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EVERY AT MOST FOUR ELEMENT ALGEBRA HAS A MAL'CEV THEORY FOR PERMUTABILITY

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ABSTRACT. It is proven that every at most four element algebra A has permutable congruences if and only if there exists a ternary Mal'cev function compatible with all congruences on A.

An algebra A is *permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for every two congruences Θ, $\Phi \in \operatorname{Con} A$. A variety $\mathcal{V}$ is *permutable* if each $A \in \mathcal{V}$ has this property. A is called *distributive* if Con A is a distributive lattice. A is *arithmetic* if it is permutable and distributive. A variety $\mathcal{V}$ is *distributive* (*arithmetic*) if each $A \in \mathcal{V}$ has this properyty. A. I. Mal'cev [3] has shown that a variety $\mathcal{V}$ is permutable if and only if there exists a ternary polynomial $p(x, y, z)$ satisfying

$$(*) \qquad p(x, z, z) = x \text{ and } p(x, x, z) = z .$$

A. P. Pixley [4] proved that a variety $\mathcal{V}$ is arithmetic if and only if there exists a ternary polynomial $m(x, y, z)$ satisfying

$$(**) \qquad m(x, y, y) = m(y, y, x) = m(x, y, x) = x .$$

A. F. Pixley [5] has shown that the foregoing result can be "localized" also for a single algebra:

Proposition. *Let A be an algebra with finite Con A. A is arithmetic if and only if there exists a Pixley function on A compatible with all congruences of A.*

Note that by a *Pixley* (*Mal'cev*) *function* on A is meant a mapping of A^3 into A satisfying (*) (or (**), respectively). Moreover, I. Korec [2] extended this Proposition also for algebras with countable Con A.

H.-P. Gumm [1] proved that the analogous assertion for permutability does not hold, i.e. there exists an algebra A with permutable congruences for which no Mal'cev function compatible with all congruences on A exists. In this example, $A = S \times S$, where S is a five element loop. The aim of this paper is to show that for algebras with at most four elements the Mal'cev theory can be localized.

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Lemma. *Let A be a four element permutable algebra. If the lattice Con A is not distributive, then Con* $A = M_3$ (see Fig. 1) *and, for a suitable notation of elements* $\{0, a, b, c\}$ *of* A,

$$\Theta_1 = \omega \cup \{\langle a, b\rangle, \langle b, a\rangle, \langle c, 0\rangle, \langle 0, c\rangle\}$$
$$\Theta_2 = \omega \cup \{\langle b, c\rangle, \langle c, b\rangle, \langle a, 0\rangle, \langle 0, a\rangle\}$$
$$\Theta_3 = \omega \cup \{\langle a, c\rangle, \langle c, a\rangle, \langle b, 0\rangle, \langle 0, b\rangle\}.$$

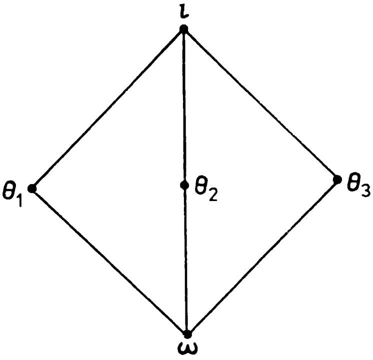

Fig. 1

Proof. Denote by $S = \{0, a, b, c\}$ the set of all elements of A. Since A has exactly four elements, it can have at most six principal congruences, namely

$$\theta(a, b), \qquad \theta(b, c), \qquad \theta(a, c), \qquad \theta(0, a), \qquad \theta(0, b), \qquad \theta(0, c).$$

Since *Con A* is not distributive, it must contain a sublattice isomorphic with M_3, thus *Con A* has at least a three element antichain. Henceforth, at least there of the foregoing six congruences must be pairwise different and nontrivial. Without loss of generality, suppose that

$$\theta(a, b) \neq \theta(b, c) \neq \theta(a, c) \neq \theta(a, b)$$

are non-trivial congruences. Denote by $\Theta_1 = \theta(a, b)$, $\Theta_2 = \theta(b, c)$, $\Theta_3 = \theta(a, c)$. Then

$$\langle a, c\rangle \in \Theta_1 \circ \Theta_2, \text{ which implies } \langle c, a\rangle \in \Theta_1 \circ \Theta_2$$

because of the permutability of congruences. Thus there must exist an element $x \in S$ with $\langle c, x\rangle \in \Theta_1$ and $\langle x, a\rangle \in \Theta_2$. Analogously, $\langle b, a\rangle \in \Theta_2 \circ \Theta_3$ implies the existence of $y \in S$ with $\langle b, y\rangle \in \Theta_3$, $\langle y, a\rangle \in \Theta_2$; and $\langle b, c\rangle \in \Theta_1 \;\; \Theta_3$ implies the existence of $z \in S$ with $\langle c, z\rangle \in \Theta_1$, $\langle z, b\rangle \in \Theta_3$.

(a) Suppose, e.g., $x = a$. Then clearly $\Theta_1 \supseteq \Theta_2$ and $\Theta_1 \supseteq \Theta_3$. If $y = b$, then we infer (by a similar argumentation) $\Theta_2 \supseteq \Theta_1$, thus $\Theta_1 = \Theta_2$ which is a contradiction. If $y = a$, we obtain $\Theta_3 \supseteq \Theta_1$ and $\Theta_3 \supseteq \Theta_2$, which give $\Theta_1 = \Theta_3$, a contradiction. If $y = c$, we obtain a contradiction from $\Theta_3 \supseteq \Theta_1$, $\Theta_2 \supseteq \Theta_1$. There remains $y = 0$, i.e.

$$\langle b, 0\rangle \in \theta(a, c),\ \langle 0, a\rangle \in \theta(b, c).$$

Since $\Theta_1 \supseteq \Theta_2$ and $\Theta_1 \supseteq \Theta_3$, also

$\langle a, 0\rangle, \langle b, 0\rangle, \langle 0, a\rangle, \langle 0, b\rangle \in \theta(a, b)$. The transitivity with $\langle c, a\rangle \in \theta(a, b)$ (for $x = a$) give also $\langle 0, c\rangle$, $\langle c, 0\rangle \in \theta(a, b)$, i.e. $\theta(a, b) = \iota$, which is a contradiction.

(b) If we suppose $x = b$ or $x = c$, we obtain a contradiction similarly as in the case (a). Hence, $x = 0$ is the only possibility. Then $\langle c, 0\rangle$, $\langle 0, c\rangle \in \theta(a, b)$.

Analogously we obtain $\langle b, 0\rangle$, $\langle 0, b\rangle \in \theta(a, c)$ and $\langle a, 0\rangle$, $\langle 0, a\rangle \in$ $\in \theta(b, c)$. Then

$$\theta(a, b) \supseteq \theta(0, c),\ \theta(b, c) \supseteq \theta(0, a),\ \theta(a, c) \supseteq (0, b).$$

(c) Since $\langle c, a\rangle \in (0, c) \circ \theta(0, a)$, we have also

$$\langle c, a\rangle \in \theta(0, a) \circ \theta(0, c),$$

i.e. there exists an element $v \in S$ with

$$\langle c, v\rangle \in \theta(0, a),\ \langle v, a\rangle \in \theta(0, c).$$

Analogously as in (a), we can proceed to prove the only possibility, namely $v = b$, whence

$$\theta(c, b) \subseteq \theta(0, a),\ \theta(a, b) \subseteq \theta(0, c).$$

Similarly, the identity $\theta(a, c) \subseteq \theta(0, b)$.

With respect to (b), $\theta(a, b)$, $\theta(b, c)$, $\theta(b, c)$, $\theta(a, c)$ are the only nontrivial congruences on A. The rest of the proof is evident.

Theorem. *Let A be an at most four element algebra. A is permutable if and only if there exists a Mal'cev function compatible with all congruences of A.*

Proof. If a such Mal'cev function in A exists, A is evidently permutable. Prove the converse implication. Suppose A is permutable.

(1) If A has the only element, the proof is trivial. If A has exactly two elements, then $Con\ A = \{\omega, \iota\}$, i.e. it is distributive. Hence, A is arithmetic and, by Pixley's result [5], there exists a Pixley function compatible with all congruences on A. However, every Pixley function is a Mal'cev function, thus the proposition holds.

(2) Let A have exactly three elements a, b, c. Suppose $Con\ A$ is not distributive. Then there exists only three nontrivial congruences, namely

$$\theta(a, b),\ \theta(b, c),\ \theta(a, c),$$

thus $Con\ A = M_3$. However, the permutability of congruences together with $Con\ A = M_3$ imply the direct decomposability of A which is impossible since card $A = 3$. Hence, A has permutable congruences if and only if A is arithmetic. Further argumentation is the same as in (1).

(3) Let A have exactly four elements. If $Con\ A$ is distributive, then A is arithmetic and the assertion is evident. Suppose A is not distributive. By the Lemma, $Con\ A = M_3$ (see FIg. 1) and for Θ_1, Θ_2, Θ_3 we have

$$\begin{aligned}\Theta_1 &= \omega \cup \{\langle a, b\rangle, \langle b, a\rangle, \langle c, 0\rangle, \langle 0, c\rangle\}\\ \Theta_2 &= \omega \cup \{\langle b, c\rangle, \langle c, b\rangle, \langle a, 0\rangle, \langle 0, a\rangle\}\\ \Theta_3 &= \omega \cup \{\langle a, c\rangle, \langle c, a\rangle, \langle b, 0\rangle, \langle 0, b\rangle\}.\end{aligned}$$

For $x \neq y \neq z \neq x$ we put $p(x, y, z) = v$, where $v \notin \{x, y, z\}$ and $\{x, y, z, v\} = \{0, a, b, c\}$, and, moreover,

$$p(x, z, z) = x,\ p(x, x, z) = z,\ p(x, y, x) = y.$$

It is easy to verify that $p(x, y, z)$ is compatible with Θ_1, Θ_2, Θ_3.

Remark 1. The Mal'cev function $p(x, y, z)$ constructed in the proof of the Theorem for a four element algebra is unique.

Remark 2. The operations on an algebra A can be:

(a) *trivial* (i.e. projections);

(b) *constant* (i.e. $f_a(x_1, \dots, x_n) = a$ for every $a_i \in A$)

(c) A can have, e.g., three unary operations:

$f_1(a) = c$	$f_2(a) = b$	$f_3(a) = 0$
$f_1(b) = 0$	$f_2(b) = a$	$f_3(b) = c$
$f_1(c) = a$	$f_2(c) = 0$	$f_3(c) = b$
$f_1(0) = b$	$f_2(0) = c$	$f_3(0) = a$.

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