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A COMPLETE METRIC ON THE SPACE OF INTEGRABLE MULTIFUNCTIONS

DUŠAN HOLÝ

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ABSTRACT. The notion of a multivalued integral was introduced by Aumann and the notion of an integrable multifunction (which we use) by Hiai. We find a complete metric on the space of integrable multifunctions with values in a Banach ·rarable space.

1. Introduction

ıotion of an integral for a multivalued function was introduced by ınn. The convergence theorems for multivalued integrals were discussed Aumann [A], Schmeidler [S], and Arstein [Ar]. These authors ained Fatou's lemma and Lebesgue's convergence theorem with the Kura- owski convergence for measurable multivalued functions having values in the losed subsets of $\mathbb{R}^n$. Fatou's lemma is of some use in mathematical economics S]).

iai [Hi] studies integrable multivalued functions with values in a Banach rable space. He proved Fatou's lemmas and Lebesgue's convergence theo- ıs for multivalued integrals mainly with the Mosco convergence but in the exive spaces.

We find a complete metric on the space of integrable multifunctions with alues in a Banach separable space, which can be a useful tool in integration ›ry.

2. Definitions and some elementary properties

Throughout the paper, Ω will denote a measurable space with σ-algebra $\mathcal{A}$. If there is a σ-finite measure defined on $\mathcal{A}$, we say that Ω is σ-finite. If there is a complete σ-finite measure defined on $\mathcal{A}$, we call Ω complete.

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Y will be a topological space, 2^Y, the space of all subsets of Y. Following Bourbaki, we will call Y: *Polish*, if Y is separable and metrizable by a complete metric, *Souslin*, if Y is metrizable and a continuous image of a Polish space.

A relation $F\colon \Omega \to Y$ is a subset of $\Omega \times Y$. Alternatively, F may be regarded as a function from Ω to 2^Y. A function $F\colon \Omega \to 2^Y - \{\emptyset\}$ is called a *multifunction.*

Let $F\colon \Omega \to Y$ be a relation and $B \subset Y$. Denote

$$F^{-1}(B) = \{\omega \in \Omega : \ F(\omega) \cap B \neq \emptyset\} .$$

A relation $F\colon \Omega \to Y$ is *measurable* (*weakly measurable*) if and only if $F^{-1}(B)$ is measurable for each closed (open) subset B of Y. We say that F is *graph measurable* if

$$\operatorname{Gr} F = \{(\omega, y) \in \Omega \times Y : \ y \in F(\omega)\} \in \mathcal{A} \times \mathcal{B} ,$$

where $\mathcal{B}$ is the σ-algebra of Borel subsets of Y, and $\mathcal{A} \times \mathcal{B}$ is understood in the usual sense.

Further we mention some properties from the papers [H], [W]:

We say that $\{f_n\}_{n\in\mathbb{Z}^+}$ is a *Castaing representation* of F if, for all $n \in \mathbb{Z}^+$, f_n is a measurable selector of F, and for all $\omega \in \Omega$

$$F(\omega) \subset \operatorname{cl}\Big\{ \bigcup_{n\geq 1} \{f_n(\omega)\} \Big\} .$$

From [W; Theorem 5.10], we know that if $(\Omega, \mathcal{A})$ is a measurable space with $\mathcal{A}$ a Souslin family, Y is a Souslin space and F is a graph measurable multifunction, then F admits a Castaing representation. Notice that $\mathcal{A}$ is a Souslin family ([KN]) if $\mathcal{A} = S(\mathcal{A})$, where $S(\mathcal{A})$ denotes the family of all sets obtained from $\mathcal{A}$ by the Souslin operation. In case that there is a σ-finite complete measure defined on the σ-algebra $\mathcal{A}$, $\mathcal{A}$ is a Souslin family ([KN]).

Further we will need the following proposition:

PROPOSITION A. ([H]) *Let J be an at most countable set, and let $F_n\colon \Omega \to Y$ be a relation for each $n \in J$. Then if each F_n is measurable (weakly measurable), so is the relation $\bigcup F_n\colon \Omega \to Y$ defined by $\Big(\bigcup\limits_n F_n\Big)(\omega) = \bigcup\limits_n F_n(\omega)$.*

PROPOSITION B. ([H]) *A relation $F\colon \Omega \to Y$ is weakly measurable if and only if the relation $\operatorname{cl} F\colon \Omega \to Y$, defined by $\operatorname{cl} F(\omega) = \operatorname{cl}\{F(\omega)\}$, is weakly measurable.*

Let $F\colon \Omega \to Y$ be a relation and $B \subset Y$. Besides the notion $F^{-1}(B)$, we need also the notion of $F^{+}(B) = \{\omega \in \Omega ; \ F(\omega) \subset B\}$

3. Main results

DEFINITION 3.1. ([HU]) Let $(\Omega, \mathcal{A})$ be complete. Let Y be a Banach separable space. Let $F\colon \Omega \to Y$ be a multifunction with a measurable graph, such that there is an integrable function $f\colon \Omega \to \mathbb{R}$ with the following property

$$\forall \omega \in \Omega \quad \|F(\omega)\| \le f(\omega)\,,$$

(i.e. $\|y\| \le f(\omega)$ for all $y \in F(\omega)$, where $\|y\|$ is a norm of y).

Then we call F an *integrable multifunction.*

Remark 3.2. The assumptions of Definition 3.1 guarantee the existence of a Castaing representation of F.

DEFINITION 3.3. Let Ω and Y be as in Definition 3.1. Denote by $\mathcal{L}$ the space of all integrable multifunctions from Ω to Y. Define the function $L\colon \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ as follows:

$$\begin{aligned} L(F,G) = \inf\Big\{\varepsilon :\ & \text{for every measurable selector } f \text{ of } F \\ & \text{there exists a measurable selector } g \text{ of } G \text{ such that} \\ & \int_\Omega |f(\omega) - g(\omega)|\,\mathrm{d}\mu \le \varepsilon \quad \text{and} \\ & \text{for every measurable selector } g \text{ of } G \\ & \text{there exists a measurable selector } f \text{ of } F \text{ such that} \\ & \int_\Omega |g(\omega) - f(\omega)|\,\mathrm{d}\mu \le \varepsilon \Big\}. \end{aligned}$$

This definition is a generalization of the definition introduced in [M].

What is a motivation for this definition? We show that a motivation for this definition is the Hausdorff metric. Since we will work with this notion further, we briefly mention some properties of this metric.

Let (W,p) be a metric space. Denote $B_\varepsilon[v] = \{z \in W :\ p(z,v) < \varepsilon\}$. If K is a subset of W and $\varepsilon > 0$, let $B_\varepsilon[K]$ denote the union of all open ε-balls whose centers run over K. If K_1 and K_2 are nonempty subsets of W and, for some $\varepsilon > 0$, both $B_\varepsilon[K_1] \supset K_2$ and $B_\varepsilon[K_2] \supset K_1$, we define the Hausdorff distance h_p between them to be

$$h_p(K_1,K_2) = \inf\{\varepsilon :\ B_\varepsilon[K_1] \supset K_2 \text{ and } B_\varepsilon[K_2] \supset K_1\}\,.$$

Otherwise, we write $h_p(K_1,K_2) = \infty$. It is easy to check that h_p defines an infinite-valued pseudometric on the nonempty subset of W, and that $h_p(K_1,K_2) = 0$ if and only if K_1 and K_2 have the same closure. Thus, if we restrict h_p to closed subsets of W, then h_p defines an infinite valued metric on such sets.

In the sequel, we shall denote the set of closed nonempty subsets of a metric space W by $\mathrm{CL}(W)$. If (W,p) is complete, then so is $\big(\mathrm{CL}(W),h_p\big)$.

If (W,p) is a pseudometric space, we can also define the function h_p on all nonempty subsets of W. Clearly h_p is also a pseudometric.

In what follows, let Y be a separable Banach space with norm $\|\cdot\|$. To simplify notation, we shall sometimes denote the norm on Y by $|\cdot|$, rather than $\|\cdot\|$.

Put further $\varrho(x,y)=\|x-y\|$, $\varrho(x,A)=\inf\{\varrho(x,a):\ a\in A\}$, and $\varrho(A,x)=\inf\{\varrho(a,x):\ a\in A\}$ for a nonempty subset A of Y. Further denote by $h_{|\cdot|}$ the Hausdorff metric on $\mathrm{CL}(Y)$ induced by ϱ.

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of Y, and $(\Omega,\mathcal{A})$ be a measurable space. A function $f\colon \Omega\to Y$ is *measurable* if it is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$.

It is easy to see that if f is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$, then $\omega\to|f(\omega)|$ is $\mathcal{A}$-measurable.

In our paper, we need the notion of an integrable function. Let $(\Omega,\mathcal{A},\mu)$ be a measurable space, and let Y be a Banach separable space. A function $f\colon\Omega\to Y$ is integrable if it is measurable and the function $\omega\to|f(\omega)|$ is integrable.

Let $\mathcal{I}(\Omega,\mathcal{A},\mu,Y)$ be the set of all integrable functions from Ω to Y. Then $\mathcal{I}(\Omega,\mathcal{A},\mu,Y)$ is a vector space. The formula

$$\|f\|=\int_{\Omega}|f(\omega)|\,\mathrm{d}\mu$$

induces a seminorm on $\mathcal{I}(\Omega,\mathcal{A},\mu,Y)$, and clearly

$$d(f,g)=\int_{\Omega}|f(\omega)-g(\omega)|\,\mathrm{d}\mu$$

induced a pseudometric on $\mathcal{I}(\Omega,\mathcal{A},\mu,Y)$.

Let $(\Omega,\mathcal{A},\mu)$ be a complete space, and let $(Y,\mathcal{B})$ be a Banach separable space. Let $F\colon\Omega\to Y$ be an integrable multifunction. Put

$$S_F=\big\{f\in\mathcal{I}(\Omega,\mathcal{A},\mu,Y):\ f(\omega)\in F(\omega)\ \text{almost everywhere}\big\}.$$

Then $S_F\neq\emptyset$, and S_F is a closed set in $\big(\mathcal{I}(\Omega,\mathcal{A},\mu,Y),d\big)$ for every multifunction F with closed values.

We can identify F with S_F. Let F, G be two integrable multifunction. It is easy to verify that

$$L(F,G)=h_d(S_F,G_F).$$

If $F\colon\Omega\to Y$ is an integrable multifunction, then the integral or mean $E[F]$ of F is defined by

$$E[F]=\int_{\Omega}F(\omega)\,\mathrm{d}\mu=\Big\{E(f)=\int_{\Omega}f(\omega)\,\mathrm{d}\mu:\ f\in S_F\Big\},$$

where $E[f] = \int\limits_{\Omega} f(\omega)\, \mathrm{d}\mu$ is the usual Bochner integral. This multivalued integral was introduced by Aumann [A].

It is easy to verify that if F, G are two integrable multifunctions, then

$$h_{|\cdot|}\big(E[F], E[G]\big) \le L(F,G)\,.$$

THEOREM 3.4. *The function $L\colon \mathcal{L}\times\mathcal{L} \to \mathbb{R}$ defined in the Definition 3.3 is a pseudo-metric.*

Proof. The proof is similar as in [M].

THEOREM 3.5. *Let $(\Omega, \mathcal{A})$ be complete and let Y be a Banach separable space. Let F, G be integrable multifunctions from Ω to Y. Then $L(F,G) = 0$ if and only if $\mathrm{cl}\{F(\omega)\} = \mathrm{cl}\{G(\omega)\}$ almost everywhere.*

Proof.

$\Longrightarrow$: Denote by $\mathrm{CL}(Y)$ the space of all nonempty closed subsets of Y and $h_{|\cdot|}$ the Hausdorff metric on $\mathrm{CL}(Y)$. Let μ be a complete σ-finite measure on $\mathcal{A}$. We prove that

$$\Big\{\omega \in \Omega:\ h_{|\cdot|}\big(\mathrm{cl}\{F(\omega)\}, \mathrm{cl}\{G(\omega)\}\big) > 0\Big\}$$

is a measurable set with measure zero.

Let $\varepsilon > 0$. It is easy to verify that

$$\begin{aligned}
&\Big\{\omega \in \Omega : h_{|\cdot|}\big(\mathrm{cl}\{F(\omega)\}, \mathrm{cl}\{G(\omega)\}\big) > \varepsilon\Big\} \\
&\quad = \bigg(\bigcup_n \bigcup_k \Big(\mathrm{cl}\, F^{-1}\big(B_{\frac{1}{k}}[y_n]\big) \cap \mathrm{cl}\, G^{+}\big(Y \setminus B_{\varepsilon+\frac{1}{k}}[y_n]\big)\Big)\bigg) \\
&\qquad \cup \bigg(\bigcup_n \bigcup_k \Big(\mathrm{cl}\, G^{-1}\big(B_{\frac{1}{k}}[y_n]\big) \cap \mathrm{cl}\, F^{+}\big(Y \setminus B_{\varepsilon+\frac{1}{k}}[y_n]\big)\Big)\bigg)\,,
\end{aligned}$$

where $\{y_n:\ n \in \mathbb{Z}^+\}$ is a countable dense set in Y. Thus

$$\Big\{\omega \in \Omega:\ h_{|\cdot|}\big(\mathrm{cl}\{F(\omega)\}, \mathrm{cl}\{G(\omega)\}\big) > 0\Big\}$$

is measurable.

Now we show that $\mu\Big\{\omega \in \Omega:\ h_{|\cdot|}\big(\mathrm{cl}\{F(\omega)\}, \mathrm{cl}\{G(\omega)\}\big) > \varepsilon\Big\} = 0$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. Put

$$A_\varepsilon = \bigcup_n \bigcup_k \Big(\mathrm{cl}\, F^{-1}\big(B_{\frac{1}{k}}[y_n]\big) \cap \mathrm{cl}\, G^{+}\big(Y \setminus B_{\varepsilon+\frac{1}{k}}[y_n]\big)\Big)\,,$$

and

$$B_\varepsilon = \bigcup_n \bigcup_k \Big(\mathrm{cl}\, G^{-1}\big(B_{\frac{1}{k}}[y_n]\big) \cap \mathrm{cl}\, F^{+}\big(Y \setminus B_{\varepsilon+\frac{1}{k}}[y_n]\big)\Big)\,.$$

Suppose $\mu(A_\varepsilon \cup B_\varepsilon) > 0$. Then either $\mu(A_\varepsilon) > 0$ or $\mu(A_\varepsilon) > 0$. Without loss of generality we can suppose that $\mu(A_\varepsilon) > \delta$.

Define a function $f\colon \Omega \times Y \to \mathbb{R}$ by $f(\omega,y) = \varrho\big(\mathrm{cl}\{G(\omega)\}, y\big)$. The function f is measurable in ω for each $y \in Y$ ([H]) and continuous in y for every $\omega \in \Omega$. Thus f is measurable ([H]), i.e. the set $C = \big\{(\omega,y) :\ \varrho\big(\mathrm{cl}\{G(\omega)\}, y\big) \geq \varepsilon\big\}$ is measurable. Put further $D = C \cap \mathrm{Gr}\, F$. Then the set $P_\Omega(D)$ contains A_ε, where $P_\Omega(\omega,y) = \omega$ for every (ω,y).

Now define the following set $E \subset \Omega \times Y$:

$$E = \big\{(\omega,y) :\ (\omega,y) \in D \text{ and } \omega \in A_\varepsilon\big\} \cup \big\{(\omega,y) :\ (\omega,y) \in \mathrm{Gr}\, F \text{ and } \omega \notin A_\varepsilon\big\}.$$

Further define a multifunction $K\colon \Omega \to Y$ by

$$K(\omega) = E_\omega = \big\{y \in Y :\ (\omega,y) \in E\big\}.$$

Clearly the multifunction K has a measurable graph and $\mathrm{Gr}\, K \subset \mathrm{Gr}\, F$. The assumptions of the theorem guarantee the existence of a Castaing representation $\{k_n\}_{n\in\mathbb{Z}^+}$ of K.

Let k_n be a measurable selector of K from the Castaing representation of K, and let g be a measurable selector of a multifunction G. Then we have

$$\int\limits_\Omega |k_n(\omega) - g(\omega)|\ \mathrm{d}\mu = \int\limits_{\Omega\setminus A_\varepsilon} |k_n(\omega) - g(\omega)|\ \mathrm{d}\mu + \int\limits_{A_\varepsilon} |k_n(\omega) - g(\omega)|\ \mathrm{d}\mu > \delta \cdot \varepsilon,$$

and that is a contradiction.

$\Longleftarrow$: Let f be a selector of F. We show that for every $\varepsilon > 0$ there is a selector g of G such that $\int\limits_\Omega |f(\omega) - g(\omega)|\ \mathrm{d}\mu < \varepsilon$. The multifunctions F and G are integrable, and $\mathrm{cl}\, F = \mathrm{cl}\, G$ almost everywhere. Thus there is an integrable function $h\colon \Omega \to \mathbb{R}$ such that $\big\|\mathrm{cl}\{F(\omega)\}\big\| \leq h(\omega)$ and $\big\|\mathrm{cl}\{G(\omega)\}\big\| \leq h(\omega)$.

There is a measurable set A such that $\mu(A) < \infty$ and $\int\limits_{\Omega\setminus A} h(\omega)\ \mathrm{d}\mu < \frac{\varepsilon}{6}$.

Put

$$M = \left\{(\omega,y) :\ \varrho\big(f(\omega), y\big) = \frac{\varepsilon}{6\mu(A)}\right\}.$$

Then M is a measurable set. Put $N = M \cap \mathrm{Gr}\, G$ and define a multifunction $K\colon \Omega \to Y$ by

$$K(\omega) = N_\omega = \big\{y \in Y :\ (\omega,y) \in N\big\}.$$

There is a Castaing representation of K. Let g^* be a function from the Castaing representation of K. Then we have:

$$\begin{aligned}\int_{\Omega} |f(\omega) - g^*(\omega)| \, \mathrm{d}\mu &= \int_{\Omega\setminus A} |f(\omega) - g^*(\omega)| \, \mathrm{d}\mu + \int_{A} |f(\omega) - g^*(\omega)| \, \mathrm{d}\mu \\ &< \int_{\Omega\setminus A} |2h(\omega)| \, \mathrm{d}\mu + \int_{A} |f(\omega) - g^*(\omega)| \, \mathrm{d}\mu \\ &\le \frac{2\varepsilon}{6} + \frac{2\varepsilon\mu(A)}{6\mu(A)} < \varepsilon .\end{aligned}$$

□

On the space $\mathcal{L}$, define a relation $\approx$ by $F \approx G$ if and only if $\mathrm{cl}\{F(\omega)\} = \mathrm{cl}\{G(\omega)\}$ almost everywhere. Let $\mathcal{L}_1$ be a space of all integrable multifunctions with closed values; put $\mathcal{L}^{\sim} = \mathcal{L}_1/\approx$ and define

$$L^{\sim} \colon \mathcal{L}^{\sim} \times \mathcal{L}^{\sim} \to \mathbb{R} \qquad \text{by} \quad L^{\sim}(F^{\sim}, G^{\sim}) = L(F_1, G_1),$$

where $F_1, G_1 \in \mathcal{L}_1$ and $F_1 \in F^{\sim}$, $G_1 \in G^{\sim}$. The standard proof of [K] shows that $L^{\sim}$ is well defined and $L^{\sim}$ is a metric on $\mathcal{L}^{\sim}$.

THEOREM 3.6. *Let $(\Omega, \mathcal{A})$ be complete, and let Y be a Banach separable space. Then the space $(\mathcal{L}^{\sim}, L^{\sim})$, defined as above, is complete.*

Proof. Let $\{F_n\}_{n\in\mathbb{Z}^+}$ be a Cauchy sequence from $\mathcal{L}^{\sim}$. Without loss of generality we can suppose that for every $n \in \mathbb{Z}^+$ is

$$L^{\sim}(F_n^{\sim}, F_{n+1}^{\sim}) < \frac{1}{2^{n+1}} .$$

For every $n \in \mathbb{Z}^+$ choose $F_n \in F_n^{\sim}$. Clearly

$$L(F_n, F_{n+1}) < \frac{1}{2^{n+1}}$$

for every $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$ and let $\{f_{n,l}\}_{l\in\mathbb{Z}^+}$ be a Castaing representation of F_n. For every selector $f_{n,l}$ of F_n we choose a d-Cauchy sequence $\{f_{n,l,p}\}_{p\ge n}$ ($d(f,g) = \int |f-g| \, \mathrm{d}\mu$) in the following way:

Let $f_{n,l,p}$ be a selector of F_p such that

$$\int_{\Omega} |f_{n,l,p}(\omega) - f_{n,l,p+1}(\omega)| \, \mathrm{d}\mu < \frac{1}{2^p} .$$

For every sequence $\{f_{n,l,p}\}_{p\geq n}$ there is a measurable function $\overline{f}_{n,l}$ such tha $\{f_{n,l,p}\}_{p\geq n}$ d-converges to $\overline{f}_{n,l}$. Now define the multifunction F by

$$F(\omega) = \mathrm{cl}\Big\{\bigcup\{\,\overline{f}_{n,l}:\ n\in\mathbb{Z}^+\,,\ l\in\mathbb{Z}^+\}\Big\}.$$

The multifunction F has a measurable graph ([H]), and $\{\,\overline{f}_{n,l}\}_{n,l\in\mathbb{Z}^+}$ is . Castaing representation of F. Now we show that F is an integrable multifun tion. It is sufficient to prove that there is an integrable function h, $h\colon \Omega\to$. such that $|F(\omega)|\leq h(\omega)$ for every $\omega\in\Omega$.

Denote $P_K(\mathbb{R})$ the family of all compact subsets of $\mathbb{R}$. Define the family c multifunctions $\{G_n:\ n\in\mathbb{Z}^+\}$, $G_n\colon \Omega\to P_K(\mathbb{R})$ by

$$G_n(\omega) = \mathrm{cl}\Big\{\bigcup\{|f_{n,l}(\omega)|:\ l\in\mathbb{Z}^+\}\Big\}$$

for every $n\in\mathbb{Z}^+$. The multifunctions are measurable ([H]).

On the family of all multifunctions with real values and bounded by an in-tegrable function, we have, by Definition 3.3, defined a metric, which is in this real case denoted by $L_{\mathbb{R}}$.

Since

$$\int\limits_{\Omega} \big|\,|f(\omega)|-|g(\omega)|\,\big|\ \mathrm{d}\mu \leq \int\limits_{\Omega} |f(\omega)-g(\omega)|\ \mathrm{d}\mu\,,$$

we also have

$$L_{\mathbb{R}}(G_n,G_m)\leq L(F_n,F_m)\,.$$

Thus the sequence $\{G_n\}$ is $L_{\mathbb{R}}$-Cauchy, and from the proof of Theorem 6.15 [M], the assumptions of which are satisfied, it is possible to see that there is an integrable function $h\colon \Omega\to\mathbb{R}$ such that $\|G_n(\omega)\|\leq h(\omega)$ for each $n\in\mathbb{Z}^+$ and $\omega\in\Omega$.

Now we prove that $\{F_n\}$ L-converges to F. We show that for every $\varepsilon>0$ there is $N(\varepsilon)$ such that, for every $n>N(\varepsilon)$, $L(F_n,F)<\varepsilon$.

Let h be an integrable function from Ω to $\mathbb{R}$ such that, for every $n\in\mathbb{Z}^+$, $\|F_n(\omega)\|\leq h(\omega)$ and $\|F(\omega)\|\leq h(\omega)$ $\forall\omega\in\Omega$.

There is a measurable set A of finite measure such that

$$\int\limits_{\Omega-A} h(\omega)\ \mathrm{d}\mu<\frac{\varepsilon}{6}\,.$$

Let g be an arbitrary selector of F. Put

$$P(\omega)=\Big\{y\in Y:\ \varrho\big(y,g(\omega)\big)\leq\frac{\varepsilon}{3\mu(A)}\Big\}\,.$$

There is a selector $\overline{f}_{n_1,l_1}$ of a multifunction F from the above Castaing representation $\{\overline{f}_{n,l}\}_{n,l\in\mathbb{Z}^+}$ of F such that

$$\{\overline{f}_{n_1,l_1}(\omega)\}\cap P(\omega)\neq\emptyset$$

on a subset $A_1\subset A$ of nonzero measure. (This is very easy to see from the fact that $\{\overline{f}_{n,l}\}_{n,l\in\mathbb{Z}^+}$ is a Castaing representation of F and thus $F(\omega)\subset \mathrm{cl}\Big\{\bigcup\{\overline{f}_{n,l}(\omega):\ n,l\in\mathbb{Z}^+\}\Big\}$.)

Suppose $\mu(A\setminus A_1)>0$. Then by the same argument as above, there is a selector $\overline{f}_{n_2,l_2}$ from $\{\overline{f}_{n,l}\}_{n,l\in\mathbb{Z}^+}\setminus\{\overline{f}_{n_1,l_1}\}$ such that

$$\{\overline{f}_{n_2,l_2}(\omega)\}\cap P(\omega)\neq\emptyset$$

on a subset $A_2\subset A\setminus A_1$ of nonzero measure.

In this way, we obtain a sequence of disjoint subsets $\{A_n:\ n\in\mathbb{Z}^+\}$ of A such that

$$A=\bigcup\{A_n:\ n\in\mathbb{Z}^+\},$$

and a sequence $\{\overline{f}_{n_k,l_k}\}_{k\in\mathbb{Z}^+}$ of measurable selectors of F.

Since h is an integrable function, then from the absolute continuity of integral it follows, that for $\frac{\varepsilon}{6}$ there is $\delta>0$ such that for arbitrary measurable set B with $\mu(B)<\delta$ it holds

$$\int\limits_B 2h(\omega)\ \mathrm{d}\mu<\frac{\varepsilon}{6}.$$

Since $\mu(A)=\sum\limits_{k=1}^{\infty}\mu(A_k)<\infty$, then there is k_0 such that

$$\mu\left(\bigcup_{k=k_0}^{\infty}A_k\right)=\sum_{k=k_0}^{\infty}\mu(A_k)<\delta.$$

So

$$\int\limits_{\bigcup\limits_{k=k_0}^{\infty}A_k} 2h(\omega)\ \mathrm{d}\mu<\frac{\varepsilon}{6}.$$

For $k=1,\dots,k_0$, choose p_k such that

$$\int\limits_{\Omega}|\overline{f}_{n_k,l_k}(\omega)-f_{n_k,l_k,p}(\omega)|\ \mathrm{d}\mu<\frac{\varepsilon}{k_0 6}\qquad\text{for all}\quad p>p_k.$$

Let $M>\max\{p_1,\dots,p_{k_0}\}$. For $p>M$, produce a selector of the multifunction F_p as follows:

Let f_p be a measurable selector of F_p. Put

$$g_p(\omega) = \begin{cases} f_{n_k,l_k,p}(\omega) & \text{for } \omega \in A_k\,, \quad k = 1,2,\dots,k_0\,, \\ f_p(\omega) & \text{otherwise}\,. \end{cases}$$

Now we show that g_p is the needed selector of F_p.

$$\begin{aligned}
&\int_\Omega |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&= \int_A |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu + \int_{\Omega\setminus A} |g(\omega) - g(\omega)|\, \mathrm{d}\mu \\
&= \sum_{k=1}^{\infty} \int_{A_k} |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu + \int_{\Omega\setminus A} |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\le \sum_{k=1}^{\infty} \int_{A_k} |g(\omega) - \overline{f}_{n_k,l_k}(\omega)|\, \mathrm{d}\mu + \sum_{k=1}^{\infty} \int_{A_k} |\overline{f}_{n_k,l_k}(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\qquad + \int_{\Omega\setminus A} |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\le \sum_{k=1}^{\infty} \frac{\varepsilon\mu(A_k)}{3\mu(A)} + \sum_{k=1}^{k_0} \int_{A_k} |\overline{f}_{n_k,l_k}(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\qquad + \sum_{k=k_0}^{\infty} \int_{A_k} |\overline{f}_{n_k,l_k}(\omega) - g_p(\omega)|\, \mathrm{d}\mu + \int_{\Omega\setminus A} |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\le \frac{\varepsilon}{3} + \sum_{k=1}^{k_0} \int_{A_k} |\overline{f}_{n_k,l_k}(\omega) - f_{n_k,l_k,p}(\omega)|\, \mathrm{d}\mu \\
&\qquad + \sum_{k=k_0}^{\infty} \int_{A_k} |\overline{f}_{n_k,l_k}(\omega) - f_p(\omega)|\, \mathrm{d}\mu + \int_{\Omega\setminus A} |g(\omega) - g_p(\omega)|\, \mathrm{d}\mu \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \int_{\bigcup_{k=k_0}^{\infty} A_k} |\overline{f} n_k, l_k(\omega) - f_p(\omega)|\, \mathrm{d}\mu + \frac{\varepsilon}{3} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \int_{\bigcup_{k=k_0}^{\infty} A_k} 2h(\omega)\, \mathrm{d}\mu + \frac{\varepsilon}{3} \le \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

The proof of the opposite inclusion is similar. □

Let us remark (see the end of this paper) that the space $\mathcal{L}^{\sim}$ of integrable multifunctions from $\Omega \to Y$ was studied also by Hiai and Umegaki in [HU]. They consider other metric Δ on $\mathcal{L}^{\sim}$.

If A and B are two nonempty closed subsets of Y, put

$$\delta(A,B) = \max\Big\{ \sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\Big\},$$

the Hausdorff distance between A and B ([Be]), where d is the metric induced by the norm of Y.

Let $F_1, F_2 \in \mathcal{L}^{\sim}$. Taking two sequences $\{f_{1i}\}$ and $\{f_{2j}\}$ of measurable functions such that

$$\begin{aligned} F_1(\omega) &= \mathrm{cl}\big(\{f_{1i}(\omega) : \ i \in \mathbb{Z}^+\}\big) \qquad \text{and} \\ F_2(\omega) &= \mathrm{cl}\big(\{f_{2j}(\omega) : j \in \mathbb{Z}^+\}\big) \qquad \text{for all} \quad \omega \in \Omega, \end{aligned}$$

we have

$$\delta\big(F_1(\omega), F_2(\omega)\big) = \max\Big\{\sup_i \inf_j \|f_{1i}(\omega) - f_{2j}(\omega)\|,\ \sup_j \inf_i \|f_{1i}(\omega) - f_{2j}(\omega)\|\Big\},$$

so that the function $\omega \to \delta\big(F_1(\omega), F_2(\omega)\big)$ is measurable. Since

$$\delta\big(F_1(\omega), F_2(\omega)\big) \le \|F_1(\omega)\| + \|F_2(\omega)\|,$$

the function $\omega \to \delta\big(F_1(\omega), F_2(\omega)\big)$ is also integrable. Hiai and Umegaki define in [HU] the metric Δ on $\mathcal{L}^{\sim}$ as follows

$$\Delta(F_1, F_2) = \int_\Omega \delta\big(F_1(\omega), F_2(\omega)\big)\ \mathrm{d}\mu .$$

A natural question is to find relations between metrics L and Δ. First we introduce some auxiliary relations.

Let f be a measurable function from Ω to Y, and let σ be a measurable function from Ω to $[0,\infty]$. Then, by literature, there is a sequence of simple measurable functions $\{f_n\}_{n\in\mathbb{Z}^+}$ such that

$$\begin{aligned} f(\omega) &= \lim_n f_n(\omega) \qquad && \text{and} \\ \|f_n(\omega)\| &\le \|f(\omega)\|, \qquad && n = 1, 2, \dots, \ \text{ for each } \omega \in \Omega . \end{aligned}$$

Here, by a simple function, we mean a function with finitely many values.

Also there is a sequence of simple measurable functions

$$\{\sigma_n\}, \qquad \sigma_n : \Omega \to [0,\infty),$$

for every $n \in \mathbb{Z}^+$, such that

$$\sigma(\omega) = \lim_n \sigma_n(\omega) \qquad \text{for each} \quad \omega \in \Omega .$$

The function $f\sigma$ is also measurable, since

$$f(\omega)\sigma(\omega) = \lim_n f_n(\omega)\sigma_n(\omega)$$

and $f_n\sigma_n$ is a simple measurable function.

Further, let B be a unit ball in Y (i.e. $B = \{y \in Y : \ \|y\| \le 1\}$), and let $\{a_i\}$ be a countable dense set in B.

Put

$$g_i(\omega) = f(\omega) + a_i \qquad \text{for every} \quad \omega \in \Omega , \quad i = 1,2,\dots$$

Clearly

$$\|g_i(\omega) - f(\omega)\| \le 1 \qquad \text{for every} \quad \omega \in \Omega , \ \ i = 1,2,..,$$

and

$$\mathrm{cl}\big(\{g_i(\omega) : \ i \in \mathbb{Z}^+\}\big) = \{y : \ \|y - f(\omega)\| \le 1\} \qquad \text{for every} \quad \omega \in \Omega .$$

Define the multifunction $H \colon \Omega \to Y$ by

$$H(\omega) = \{y : \ \|y - f(\omega)\| \le \sigma(\omega)\} \qquad \text{for every} \quad \omega \in \Omega .$$

We show that H is a weakly measurable multifunction.

For every $i \in \mathbb{Z}^+$, let $h_i \colon \ \Omega \to Y$ be the following function:

$$h_i(\omega) = \big(g_i(\omega) - f(\omega)\big)\sigma(\omega) + f(\omega) \qquad \text{for every} \quad \omega \in \Omega .$$

Clearly, the function h_i is measurable for every $i \in \mathbb{Z}^+$. It is very easy to verify that $\|h_i(\omega) - f(\omega)\| \le \sigma(\omega)$ for every $\omega \in \Omega$ and every $i \in \mathbb{Z}^+$.

Now we show that

$$\mathrm{cl}\big(\{h_i(\omega) : \ i \in \mathbb{Z}^+\}\big) = H(\omega) \qquad \text{for every} \quad \omega \in \Omega .$$

If $\sigma(\omega) = 0$, then clearly $H(\omega) = \mathrm{cl}\big(\{h_i(\omega) : \ i \in \mathbb{Z}^+\}\big)$. Now let $\omega \in \Omega$ be such that $\sigma(\omega) \ne 0$. It is sufficient to prove that

$$H(\omega) \subset \mathrm{cl}\big(\{h_i(\omega) : \ i \in \mathbb{Z}^+\}\big) .$$

Let $y \in H(\omega)$ and $\varepsilon > 0$. We show that for the set

$$O_y = \{z \in Y : \ \|y - z\| < \varepsilon\}$$

the following relation holds:

$$O_y \cap \big(\{h_i(\omega) : \ i \in \mathbb{Z}^+\}\big) \ne \emptyset .$$

Clearly, we can write y as $f(\omega)+c$, where c is an element from Y with $\|c\| \leq \sigma(\omega)$. Further, put

$$y_1 = \frac{y}{\sigma(\omega)} + f(\omega) - \frac{f(\omega)}{\sigma(\omega)}.$$

Then we have

$$\|y_1 - f(\omega)\| = \left\| \frac{y}{\sigma(\omega)} + f(\omega) - \frac{f(\omega)}{\sigma(\omega)} - f(\omega) \right\| = \frac{1}{\sigma(\omega)} \|y - f(\omega)\| \leq 1.$$

Put

$$O_{y_1} = \left\{ z \in Y : \ \|z - y_1\| < \frac{\varepsilon}{\sigma(\omega)} \right\}.$$

There is $i \in \mathbb{Z}^+$ such that $g_i(\omega) \in O_{y_1}$. We show that $\|h_i(\omega) - y\| < \varepsilon$.

$$\begin{aligned} &\big\| \big(g_i(\omega) - f(\omega)\big)\sigma(\omega) + f(\omega) - \big(\big(y_1 - f(\omega)\big)\sigma(\omega) + f(\omega)\big) \big\| \\ &\qquad = \|g_i(\omega)\sigma(\omega) - y_1\sigma(\omega)\| = \|g_i(\omega) - y_1\|\sigma(\omega) < \varepsilon. \end{aligned}$$

The multifunction $H\colon \Omega \to Y$ is weakly measurable because the multifunction $P\colon \Omega \to Y$ defined by $P(\omega) = \big\{h_i(\omega) : \ i \in \mathbb{Z}^+\big\}$ is weakly measurable ([H]).

The following example shows that there are two multifunctions F and G, for which $L^{\sim}(F,G) < \Delta(F,G)$.

Example. Let $\Omega = Y = \mathbb{R}$ with the usual metric. Put

$$\begin{aligned} F(\omega) &= 0 && \text{if} \quad \omega \in (-\infty,-1) \cup (0,\infty), \\ F(\omega) &= \{1,2\} && \text{if} \quad \omega \in \langle -1,0 \rangle, && \text{and} \\ G(\omega) &= 0 && \text{if} \quad \omega \in (-\infty,0) \cup (1,\infty), \\ G(\omega) &= \{0,-2\} && \text{if} \quad \omega \in \langle 0,1 \rangle. \end{aligned}$$

It is very easy to verify that $\Delta(F,G) = 4$ and $L^{\sim}(F,G) = 3$.

PROPOSITION 3.7. *$L^{\sim}(F,G) \leq \Delta(F,G)$ for all multifunctions $F,G\colon \Omega \to Y$.*

Proof. Suppose that there are multifunctions F, G for which

$$L^{\sim}(F,G) > \Delta(F,G), \qquad \text{where} \quad \Delta(F,G) = \int_{\Omega} \sigma(\omega)\, \mathrm{d}\mu = a,$$

and $\sigma(\omega)$ is the Hausdorff distance between $F(\omega)$ and $G(\omega)$.

Hence, one of the following possibilities is true:

1. There is f, a selector of the multifunction F such that

$$\int_{\Omega} \|g(\omega) - f(\omega)\| \, \mathrm{d}\mu > a$$

for every selector of the multifunction G.

2. There is g, a selector of the multifunction G such that

$$\int_{\Omega} \|g(\omega) - f(\omega)\| \, \mathrm{d}\mu > a$$

for every selector f of the multifunction F.

Suppose condition 1 is true. The multifunction

$$H(\omega) = \{y : \ \|f(\omega) - y\| \le \sigma(\omega)\}$$

is weakly measurable, as we proved above; so H has a measurable graph. Hence

$$H(\omega) \cap G(\omega) \ne \emptyset \qquad \text{for every} \quad \omega \in \Omega$$

because $\sigma(\omega)$ is the Hausdorff distance between the sets $F(\omega)$ and $G(\omega)$ and $f(\omega) \in F(\omega)$. Put

$$P(\omega) = H(\omega) \cap G(\omega) \qquad \text{for every} \quad \omega \in \Omega .$$

Then P is a graph measurable multifunction. There is a selector p of the multifunction P for which

$$\int_{\Omega} \|f(\omega) - p(\omega)\| \, \mathrm{d}\mu \le \int_{\Omega} \sigma(\omega) \, \mathrm{d}\mu = a$$

because p is a selector of the multifunction H. But that is a contradiction because p is also a selector of the multifunction G. □

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