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ON LEXICO EXTENSIONS OF LATTICE ORDERED GROUPS

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This note was inspired by Conrad's paper [1], namely by the main theorem of the paper, concerning the structure of lattice ordered groups G having the property that each orthogonal subset of G is finite. (Cf. Theorem (A) below.) While the 'if' part of this theorem is obvious, by proving the 'only if' part several non-trivial steps and auxiliary results were used (cf. also Conrad [3] and Fuchs [4]).

By searching for a more simple and self-contained version of the proof of the 'only if' part of this theorem it turned out that for obtaining a good basis enabling one to perform some rather simple induction steps we have at first to generalize the theorem under consideration.

The case of lattice ordered groups G such that each upper-bounded orthogonal subset of G is finite (cf. [2]) will not be investigated in the present note.

1. Preliminaries

For the terminology and denotations cf. Fuchs [4] and Conrad [3]. The group operation in a lattice ordered group will be written additively. Let us recall the following notions.

Let G be a lattice ordered group. A subset X of G will be said to be orthogonal if $0 \leqq x$ for each $x \in X$, and $x_1 \wedge x_2 = 0$ for each pair of distinct elements $x_1, x_2 \in X$. For $Y \subset G$ we put

$$Y^\delta = \{g \in G \colon |g| \wedge |y| = 0 \text{ for each } y \in Y\};$$

Y^δ is a polar of G. Instead of $g \in Y^\delta$ we also write $g \perp Y$.

An element $g \in G$ with $g > 0$ is called a non-unit if $\{g\}^\delta \neq \{0\}$.

Let $H \neq \{0\}$ be a convex l-subgroup of G and let $g \in G$. If $h < g$ is valid for each $h \in H$, then we say that g exceeds H. If from the relations $0 < g_1 \in G \backslash H$ it follows that g_1 exceeds H, then we write $G = \langle H \rangle$; in such a case G is said to be a lexico extension of H; if, moreover, $H \neq G$, then G is called a proper lexico extension (of H). If $G = \langle H \rangle$ and H is normal in G, then G is called a normal lexico extension of H.

The following theorem is the main result of [1]:

(A) *Let G be a lattice ordered group. Each orthogonal subset of G is finite if and only if G can be obtained from a finite number of linearly ordered groups by the operations of direct product and normal lexico extension.*

Since in the case of $G=\{0\}$ our investigations would be trivial, we shall always assume that $G\neq\{0\}$. Let us consider the following conditions for G:

(F_1) Each orthogonal subset of G is finite.

(F_2) There are elements $0<e_i\in G$ $(i=1, 2, \ldots, n)$ such that each interval $[0, e_i]$ is a chain and $\{e_1, e_2, \ldots, e_n\}$ is a maximal orthogonal set of non-zero elements of G.

(F_3) There are convex l-subgroups $A_1, \ldots, A_n$ of G with $A_i\neq\{0\}$ such that (i) $\{A_1\cup A_2\cup\ldots\cup A_n\}^\delta=\{0\}$, and (ii) $A_1, \ldots, A_n$ are mutually orthogonal and linearly ordered.

We have obviously $(F_1)\Leftrightarrow(F_2)$, and $(F_3)\Rightarrow(F_2)$. Let (F_2) be valid; since the convex l-subgroup of G generated by e_i is linearly ordered, we infer that $(F_2)\Rightarrow(F_3)$; thus (F_1), (F_2) and (F_3) are mutually equivalent.

Again, let H be a convex l-subgroup of G, $0<x\in G$. If the set $\{h\in H\colon h\leqq x\}$ has a greatest element h_0, then we denote $h_0=x(H)$; h_0 will be called the component of x in H. We shall say that H is an *el*-subgroup of G if for each positive element x of G not exceeding H the component $x(H)$ does exist.

If H, G are as above and $H=\langle K\rangle$, $K\neq H$, then H is an *el*-subgroup of G (cf. [2]). In fact, let $0<x\in G$ such that x does not exceed H. Then there is $y\in H\setminus K$ such that y is incomparable with x; it is easy to verify that $2y\wedge x=x(H)$. Also, each non-zero linearly ordered group is a proper lexico extension. If A is a convex l-subgroup of G and if A is linearly ordered, then $A^{\delta\delta}$ is linearly ordered as well. Hence and in view of the equivalence $(F_1)\Leftrightarrow(F_3)$ the following theorem is a generalization of the 'only if' part of (A):

(B) *Let $A_1, A_2, \ldots, A_n$ be pairwise orthogonal el-subgroups of a lattice ordered group G such that $\{A_1\cup A_2\cup\ldots\cup A_n\}^\delta=\{0\}$. Then* (i) *G can be constructed from the system $\{A_1, \ldots, A_n\}$ by means of the operations of direct product and lexico extension, and* (ii) *G can be constructed from the system $\{A_1^{\delta\delta}, \ldots, A_n^{\delta\delta}\}$ by means of the operations of direct product and normal lexico extension.*

2. Proof of theorem (B)

Without loss of generality we can assume that all A_i are non-zero. If $0<g\in G$ and if $g(A_i)$ exists, then $g-g(A_i)$ is orthogonal to A_i (in fact, if $0<a_i\in A_i$, $a_i\leqq g-g(A_i)$, then $g(A_i)<a_i+g(A_i)\in A_i$ and $a_i+g(A_i)\leqq g$, which is impossible).

Let $i\in\{1, 2, \ldots, n\}$, $B_i=A_i^{\delta\delta}$. If $0<y\in B_i$ and if y does not exceed A_i, then y is

orthogonal to each A_j with $j \neq i$, hence so is $y' = y - y(A_i)$, and, moreover, y' is orthogonal to A_i; therefore $y' = 0$, implying $y \in A_i$. We have shown that either $B_i = A_i$ or B_i is a proper lexico extension of A_i. In both cases B_i is an el-subgroup of G. The lattice ordered groups $B_1, \ldots, B_n$ are clearly mutually orthogonal. We have also verified that the assertion of the lemma is valid for $n = 1$. Assume this assertion to be valid for each $m < n$.

Let M be the set of all $0 < g \in G$ such that g exceeds at least one B_i. Let us distinguish two cases. At first suppose that M is empty. Let $0 < g \in G$. Then $g(B_i)$ does exist for each $i \in \{1, \ldots, n\}$ and we have $0 \leqq g' = g - \Sigma g(B_i)$, $g' \perp B_i$ for $i = 1, 2, \ldots, n$, hence $g' \perp A_i$ for $i = 1, 2, \ldots, n$ and therefore $g' = 0$, $g = \Sigma g(B_i)$. From this we infer that $G = B_1 \times B_2 \times \ldots \times B_n$ is valid.

Now assume that M is non-empty. For $g \in M$ we put $f_i(g) = 1$ if g exceeds A_i, and $f_i(g) = 0$ otherwise; next we put $f(g) = (f_1(g), \ldots, f_n(g))$, $F = \{f(g)\colon g \in M\}$. The set M is partially ordered coordinate-wise. Since F is finite, there exists $g \in M$ such that $f(g)$ is a minimal element in F. Without loss of generality we can assume that there is a positive integer $k \leqq n$ such that $f_i(g) = 1$ for $i \leqq k$ and $f_i(g) = 0$ for $k < i \leqq n$. We must have $k \geqq 2$; in fact, if $k = 1$, then for $g' = g - \Sigma g(B_i)$ $(i = 2, \ldots, n)$ we would have $g' \perp B_i$ for $i = 2, \ldots, n$, hence $g' \in B_1$ and therefore for each $g'' \in B_1$ with $g'' > g'$ we would have $g \not\geqq g''$, which is a contradiction. Put $C = (B_1 \cup B_2 \cup \ldots \cup B_k)^{\delta\delta}$.

Since $B_1, B_2, \ldots, B_k$ are mutually orthogonal, the convex l-subgroup of G generated by the set $B_1 \cup B_2 \cup \ldots \cup B_k$ is a direct product $B_1 \times B_2 \times \ldots \times B_k$. We have $g' \in C$; if $g' \in B_1 \times B_2 \times \ldots \times B_k$, then g does not exceed B_i for $i = 1, 2, \ldots, k$, which is impossible. Thus $g' \notin B_1 \times \ldots \times B_k$ and therefore $C \setminus (B_1 \times B_2 \times \ldots \times B_k) = \emptyset$.

Let $0 < x \in C \setminus (B_1 \times B_2 \times \ldots \times B_k)$. Hence $x(B_j) = 0$ for $j = k+1, \ldots, n$. If $x \notin M$, then $x(B_i)$ exists for each $i \in \{1, \ldots, n\}$, thus for $x' = x - \Sigma x(B_i)$ $(i = 1, \ldots, n)$ we have $x' \perp B_i$ for $i = 1, \ldots, n$ implying $x' = 0$ and so $x = \Sigma x(B_i)$ $(i = 1, 2, \ldots, k)$, hence $x \in B_1 \times \ldots \times B_k$, which is a contradiction. Thus $x \in M$. We want to verify that x exceeds $B_1 \times \ldots \times B_k$. By way of contradiction, assume that x does not exceed $B_1 \times \ldots \times B_k$. Then there is $i \in \{1, 2, \ldots, k\}$ such that x does not exceed B_i. Without loss of generality we can take $i = 1$. Thus $x(B_1)$ does exist; put $y = x - x(B_1)$. The relation $y \notin M$ would imply $x \notin M$; therefore $y \in M$. We have $y \in C$, hence $f_j(y) = 0$ for $j = k+1, \ldots, n$; moreover, $f_1(y) = 0$. Hence $f(y) < f(g)$, which is a contradiction with the minimality of $f(g)$. Therefore C is a proper lexico extension of $B_1 \times B_2 \times \ldots \times B_k$.

Moreover, C is a normal lexico extension of $B_1 \times \ldots \times B_n = G_1$. In fact, because $B_1 \times \ldots \times B_n$ is a non-trivial direct product, it is generated by non-units of C. From $C = \langle G_1 \rangle$ it follows that all non-units of C belong to G_1. Next, if c is a non-unit of C and $c_1 \in C$, then clearly $-c_1 + c + c_1$ is a non-unit of C. Hence G_1 is a normal subgroup of C.

To finish the proof it suffices to apply the induction assumption to the system

$$\{C, B_{k+1}, \ldots, B_n\}$$

which has less than n elements (notice that each element of this system is a polar of G).

Corollary. *Let* $A_1, A_2, \ldots, A_n$ *be convex* l*-subgroups of a lattice ordered group* G *such that* (i) *all* A_i *are proper lexico extensions;* (ii) *they are mutually orthogonal, and* (iii) $(\cup A_i)^\delta = \{0\}$. *Then* (a) G *can be constructed from the system* $A_1, \ldots, A_n$ *by the operations of direct product and lexico extension, and* (b) G *can be constructed from the system* $A_1^{\delta\delta}, \ldots, A_n^{\delta\delta}$ *by the operations of direct product and normal lexico extension.*

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О ЛЕКСИКО-РАСШИРЕНИЯХ РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

В этой заметке описана структура решеточно упорядоченной группы G при условии, что в G существуют выпуклые l-подгруппы $A_1, \ldots, A_n$ такие, что а) система $\{A_1, \ldots, A_n\}$ — ортогональная; б) каждое A_i является собственным лексикорасширением, и в) $(\cup A_i)^\delta = \{0\}$.