ \alpha\Lambda\}$$

equipped with the norm

$$\|x\| = \left[\sum_{\alpha\in\Lambda} \left(\|x_\alpha\|_{p,\alpha}\right)^q\right]^{1/q},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \Sigma_1, \mu_1)$ and $L_q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < +\infty$, $q = \max(2,p)$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space of all measurable L_p-value functions x on S_2 such that

$$\|x\| = \left(\int_{S_2} \left(\|x(s)\|_p\right)^q \, \mu_2(\mathrm{d}s) \right)^{1/q} .$$

These spaces are q-uniformly convex with $q = \max(2,p)$, [12], [16] and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - d \cdot W_q(\lambda) \cdot \|x-y\|^q$$

with a constant given by

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{if } 1 < p \le 2, \\ \dfrac{1}{p \cdot 2^p} & \text{if } 2 < p < +\infty. \end{cases}$$

Hence, from Theorem 2, we have the following:

COROLLARY 5. *Let C be a nonempty bounded closed convex subset of the space X, where $X = H^p$, or $X = H^{r,p}(\Omega)$, or $X = L_{q,p}$, or $X = L_q(L_p)$, and $1 < p < +\infty$, $q = \max(2,p)$, $r \ge 0$. If $T\colon C \to C$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} \|T^n\| < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot d \cdot N^q}\right)\right]^{1/q},$$

where $q = \max(2,p)$, then T has a fixed point in C. □

Problem. It is not known whether the estimate of the expression "$\liminf_{n\to\infty} \|T^n\|$" is sharp.

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Received April 15, 1991

Revised July 6, 1992

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