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ON THE α-COMPLETENESS OF PSEUDO MV-ALGEBRAS

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ABSTRACT. Let α be an infinite cardinal. In this paper we prove that if $\mathcal{A}$ is a pseudo MV-algebra such that the corresponding lattice $\ell(\mathcal{A})$ is α-complete, then $\mathcal{A}$ is an MV-algebra. The collection of all α-complete pseudo MV-algebras is a radical class.

1. Introduction

The investigation of pseudo MV-algebras was begun in [5], [6], [10] (in [10], the term "generalized MV-algebra" has been applied).

According to the result of [4], each pseudo MV-algebra $\mathcal{A}$ can be constructed from a lattice ordered group G with a strong unit u; in this situation we write $\mathcal{A} = \Gamma(G, u)$. Then the underlying set A of $\mathcal{A}$ is the interval $[0, u]$ of G. The pseudo MV-algebra $\mathcal{A}$ is an MV-algebra if the lattice ordered group G is abelian.

The mentioned result from [4] is a generalization of the well-known theorem (cf. [9], [1]) concerning the relation between MV-algebras and abelian lattice ordered groups.

The partial order $\leqq$ on G induces a partial order on A; we obtain a distributive lattice $(A; \leqq)$ which will be denoted by $\ell(\mathcal{A})$.

Let α be an infinite cardinal and let $\mathcal{A}$, G be as above. We say that $\mathcal{A}$ is *α-complete* if the lattice $\ell(\mathcal{A})$ is α-complete.

We prove that $\mathcal{A}$ is α-complete if and only if G is conditionally α-complete. It is well-known that each σ-complete lattice ordered group is abelian. We infer that if $\mathcal{A}$ is α-complete, then it is an MV-algebra. Hence we obtain a generalization of [3; Theorem 3.3].

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Radical classes of MV-algebras have been investigated in [8]. We prove that the collection of all α-complete MV-algebras is a radical class.

2. Preliminaries

We recall the definition of a pseudo MV-algebra.

Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be an algebra of type $(2,1,1,0,0)$. For $x, y \in A$ we put

$$x \odot y = \sim(\neg x \oplus \neg y)\,.$$

$\mathcal{A}$ is called a *pseudo MV-algebra* if the axioms (A1)–(A8) from [3] are satisfied. (Cf. also the references given in Section 1 above.)

Let G be a lattice ordered group with a strong unit u. Put $A = [0, u]$; for each $x, y \in A$ we set

$$x \oplus y = (x + y) \wedge u\,, \quad \neg x = u - x\,, \quad \sim x = -x + u\,, \quad 1 = u\,.$$

Denote $\Gamma(G, u) = (A; \oplus, \neg, \sim, 0, u)$. Then $\Gamma(G, u)$ is a pseudo MV-algebra; moreover, according to [4], for each pseudo MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$. The meaning of $\ell(\mathcal{A})$ has been defined in Section 1 above.

An element $a \in A$ is an *atom* of $\mathcal{A}$ if $a > 0$ and the interval $[0, a]$ of $\ell(\mathcal{A})$ is a two-element set. $\mathcal{A}$ is *atomic* if for each $0 \neq x \in A$ there exists an atom a of $\mathcal{A}$ such that $a \leqq x$.

Let α be an infinite cardinal and let L be a lattice. If for each nonempty (bounded) subset X of L with $\operatorname{card} X \leqq \alpha$ there exist $\sup X$ and $\inf X$ in L, then L is said to be (*conditionally*) *α-complete*. In the case $\alpha = \aleph_0$ we speak about *σ-completeness* or *conditional σ-completeness*.

The lattice L is (*conditionally*) *complete* if it is (conditionally) α-complete for each infinite cardinal α. Recall that in the literature on lattice ordered groups a somewhat modified terminology is applied. Namely, a lattice ordered group is called complete if it is (in our terminology) conditionally complete.

3. Conditional α-completeness of a lattice ordered group; radical classes

Again, let α be an infinite cardinal; let G be a lattice ordered group.

LEMMA 3.1. *Let $a, b, c \in G$, $a < b < c$. Assume that both the intervals $[a, b]$ and $[b, c]$ are α-complete. Then the interval $[a, c]$ is α-complete as well.*

Proof. Let $X = \{x_i\}_{i\in I}$ be a nonempty subset of $[a,c]$ such that $\operatorname{card} I \leqq \alpha$. For each $i \in I$ we put

$$x_i^1 = x_i \wedge b\,, \qquad x_i^2 = x_i \vee b\,.$$

Then we have

$$-x_i^1 + x_i = -b + x_i^2\,.$$

Since the lattice $[a,b]$ is α-complete, there exists the element

$$x^1 = \bigvee_{i\in I} x_i^1$$

in $[a,b]$. Analogously, there exists the element

$$x^2 = \bigvee_{i\in I} x_i^2$$

in $[b,c]$. For each $i \in I$ the relation

$$x_i = x_i^1 + (-x_i^1 + x_i) = x_i^1 + (-b + x_i^2)$$

is valid. We denote

$$x = x^1 + (-b + x^2)\,.$$

Thus we have $x \geqq x_i$ for each $i \in I$. Hence $x \geqq a$. Also, since $x^1 \leqq b$ and $x^2 \leqq c$, we get $x \leqq b + (-b + c) = c$; thus $x \in [a,b]$.

Assume that z is an element of G such that $z \geqq x_i$ for each $i \in I$. Put $y = z \wedge c$. Then, clearly, $y \in [a,c]$ and $y \geqq x_i$ for each $i \in I$. We set

$$y^1 = y \wedge b\,, \qquad y^2 = y \vee b\,.$$

We have

$$y^1 \geqq x_i^1\,, \quad y^2 \geqq x_i^2 \qquad \text{for each} \quad i \in I\,.$$

Further, similarly as for the element x, we obtain

$$y = y^1 + (-b + y^2)\,.$$

Therefore in view of the definition of x^1 and x^2 we get $y \geqq x$. Thus $z \geqq x$. This yields that the set X possesses the supremum in G and that this supremum belongs to the interval $[a,c]$. Analogously we obtain the dual result concerning the infimum of the set X. Hence the lattice $[a,c]$ is α-complete. □

PROPOSITION 3.2. *Let G be a lattice ordered group with a strong unit u. Let α be an infinite cardinal. Suppose that the interval $[0,u]$ is α-complete. Then G is conditionally α-complete.*

Proof. For $u = 0$, the assertion is trivial. Assume that $u \neq 0$ and that $x, v \in G$, $x \leqq v$. We have to verify that the interval $[x,v]$ is α-complete.

Since u is a strong unit of G, there exist integers n_1, n_2 such that $n_1 < n_2$ and

$$[x, v] \subseteq [n_1 u, n_2 u].$$

Put $m = n_2 - n_1$. Then the interval $[n_1 u, n_2 u]$ is isomorphic to the interval $[0, mu]$.

By applying 3.1 and by using the obvious induction we obtain that the interval $[0, mu]$ is α-complete. Therefore the interval $[x, v]$ is α-complete as well. □

COROLLARY 3.3. *Let α be an infinite cardinal. Let $\mathcal{A}$ be a pseudo MV-algebra, $\mathcal{A} = \Gamma(G, u)$. Then the following conditions are equivalent:*

(i) *G is conditionally α-complete;*

(ii) *$\ell(\mathcal{A})$ is α-complete.*

Proof. The implication (i) $\implies$ (ii) is obvious. The converse implication is a consequence of 3.2. □

Since each conditionally σ-complete lattice ordered group is abelian, we obtain:

COROLLARY 3.4. *Let $\mathcal{A}$ be a pseudo MV-algebra. If the lattice $\ell(\mathcal{A})$ is σ-complete, then $\mathcal{A}$ is an MV-algebra.*

COROLLARY 3.5. ([3; Theorem 3.3]) *Let $\mathcal{A}$ be a pseudo MV-algebra. Assume that $\mathcal{A}$ is atomic and that the lattice $\ell(\mathcal{A})$ is complete. Then $\mathcal{A}$ is an MV-algebra.*

The notion of radical class of MV-algebras has been defined and investigated in [8]; it has been shown that there exists a one-to-one correspondence between radical classes of MV-algebras and radical classes of abelian lattice ordered groups (these have been dealt with in several papers; cf. e.g., [7], [2]).

We apply the terminology and notation from [8]. We recall the definition of the radical class of MV-algebras.

DEFINITION 3.6. A nonempty class Y of MV-algebras which is closed with respect to isomorphisms is called a *radical class* if the following conditions are satisfied:

1) Whenever $\mathcal{A}_1 \in Y$ and $\mathcal{A}_2$ is a substructure of $\mathcal{A}_1$, then $\mathcal{A}_2 \in Y$.

2) If $\mathcal{B}$ is an MV-algebra and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are substructures of $\mathcal{B}$ such that $\mathcal{A}_i \in Y$ for $i = 1, 2, \dots, n$, then $\bigvee\limits_{i=1}^{n} \mathcal{A}_i$ belongs to Y.

Let α be an infinite cardinal. We denote by $\mathcal{C}_\alpha$ the class of all MV-algebras which are α-complete.

PROPOSITION 3.7. *For each infinite cardinal α, $\mathcal{C}_\alpha$ is a radical class of MV-algebras.*

P r o o f . It is obvious that $\mathcal{C}_\alpha$ is closed with respect to isomorphisms. There exists an MV-algebra $\mathcal{A}$ which is complete; thus $\mathcal{A} \in \mathcal{C}_\alpha$, whence $\mathcal{C}_\alpha \neq \emptyset$.

Let $\mathcal{A}_1 \in \mathcal{C}_\alpha$ and let $\mathcal{A}_2$ be a substructure of $\mathcal{A}_1$. Then the lattice $\ell(\mathcal{A}_2)$ is an interval of the lattice $\ell(\mathcal{A}_1)$. Therefore $\ell(\mathcal{A}_2)$ is α-complete and thus $\mathcal{A}_2 \in \mathcal{C}_\alpha$.

Let $\mathcal{B}$ be an MV-algebra and suppose that $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are substructures of $\mathcal{B}$ such that all $\mathcal{A}_i$ $(i = 1, 2, \dots, n)$ belong to $\mathcal{C}_\alpha$. Let B be the underlying set of $\mathcal{B}$. For each $i \in \{1, 2, \dots, n\}$ there exists $b_i \in B$ such that the underlying set of $\mathcal{A}_i$ is the interval $[0, b_i]$ of the lattice $\ell(\mathcal{B})$. Since $\mathcal{A}_i \in \mathcal{C}_\alpha$, the interval $[0, b_i]$ is α-complete.

There exists an abelian lattice ordered group G_1 with a strong unit u_1 such that $\mathcal{B} = \Gamma(G_1, u_1)$. By applying 3.1 and by using induction on n we obtain that the interval $[0, b_1 + b_2 + \dots + b_n]$ of G_1 is α-complete. Put $b = b_1 \vee b_2 \vee \dots \vee b_n$. Then $[0, b] \subseteqq [0, b_1 + b_2 + \dots + b_n]$, whence the interval $[0, b]$ of G_1 is α-complete as well.

Denote $\bigvee\limits_{i=1}^{n} \mathcal{A}_i = \mathcal{A}^0$. We have $b \in \mathcal{B}$ and in view of the definition of $\mathcal{A}^0$ we conclude that the interval $[0, b]$ of $\ell(\mathcal{B})$ is the underlying set of $\mathcal{A}^0$. Hence $\mathcal{A}^0$ is α-complete. $\square$

COROLLARY 3.7.1. *The collection of all complete MV-algebras is a radical class.*

Let α be as above. We denote by $\mathcal{C}_\alpha^-$ the class of all MV-algebras which are β-complete for each infinite cardinal β with $\beta < \alpha$.

By analogous argument as in 3.7 we obtain:

PROPOSITION 3.8. *For each infinite cardinal α, $\mathcal{C}_\alpha^-$ is a radical class of MV-algebras.*

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