Mathematica Slovaca

Peter Capek

Decomposition theorems in measure theory

Mathematica Slovaca, Vol. 31 (1981), No. 1, 53--69

Persistent URL: http://dml.cz/dmlcz/131066

Math. Slovaca 31, 1981, No. 1, 53—69

DECOMPOSITION THEOREMS IN MEASURE THEORY

PETER CAPEK

In [1] T. Neubrunn suggested to formulate and prove some theorems of measure theory in terms of "null sets" without using the concept of a measure.

In the present paper we study decomposition theorems from this point of view. It appears to be a unifying approach concerning various decompositions.

The paper consists of two chapters. In the first one decomposition theorems are formulated and proved abstractly while in the second their consequences are presented.

Throughout the paper $(X, \mathcal{S})$ will denote a measurable space with a σ-ring $\mathcal{S}$ of subsets of X. A subset A of X will be called locally measurable if $A \cap E \in \mathcal{S}$ for all $E \in \mathcal{S}$. The family of all locally measurable sets will be denoted by $\mathcal{S}_\lambda$. We have: $\mathcal{S} \subset \mathcal{S}_\lambda$; $\mathcal{S}_\lambda$ is a σ-algebra [3, p. 35].

Let $\mathscr{E}$ be a family of subsets of X. In what follows the symbol "$\mathscr{E}C$" is used in the Ficker's sense ([10]) and means that every family of pairwise disjoint elements from $\mathscr{E}$ is at most countable (therefore $\emptyset \notin \mathscr{E}$). If $A \subset X$, then we use the symbol $A \mid \mathscr{E}$ in the Hahn's sense ([12]) i.e. $A \mid \mathscr{E} = \{E \in \mathscr{E}: E \subset A\}$. The symbol $A^\perp$ stands for $X - A$, N denotes the set of positive integers and R, R_0 denote the sets of real and rational numbers, respectively.

Some parts in the first chapter (i.e. sections 1—4) can be read independently, to read them it is sufficient to know preliminaries. To read the sections in the second chapter (i.e. sections 5—9) it is necessary to know preliminaries to the second chapter and then the order of possible reading is as follows: [1, 5] (i.e. section 5 can be studied immediately after section 1) [2, 6], [3, 7], [4, 8, 9].

In June 1976 during my study stay in Brest at Prof. M$^{\text{me}}$ Godet-Thobie I have reported in a seminar most of the results appearing in the present paper. See [5], [6].

I. Abstract formulation of decomposition theorems

Preliminaries

First we list some properties of a family $\mathcal{M} \subset \mathcal{S}$ we shall work with:

(i) $\mathcal{M} \neq \emptyset$

(ii) $E \in \mathcal{M},\ F \in \mathcal{S}_\lambda \Rightarrow E \cap F \in \mathcal{M}$

(iii) $E_1,\ E_2 \in \mathcal{M} \Rightarrow E_1 \cup E_2 \in \mathcal{M}$

(iv) $E_k \in \mathcal{M},\ k \in N \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$

(v) $E_1, E_2 \in \mathcal{M},\ E_1 \cap E_2 = \emptyset \Rightarrow E_1 \cup E_2 \in \mathcal{M}$

(vi) $E_k \in \mathcal{M},\ k \in N$, where E_k are pairwise disjoint $\Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$

(vii) $\emptyset \in \mathcal{M}$

Definition 1. *A subfamily $\mathcal{M}$ of a σ-ring $\mathcal{S}$ is called:*

(j) *hereditary in $\mathcal{S}$ if it satisfies* (ii)

(jj) *an ideal if it satisfies* (i), (ii), (iii)

(jjj) *a σ-ideal if it satisfies* (i), (ii), (iv)

(jw) *a generalized ideal (briefly g-ideal) if it satisfies* (v), (vii)

(w) *a generalized σ-ideal (briefly g-σ-ideal) if it satisfies* (vi), (vii)

Example 1. Let μ be a measure or an outer measure on $\mathcal{S}$ and ν a signed measure on $\mathcal{S}$, then putting $\mathcal{M} = \{E \in \mathcal{S}: \mu(E) = 0\}$, $\mathcal{N} = \{E \in \mathcal{S}: \nu(E) = 0\}$, $\mathcal{N}_0 = \{E \in \mathcal{S}: \nu(F) = 0$ for all $F \in E \mid \mathcal{S}\}$ we obtain σ-ideals $\mathcal{M}$, $\mathcal{N}_0$ in $\mathcal{S}$ and a g-σ-ideal $\mathcal{N}$ in $\mathcal{S}$.

Definition 2. *Let $\mathcal{N}$ be a subfamily of a σ-ring $\mathcal{S}$ and $E \in \mathcal{S}_\lambda$. Then the family $\mathcal{N}_E = \{A \in \mathcal{S}: E \cap A \in \mathcal{N}\}$ is called "the contraction of the family $\mathcal{N}$ by E".*

Remark 1. If $\mathcal{N}$ is hereditary then $\mathcal{N}_E \supset \mathcal{N}$ (Lemma 1) and the term "contraction" seems to be inconvenient. But the notation was motivated by that of the contraction of a measure by E. For if ν is a positive measure defined on $\mathcal{S}$ and $E \in \mathcal{S}_\lambda$, then putting $\mathcal{N} = \{G \in \mathcal{S}: \nu(G) = 0\}$ we obtain $\mathcal{N}_E = \{G \in \mathcal{S}: \nu(E \cap G) = 0\}$.

Definition 3. *Let $\mathcal{M}$ be a subfamily of a σ-ring $\mathcal{S}$. Then the subfamily $\mathcal{M}_0 = \{E \in \mathcal{S}: E \mid \mathcal{S} \subset \mathcal{M}\}$ is called a subfamily derived from the family $\mathcal{M}$.*

Lemma 1. *Let $\mathcal{M}$, $\mathcal{N}$ be subfamilies of $\mathcal{S}$ and E, $F \in \mathcal{S}_\lambda$.*

We have:

(a) $(\mathcal{N}_E)_F = \mathcal{N}_{E \cap F}$

(b) $\mathcal{N} \subset \mathcal{M} \Rightarrow \mathcal{N}_E \subset \mathcal{M}_E$

(c) $\mathcal{N}_X = \mathcal{N}$

(d) $(\mathcal{M} - \mathcal{N})C \Rightarrow (\mathcal{M}_F - \mathcal{N}_F)C$

If $\mathcal{N}$ is a g-ideal, then we have:

(e) $\mathcal{N}_\emptyset = \mathcal{S}$

(f) $\mathcal{N}_E = \mathcal{N} \Rightarrow \mathcal{N}_{E^\perp} = \mathcal{S}$

(g) $E \in (\mathcal{S} - \mathcal{N}),\ F \in E|\mathcal{N} \Rightarrow E - F \notin \mathcal{N}$

If $\mathcal{N}$ is hereditary, then we have:

(h) $\mathcal{N} \subset \mathcal{N}_E$

(k) $\mathcal{N}_E$ *is hereditary*

(l) $\mathcal{N}_0 = \mathcal{N}$

If $\mathcal{N}$ is an ideal, we obtain:

(m) $\mathcal{N}_{E \cup F} = \mathcal{N}_E \cap \mathcal{N}_F$

(n) $\mathcal{N}_E$ *is an ideal*

(o) $F \in \mathcal{N} \Rightarrow \mathcal{N}_F = \mathcal{S} \Leftrightarrow \mathcal{N}_{F^\perp} = \mathcal{N}$

We omit the straightforward proof of the above Lemma. For the rest of the paper the letters (a)—(o) will be reserved for the above stated properties.

Definition 4. *Let $\mathcal{M}$, $\mathcal{N}$ be subfamilies of $\mathcal{S}$. Then $\mathcal{N}$ is said to be $\mathcal{M}$-singular (denoted $\mathcal{N} \perp \mathcal{M}$) if there exists $A \in \mathcal{M}_0$ such that $\mathcal{N}_{A^\perp} = \mathcal{S}$.*

Remark 2. It can be easily checked that if $\mathcal{M}$, $\mathcal{N}$ are σ-ideals and $\mathcal{S}$ is a σ-algebra, then definition 4 agrees with Ficker's definition of singularity of σ-ideals given in [8], (def. 15) and in [9].

1. The Lebesgue decomposition theorem

Lemma 2. *Let $\mathcal{M}$ be a σ-ideal, $\emptyset \in \mathcal{N} \subset \mathcal{M}$ and $(\mathcal{M} - \mathcal{N})C$. Then there exists $F \in \mathcal{M}$ with $\mathcal{N}_{F^\perp} = \mathcal{M}$.*

Proof. If $\mathcal{M} = \mathcal{N}$ it is sufficient to put $F = \emptyset$. If $\mathcal{M} \neq \mathcal{N}$ then $\mathcal{M} - \mathcal{N} \neq \emptyset$ and according to Zorn's lemma, there exists a maximal family $(F_i)_{i \in I}$ of pairwise disjoint sets from $(\mathcal{M} - \mathcal{N})$. Since $(\mathcal{M} - \mathcal{N})C$, $(F_i)_{i \in I}$ is at most countable and by (iv) $F = \bigcup_{i \in I} F_i \in \mathcal{M}$ which by (o) implies $\mathcal{M}_{F^\perp} = \mathcal{M}$. It remains to show that $\mathcal{M}_{F^\perp} = \mathcal{N}_{F^\perp}$, but with respect to (b), it sufficies to show that $\mathcal{M}_{F^\perp} \subset \mathcal{N}_{F^\perp}$. If this were not true, there would exist a $G \in (\mathcal{M}_{F^\perp} - \mathcal{N}_{F^\perp})$. Then we would have $G \cap F^\perp \in (\mathcal{M} - \mathcal{N})$, which contradicts the maximality of $(F_i)_{i \in I}$.

Applications of this lemma and the following theorem are presented in Section 5.

Theorem 1. (The Lebesgue decomposition theorem) *Let $\mathcal{M}$ be a σ-ideal, $\emptyset \in \mathcal{N}$ and let $(\mathcal{M} - \mathcal{N})C$. Then there exists $F \in \mathcal{M}$ such that $\mathcal{N}_{F^\perp} \supset \mathcal{M}$ and $\mathcal{N}_F \perp \mathcal{M}$.*

Proof. Since $\mathcal{M} - \mathcal{M} \cap \mathcal{N} = \mathcal{M} - \mathcal{N}$ and $(\mathcal{M} - \mathcal{N}) C$ it follows from lemma 2 that there exists $F \in \mathcal{M}$ with $(\mathcal{M} \cap \mathcal{N})_{F^\perp} = \mathcal{M}$. Hence by using (b), we obtain $\mathcal{N}_{F^\perp} \supset \mathcal{M}$. On the other hand $\mathcal{N}_F \perp \mathcal{M}$ because $F \in \mathcal{M} = \mathcal{M}_0$ and by (a) and (e) we have $(\mathcal{N}_F)_{F^\perp} = \mathcal{S}$.

2. Decomposition of an ideal into atomless and totally atomic parts and its relation to the Lebesgue theorem

The proof of the main result of this section has been inspired by Remark 3.1 in Traynor's paper [20].

Definition 5. *For $\mathcal{N} \subset \mathcal{S}$ we denote $\mathcal{A}(\mathcal{N}) = \bigcap_{F \in \mathcal{S}} (\mathcal{N}_F \cup \mathcal{N}_{F^\perp}) - \mathcal{N}$. Then any element of $\mathcal{A}(\mathcal{N})$ is called an $\mathcal{N}$-atom.*

We remark that this concept is equivalent with the concept of an atom in [18]. The proof of the following lemma follows directly from the definition on an atom.

Lemma 3. *Let $\mathcal{N} \subset \mathcal{S}$, $F \in \mathcal{S}_\lambda$, $G \in \mathcal{S}$. Then $G \in \mathcal{A}(\mathcal{N}_F)$ iff $G \cap F \in \mathcal{A}(\mathcal{N})$.*

Lemma 4. *Let $\mathcal{N}$ be a hereditary subfamily of $\mathcal{S}$ and $F \in \mathcal{S}_\lambda$ Then $\mathcal{A}(\mathcal{N}) \subset \mathcal{A}(\mathcal{N}_F) \cup \mathcal{N}_F$. Moreover if $\mathcal{N}$ is an ideal and $\mathcal{A}(\mathcal{N}) \subset \mathcal{N}_{F^\perp}$, then $\mathcal{A}(\mathcal{N}) \subset \mathcal{A}(\mathcal{N}_F)$.*

Proof. Let $B \in \mathcal{A}(\mathcal{N})$, then $B \in \mathcal{N}_E \cup \mathcal{N}_{E^\perp}$ for all $E \in \mathcal{S}$. Because of (h) $\mathcal{N} \subset \mathcal{N}_F$ we get from (b) that $B \in (\mathcal{N}_F)_E \cup (\mathcal{N}_F)_{E^\perp}$. Hence we have that if $B \notin \mathcal{N}_F$, then $B \in \mathcal{A}(\mathcal{N}_F)$.

Let $\mathcal{N}$ be an ideal and $A \in \mathcal{A}(\mathcal{N}) \subset \mathcal{N}_{F^\perp}$. Then $A \in \mathcal{N}_F \cap \mathcal{N}_{F^\perp} = \mathcal{N}_{F \cup F^\perp} = \mathcal{N}_X = \mathcal{N}$ due to (m), (c) which is a contradiction. Thus $A \notin \mathcal{N}_F$ and by the first part of the lemma $A \in \mathcal{A}(\mathcal{N}_F)$.

Lemma 5. *Let $\mathcal{N}$ be an ideal and $M \in \mathcal{N}$. Then: $A \in \mathcal{A}(\mathcal{N}) \Leftrightarrow A \cup M \in \mathcal{A}(\mathcal{N})$.*

Proof. Let $A \in \mathcal{A}(\mathcal{N})$, then $A \notin \mathcal{N}$ and from the heredity of $\mathcal{N}$, $A \cup M \notin \mathcal{N}$. Since $A \in \mathcal{N}_E \cup \mathcal{N}_{E^\perp}$ and since by (h) $M \in \mathcal{N}_E \cap \mathcal{N}_{E^\perp}$ from (n) we get that for all $E \in \mathcal{S}$, $A \cup M \in \mathcal{N}_E \cup \mathcal{N}_{E^\perp}$.

Conversely. Let $A \cup M \in \mathcal{A}(\mathcal{N})$, then $A \cup M \notin \mathcal{N}$. Since $(M - A) \in \mathcal{N}$, then from (g) $A = (A \cup M) - (M - A) \notin \mathcal{N}$. Since $A \cup M \in \mathcal{N}_E \cup \mathcal{N}_{E^\perp}$ and by (k), $\mathcal{N}_E \cup \mathcal{N}_{E^\perp}$ is hereditary, we get that for all $E \in \mathcal{S}$, $A \in \mathcal{N}_E \cup \mathcal{N}_{E^\perp}$. Hence $A \in \mathcal{A}(\mathcal{N})$.

Notation. *Let us denote by $\mathcal{A}(\mathcal{N})^\sigma$ a family of all $E \in \mathcal{S}$ which can be expressed in the form $E = \bigcup_{i \in I} A_i$ where $(A_i)_{i \in I}$ is an at most countable subfamily of $\mathcal{A}(\mathcal{N})$. Let us denote by $\mathcal{A}(\mathcal{N})^*$ the σ-ideal generated by the family $\mathcal{A}(\mathcal{N})$.*

Remark 2. Obviously if $\mathcal{A}(\mathcal{N}) = \emptyset$, then $\mathcal{A}(\mathcal{N})^\sigma = \emptyset$ and $\mathcal{A}(\mathcal{N})^* = \{\emptyset\}$. If $\mathcal{A}(\mathcal{N}) \neq \emptyset$, then $\mathcal{A}(\mathcal{N})^*$ consists of all $B \in \mathcal{S}$ for which there exists an at most countable subfamily $(A_i)_{i \in I}$ of $\mathcal{A}(\mathcal{N})$ such that $B \subset \bigcup_{i \in I} A_i$.

Definition 6. *A subfamily $\mathcal{N}$ of a σ-ring $\mathcal{S}$ is called*

(1) *atomless if $\mathcal{A}(\mathcal{N})=\emptyset$,*
(2) *atomic if $\mathcal{A}(\mathcal{N})\neq\emptyset$,*
(3) *totally atomic if for all $E\in\mathcal{S}$ there exists $A\in\mathcal{A}(\mathcal{N})^\sigma$ such that $E-A\in\mathcal{N}$,*
(4) *uniformly totally atomic if there exists $A\in\mathcal{A}(\mathcal{N})^\sigma$ such that for all $E\in\mathcal{S}$, there hold $E-A\in\mathcal{N}$.*

Lemma 6. *Let $\mathcal{N}$ be an atomic ideal, then $\mathcal{A}(\mathcal{N})^*\supset\mathcal{N}$. Moreover if $\mathcal{N}$ is a σ-ideal, then $\mathcal{A}(\mathcal{N})^\sigma = \mathcal{A}(\mathcal{N})^*-\mathcal{N}$.*

Proof. Let $\mathcal{N}$ be an ideal. If $A\in\mathcal{A}(\mathcal{N})$, then by Lemma 5, for all $M\in\mathcal{N}$ we have $A\cup M\in\mathcal{A}(\mathcal{N})\subset\mathcal{A}(\mathcal{N})^*$. Hence from heredity of $\mathcal{A}(\mathcal{N})^*$ we get $\mathcal{N}\subset\mathcal{A}(\mathcal{N})^*$.

Let $\mathcal{N}$ be a σ-ideal and $B\in\mathcal{A}(\mathcal{N})^*-\mathcal{N}$. Then by Remark 2 there exists an at most countable family $(B_n)_{n\in K}$ of elements $\mathcal{A}(\mathcal{N})$ such that $B\subset\bigcup_{n\in K}B_n$. We denote $J=\{n\in K\colon B_n\in\mathcal{N}_B\}$, $I=\{n\in K\colon B_n\in\mathcal{N}_{B^\perp}\}$. By definition of an atom $I\cup J=K$. Since $\mathcal{N}$ is a σ-ideal, $C=\bigcup_{n\in J}(B_n\cap B)\in\mathcal{N}$. If we had $I=\emptyset$, then we would have $B=\bigcup_{n\in J}(B_n\cap B)=C\in\mathcal{N}$, which contradicts the hypothesis.

Thus $I\neq\emptyset$, Since $B_n=(B_n\cap B)\cup(B_n-B)$ and for $n\in I$ $(B_n-B)\in\mathcal{N}$ we get from Lemma 5 that for all $n\in I$, $(B_n\cap B)\in\mathcal{A}(\mathcal{N})$. We choose $n_0\in I$. Then $\{[(B_{n_0}\cap B)\cup C]\}\cup\{(B_n\cap B)\colon n\in(I-\{n_0\})\}$ is a countable subfamily of $\mathcal{A}(\mathcal{N})$, the union of which is B.

Lemma 7. *Let $\mathcal{N}$ be an ideal and $F\in\mathcal{S}_\lambda$. Then $\mathcal{N}_F$ is atomless iff $\mathcal{A}(\mathcal{N})\subset\mathcal{N}_F$.*

Proof. Let $\mathcal{N}_F\supset\mathcal{A}(\mathcal{N})$ and assume that there is $G\in\mathcal{A}(\mathcal{N}_F)$. Then I emma 3 implies $G\cap F\in\mathcal{A}(\mathcal{N})$. Since $\mathcal{A}(\mathcal{N})\subset\mathcal{N}_F$, $G\cap F\in\mathcal{N}_F$. Hence $G\in\mathcal{N}_F$ wh'ch is in contradiction with $G\in\mathcal{A}(\mathcal{N}_F)$. The converse implication follows from the first part of Lemma 4.

Lemma 8. *Let $\mathcal{N}$ be an atomic ideal and $F\in\mathcal{S}_\lambda$. Let $\mathcal{N}_{F^\perp}\supset\mathcal{A}(\mathcal{N})$ and $\mathcal{N}_F\perp\mathcal{A}(\mathcal{N})^*$, then $\mathcal{N}_F$ is uniformly totally atomic. Conversely, if $\mathcal{N}_F$ is uniformly totally atomic, then $\mathcal{N}_F\perp\mathcal{A}(\mathcal{N})$.*

Proof. If $\mathcal{N}_F\perp\mathcal{A}(\mathcal{N})^*$, then there exists $B\in\mathcal{A}(\mathcal{N})^*$ such that for all $E\in\mathcal{S}$ we have $E-B\in\mathcal{N}_F$. Then $B\subset\bigcup_{i\in I}A_i$, where $(A_i)_{i\in I}$ is a nonempty at most countable subfamily of $\mathcal{A}(\mathcal{N})$. Since $\mathcal{N}_{F^\perp}\supset\mathcal{A}(\mathcal{N})$, according to Lemma 4 $(A_i)_{i\in I}\subset\mathcal{A}(\mathcal{N}_F)$. Then $\bigcup_{i\in I}A_i\in\mathcal{A}(\mathcal{N}_F)^\sigma$ and for all $E\in\mathcal{A}$ one has $E-\bigcup_{i\in I}A_i\in\mathcal{N}_F$, thus $\mathcal{N}_F$ is uniformly totally atomic.

Proof of the converse inclusion is analogous.

Lemma 9. *Let $\mathcal{N}$ be a σ-ideal. Then $(\mathcal{A}(\mathcal{N})^*-\mathcal{N})C$ holds iff $\mathcal{A}(\mathcal{N})C$.*

Proof. Sufficiency is evident from the inclusion $\mathcal{A}(\mathcal{N}) \subset \mathcal{A}(\mathcal{N})^* - \mathcal{N}$. Necessity follows by Lemma 6, according to which for all $B \in (\mathcal{A}(\mathcal{N})^* - \mathcal{N})$ there exists $A \in B \mid \mathcal{A}(\mathcal{N})$.

Theorem 2. *(Theorem on the decomposition of an ideal into totally atomic and atomless parts).*

Let $\mathcal{N}$ be an atomic ideal such that one of the following two conditions holds: (a) $(\mathcal{A}(\mathcal{N})^* - \mathcal{N})C$ *or* (b) *$\mathcal{N}$ is a σ-ideal satisfying $\mathcal{A}(\mathcal{N})C$.*

Then there exists $F \in \mathcal{A}(\mathcal{N})^*$ such that $\mathcal{N}_{F^\perp}$ is atomless and $\mathcal{N}_F$ is uniformly totally atomic.

Proof. We apply Theorem 1 to the ideal $\mathcal{N}$ and σ-ideal $\mathcal{M} = \mathcal{A}(\mathcal{N})^*$. By Lemma 9 in case (b) the assumption $(\mathcal{M} - \mathcal{N})C$ is satisfied too. Thus by Theorem 1 there exists $F \in \mathcal{A}(\mathcal{N})^*$ such that $\mathcal{N}_{F^\perp} \supset \mathcal{A}(\mathcal{N})^*$ and $\mathcal{N}_F \perp \mathcal{A}(\mathcal{N})^*$. Then $\mathcal{N}_{F^\perp} \supset \mathcal{A}(\mathcal{N})$ and by Lemma 7 $\mathcal{N}_{F^\perp}$ is atomless and by Lemma 8 $\mathcal{N}_F$ is uniformly totally atomic.

Lemma 10. *Let $(\mathcal{N}_k)_{k=1}^{\infty}$ be a sequence of subfamilies of $\mathcal{S}$. Then $\mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right) \subset \bigcup_{k=1}^{\infty} \mathcal{A}(\mathcal{N}_k)$.*

Proof. Let $A \in \mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)$. Then $A \notin \left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)$. Thus there exists $q \in N$ such that $A \notin \mathcal{N}_q$, we put $\mathcal{N}_q = \mathcal{M}$. From the inclusions $\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)_E \cup \left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)_{E^\perp} \subset \mathcal{M}_E \cup \mathcal{M}_{E^\perp}$ it follows that for all $E \in \mathcal{S}$ we have $A \in \mathcal{M}_E \cup \mathcal{M}_{E^\perp}$; which proves that $A \in \mathcal{A}(\mathcal{N}_q)$.

Theorem 3. *Let $\mathcal{N}_k$ be a sequence of subfamilies of $\mathcal{S}$ and let $\mathcal{N}$ be a σ-ideal such that $\bigcap_{k=1}^{\infty} \mathcal{N}_k \subset \mathcal{N}$ and let $(\mathcal{N} - \mathcal{N}_k)C$ for all $k \in N$. Then there exists $F \in \mathcal{N}$ such that*

$$\{A - F \colon A \in \mathcal{A}(\mathcal{N})\} \subset \bigcup_{k=1}^{\infty} \mathcal{A}(\mathcal{N}_k).$$

Proof. It holds $\mathcal{N} - \bigcap_{k=1}^{\infty} \mathcal{N}_k = \bigcup_{k=1}^{\infty} (\mathcal{N} - \mathcal{N}_k)$. Hence by using the assumptions $(\mathcal{N} - \mathcal{N}_k)C$ we get that $\left(\mathcal{N} - \bigcap_{k=1}^{\infty} \mathcal{N}_k\right)C$ is satisfied. By Lemma 2 there exists $F \in \mathcal{N}$ such that $\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)_{F^\perp} = \mathcal{S}$. Then by Lemma 3 $\{A - F \colon A \in \mathcal{A}(\mathcal{N})\} \subset \mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)$.

3. The Hahn decomposition theorem

Lemma 11. *Let $\mathcal{N}$ be a g-σ-ideal in $\mathcal{S}$. Then the family $\mathcal{N}_0$ derived from $\mathcal{N}$ is a σ-ideal in $\mathcal{S}$.*

In the rest of this section $\mathcal{P}$, $\mathcal{N}$ will be g-σ-ideals in $\mathcal{S}$ such that

(viii) $\mathcal{P} \cup \mathcal{N} = \mathcal{S}$

(ix) $(\mathcal{N} - \mathcal{P})C$ and $[N \mid (\mathcal{P} - \mathcal{N})]C$ for all $N \in (\mathcal{N} - \mathcal{P})$

Lemma 12. *If $E \in (\mathcal{N} - \mathcal{P})$ then there exists $E_0 \in E \mid (\mathcal{N}_0 - \mathcal{P})$.*

Proof. Let $E \in (\mathcal{N} - \mathcal{P})$. If $E \mid \mathcal{S} \subset \mathcal{N}$, it is sufficient to put $E_0 = E$. Otherwise by Zorn Lemma there is a maximal family $(F_i)_{i\in I}$ of pairwise disjoint elements of $E \mid (\mathcal{P} - \mathcal{N})$. In view of (ix) $(F_i)_{i\in I}$ is countable and hence by (vi) $F = \bigcup_{i\in I} F_i \in \mathcal{P}$.

Put $E_0 = E - F$. Since $E \notin \mathcal{P}$, $F \in E \mid \mathcal{P}$ it follows from (g) that $E_0 \notin \mathcal{P}$. It remains to show that $E_0 \in \mathcal{N}_0$. In fact if $G \in E_0 \mid \mathcal{S}$ then G is disjoint with each element of $(F_i)_{i\in I}$ and since $(F_i)_{i\in I}$ is maximal we have $G \notin (\mathcal{P} - \mathcal{N})$. Now by (viii) we obtain $G \in \mathcal{N}$ and thus $E_0 \in \mathcal{N}_0$.

Theorem 4. (The Hahn decomposition theorem) *Let $\mathcal{P}$, $\mathcal{N}$ be g-σ-ideals satisfying* (viii), (ix). *Then there exists $A \in \mathcal{S}$ such that $A \mid \mathcal{S} \subset \mathcal{N}$ and $A^{\perp} \mid \mathcal{S} \subset \mathcal{P}$.*

Proof. If $(\mathcal{N} - \mathcal{P}) = \emptyset$, then $A = \emptyset$. Suppose therefore that $\mathcal{N} - \mathcal{P} \neq \emptyset$. Then by Lemma 12 $\mathcal{N}_0 - \mathcal{P} \neq \emptyset$ and by Zorn Lemma there exists a maximal family $(A_i)_{i\in I}$ of pairwise disjoint elements of $(\mathcal{N}_0 - \mathcal{P})$. Since $(\mathcal{N} - \mathcal{P})C$, the family $(A_i)_{i\in I}$ is countable and by Lemma 11 $A = \bigcup_{i\in I} A_i \in \mathcal{N}_0$. We have proved $A \mid \mathcal{S} \subset \mathcal{N}$.

It remains to be proved that $A^{\perp} \mid \mathcal{S} \subset \mathcal{P}$. In the opposite case we have by Lemma 12 that there exists $E \in A^{\perp} \mid (\mathcal{N}_0 - \mathcal{P})$ in contradiction to the maximality of $(A_i)_{i\in I}$.

4. The Hahn-Herer decomposition for measures with values in a topological group

If we have a signed measure μ defined on $\mathcal{S}$ and $\mathcal{M} = \{E \in \mathcal{S}: \mu(E) = 0\}$, then the Hahn decomposition gives two significant sets A, X-$A \in \mathcal{S}_\lambda$ having the following property:

(P) $\quad B \in [(A \mid \mathcal{M}) \cup (X - A \mid \mathcal{M})] \Rightarrow B \mid \mathcal{S} \subset \mathcal{M}.$

As it has been shown by Herer in [14] in the case of group-valued measures the decomposition satisfying (P) in general need not consist of two elements. In this section we give a generalization of Herer's results concerning such a decomposition.

Lemma 13. *Let $\mathcal{M} \subset \mathcal{S}$, then $(\mathcal{S} - \mathcal{M})C \Rightarrow (\mathcal{S} - \mathcal{M}_0)C$.*

The proof is obvious because for all $A \in (\mathcal{S} - \mathcal{M}_0)$ there exists $B \in A \mid (\mathcal{S} - \mathcal{M})$.

Troughout this section $\mathcal{M}$ will be a g-σ-ideal (then $\mathcal{M}_0$ will be a σ-ideal). Put: $\mathcal{M}^* = \{\mathrm{A} \in (\mathcal{S} - \mathcal{M}): \mathrm{A} \mid \mathcal{M} \subset \mathcal{M}_0\}$.

Definition 7. *Let K be a nonempty finite initial segment of the set of positive integers. Let $B_1 \supset B_2 \supset \ldots$ be a finite (or infinite) chain with the indices in K such* *that $B_n \in (\mathscr{S} - \mathscr{M})$ for n even and $B_n \in \mathscr{M}$ for n odd. Then we say that $(B_k)_{k \in K}$ is a finite (infinite) decreasing alternating chain and the set B_1 is called its beginning.*

Definition 8. *We say that a g-σ-ideal $\mathscr{M}$ is pathological if there exists $A \in (\mathscr{S} - \mathscr{M})$ such that each set $B \in A \,|\, (\mathscr{S} - \mathscr{M})$ is the beginning of some infinite alternating chain.*

Lemma 14. *Let $\mathscr{M}$ be a g-σ-ideal in $\mathscr{S}$. Then $\mathscr{M}$ is not pathological iff $A \,|\, \mathscr{M}^* \neq \emptyset$ for all $A \in (\mathscr{S} - \mathscr{M})$.*

Proof. Let $\mathscr{M}$ not be pathological and let $A \in (\mathscr{S} - \mathscr{M})$. Then there exists $B \in A \,|\, (\mathscr{S} - \mathscr{M})$ such that B is not the beginning of any infinite alternating chain. If $B \,|\, \mathscr{M} \subset \mathscr{M}_0$, then $B \in \mathscr{M}^*$ and thus $A \,|\, \mathscr{M}^* \neq \emptyset$. If $B \,|\, \mathscr{M}$ is not a subfamily of $\mathscr{M}_0$, then there exists B_2, B_3 such that $B_2 \in \mathscr{M}$, $B_3 \in (\mathscr{S} - \mathscr{M})$, $B = B_1 \supset B_2 \supset B_3$. If $B_3 \,|\, \mathscr{M} \subset \mathscr{M}_0$, then $B_3 \in {}^*$ and thus $A \,|\, \mathscr{M}^* \neq \emptyset$. If $B_2 \,|\, \mathscr{M}$ is not a subfamily of $\mathscr{M}_0$, we may proceed in the construction of the chain. Since B is not the beginning of any infinite decreasing alternating chain, the construction is finished after a finite number of steps, i.e., there exists a finite sequence $(B_k)_{k=1}^n$, such that $B = B_1 \supset B_2 \supset B_3 \supset \ldots \supset B_n$, where $B_k \in (\mathscr{S} - \mathscr{M})$ for k odd and $B_k \in \mathscr{M}$ for k even. According to the method of construction we get that n is odd and thus $B_n \in (\mathscr{S} - \mathscr{M})$ and that the chain $(B_k)_{k=1}^n$ is maximal in the following sense: there do not exist B', B'' with $B_n \supset B'B''$, $B' \in \mathscr{M}$ and $B'' \in (\mathscr{S} - \mathscr{M})$. Hence $B_n \in \mathscr{M}^*$.

Let us suppose now that $A \,|\, \mathscr{M}^* \neq \emptyset$ for all $A \in (\mathscr{S} - \mathscr{M})$. Then for all $A \in (\mathscr{S} - \mathscr{M})$ there exists $B \in A \,|\, (\mathscr{S} - \mathscr{N})$ such that $B \,|\, \mathscr{M} \subset \mathscr{M}_0$. We can easily see that B is not the beginning of any infinite decreasing alternating chain.

Lemma 15. *Let $\mathscr{M}$ be a g-σ-ideal on S such that $(\mathscr{S} - \mathscr{M})C$. Then $\mathscr{M}$ is not pathological.*

Proof. According to Lemma 14 it is sufficient to show that for all $F \in (\mathscr{S} - \mathscr{M})$ we have $F \,|\, \mathscr{M}^* \neq \emptyset$. Let $F \in (\mathscr{S} - \mathscr{M})$. If $F \,|\, \mathscr{M} \subset \mathscr{M}_0$, $F|\, \mathscr{M}^* \neq \emptyset$ holds. If $F \,|\, (\mathscr{M} - \mathscr{M}_0) \neq \emptyset$, then according to Zorn Lemma there exists a maximal family $(E_i)_{i \in I}$ of pairwise disjoint elements from $F \,|\, (\mathscr{M} - \mathscr{M}_0)$. According to Lemma 13, the family $(E_i)_{i \in I}$ is at most countable and thus $E = \bigcup_{i \in I} E_i \in \mathscr{M}$. From $F \notin \mathscr{M}$ and $E \in F \,|\, \mathscr{M}$ it follows by (g) that $F - E \notin \mathscr{M}$. From the maximality of $(E_i)_{i \in I}$ it follows that $F - E$ does not contain any element of $\mathscr{M} - \mathscr{M}_0$ and thus $(F - E) \,|\, \mathscr{M} \subset \mathscr{M}_0$, i.e. $F - E \in \mathscr{M}^*$.

Theorem 5. (The Hahn—Herer decomposition theorem) *Let $\mathscr{M}$ be a g-σ-ideal in $\mathscr{S}$. Let $\mathscr{M}$ be not pathological. Then there exists a family $(A_t)_{t \in T}$ of pairwise disjoint elements of $\mathscr{S}$ such that $\left[\bigcup_{t \in T} (A_t \,|\, \mathscr{M})\right] \subset \mathscr{M}_0$ and $\left(x - \bigcup_{t \in T} A_t\right) \,|\, \mathscr{S} \subset \mathscr{M}$.*

Proof. If $\mathcal{M}=\mathcal{S}$, then the singleton $\{A\}$ where $A\in\mathcal{S}$ satisfies the conditions in the conclusion of the theorem. Let $(\mathcal{S}-\mathcal{M})\neq\emptyset$. Since $\mathcal{M}$ is not pathological, $\mathcal{M}^*\neq\emptyset$ by Lemma 14. Then by Zorn Lemma there exists a maximal family $(A_t)_{t\in T}$ of pairwise disjoint elements of $\mathcal{M}^*$. Since $A_t\in\mathcal{M}^*$, $A_t\,|\,\mathcal{M}\subset\mathcal{M}_0$ holds for all $t\in T$ and thus $\bigcup_{t\in T}(A_t\,|\,\mathcal{M})\subset\mathcal{M}_0$.

It remains to prove $\left(X-\bigcup_{t\in T}A_t\right)|\,\mathcal{S}\subset\mathcal{M}$. If this were not true, there would exist $B\in\left(X-\bigcup_{t\in T}A_t\right)\in(\mathcal{S}-\mathcal{M})$. Then by Lemma 14 $B\,|\,\mathcal{M}^*\neq\emptyset$ which contradicts the maximality of the family $(A_t)_{t\in T}$.

II. Applications and connections with other results

In the second chapter we shall deal with applications of the results obtained in sections 1 to 4. These applications will first concern set functions with values in an Abelian semigroup with a neutral element. This includes, of course, vector spaces as a special case.

The extended real numbers $\bar{R}=R\cup\{\infty\}$ are included as a range of our measure too. It is sufficient to take the usual addition in R and to put $r+\infty=\infty+r=\infty+\infty=\infty$.

The results of the present paper can further be applied to multimeasures (see the paper [24] by Christianne Godet—Thobie) and to submeasures (see e.g. the paper [23] by Ivan Dobrakov)

Preliminaries

Let us now introduce definitions for all the remaining sections.

Definition 9. *Let μ be a set function defined on $\mathcal{S}$. Then the system $\mathcal{M}=\{E\in\mathcal{S}\colon \mu(F)=0$ for all $F\in E\,|\,\mathcal{S}\}$ is called the null system of μ.*

Obviously "0" from the definition 9 indicates the neutral element of the semigroup which contains the values of the set function μ. According to definition 9 we can easily verify that the null system will be hereditary. Note that in the rest of the paper only such set functions μ for which $\mu(\emptyset)=0$ are considered.

Definition 10. *Let μ, ν be set functions defined on $\mathcal{S}$ and $\mathcal{M}$, $\mathcal{N}$ be their null systems. Then we say that μ is ν-singular (we denote $\mu\perp\nu$) iff $\mathcal{M}\perp\mathcal{N}$.*

We say that μ is dominated by a set function ν (we denote $\mu\ll\nu$) iff $\mathcal{M}\supset\mathcal{N}$.

We say that μ is equivalent to ν (we denote $\mu\equiv\nu$) iff $\mathcal{M}=\mathcal{N}$.

Remark 3. If μ is a set function and F a locally measurable set, then we can define the contraction of μ by F in the following way: $\mu_F(E)=\mu(F\cap E)$ for all $E\in\mathcal{S}$.

The following proposition plays an important role in the proofs of the corollaries in Chapter II. It is significant because with its help the applications of abstract results to set functions are rather straighforward. The proof of the proposition is not difficult.

Proposition 1. *Let $\mathcal{M}$ be the null system of a set function μ, then $\mathcal{M}_F$ is the null system of μ_F i.e. $\mathcal{M}_F = \{G \in \mathcal{S}\colon \mu_F(E) = 0$ for all $E \in G \mid \mathcal{S}\}$.*

Definition 11. *Let μ be a set function defined on $\mathcal{S}$ and $\mathcal{M}$ be its null system. Then we say that μ satisfies CCC (i.e. the countable chain condition, see for example [19]) if it satisfies $(\mathcal{S} - \mathcal{M})C$.*

5. Applications of the Lebesgue decomposition theorem

Applications of abstract formulations will begin with some corollaries of Lemma 2.

Corollary 1. *Let μ, v be set functions defined on $\mathcal{S}$ and let $\mathcal{M}$, $\mathcal{N}$ be their null systems, let μ be additive, $\mathcal{M}$ be a σ-ideal such that $\mathcal{N} \subset \mathcal{M}$ and let $(\mathcal{M} - \mathcal{N})C$ hold. Then there exists $F \in \mathcal{M}$ such that $v_F \ = \mu$ and $\mu - \mu_F$.*

Corollary 2. *Let v be an additive set function defined on $\mathcal{S}$ satisfying CCC. Then there exists $F \in \mathcal{S}$ such that $v = v_F$.*

The proof of the Corollary 1 follows from Lemma 2 and Proposition 1. According to Lemma 1, $F \in \mathcal{M}$ implies $\mathcal{M}_F = \mathcal{S}$ and thus $\mu_F = 0$ (i.e. $\mu_F(E) = 0$ for all $E \in \mathcal{S}$). Hence $\mu = \mu_{F^\perp}$ by additivity.

Corollary 2 follows from Lemma 2 for $\mathcal{M} = \mathcal{S}$.

Corollary 3. *Let μ, v be set functions and $\mathcal{M}$, $\mathcal{N}$ their null systems. Let $\mathcal{M}$ be a σ-ideal and let $(\mathcal{M} - \mathcal{N})C$ hold. Then there exists $F \in \mathcal{M}$ such that $v_{E^\perp} \ll \mu$ and $v_F \perp \mu$.*

If moreover v is additive, then $v = v_E + v_{E^\perp}$.

The proof is evident from Theorem 1 and Proposition 1. (Note that $v(\emptyset) = 0$ as was supposed at the beginning of the chapter)

The questions of the Lebesgue decomposition theorem and propositions concerning it were discussed by many authors.

Omitting the most classical results, the standard formulations of the Lebesgue decomposition theorem may be found e.g. in [13, Theorem 32.C], [3, Theorem 47.3]. Other proofs are in papers [4], [22]. Even the following very simple example shows that our result is stronger than the results where totally σ-finitness or finiteness is required for v. (These assumptions are stronger than the assumption that v satisfies *CCC*).

Let v be defined on $\mathcal{S} = \{\emptyset, X\}$ where $x \neq \emptyset$ in the following way: $v(\emptyset) = 0$ and $v(X) = \infty$, and let μ be an arbitrary measure. Then the hypotheses of Theorem 1 and of the Corollary 3 are satisfied, but v is not σ-finite.

Problems of the formulation of the Lebesgue theorem in terms of the null systems were discussed by Ficker in [8], [9]. He uses the assumptions $X \in \mathscr{S}$ and $(\mathscr{S}-\mathscr{N})C$. His result can be obtained directly from Theorem 1. The result of Musiał [19; Theorem 9] which assumes $\nu \ll \mu$ also follows from Theorem 1 and Corrolary 3.

With the aim to compare the result mentioned above I give an example which shows that the used assumption $(\mathscr{M}-\mathscr{N})C$, is weaker than the assumption *CCC*. This example shows that Theorem 1 is stronger than all formulations of the Lebesgue theorem mentioned above.

It also shows that Ficker's assumptions $X \in \mathscr{S}$ and $(\mathscr{S}-\mathscr{N})C$ and Musiał's assumption $\nu \ll \mu$ are not necessary.

Example 3. Let $\mathscr{S}$ be the σ-ring of all countable subsets of R. Let $(2^R, \Delta)$ be the group of all subsets of the set R with symmetric difference as the group operation. The convergence $A_n \to A$ will be used in the usual sense (i.e. $A_n \to A$ iff $A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big)$. We can easily see that $(2^R, \Delta)$ with the convergence topology introduced in such a way forms a Hausdorff topological group. Let ν and μ be measures defined on $\mathscr{S}$ as follows:

$\nu(A) = A \cap R_0$ and $\mu(A) = A \cap I$ for $A \in \mathscr{S}$. Since $\emptyset$ is the neutral element in the group $(2^R, \Delta)$ we have $\mathscr{N} = \{A \in \mathscr{S}\colon A \subset I\}$ and $\mathscr{M} = \{A \in \mathscr{S}\colon A \subset R_0\}$. We can easily verify that the assumption $(\mathscr{M}-\mathscr{N})C$ of the Theorem 1 is fulfilled and that the required decomposition exists, but the measure ν with values in the group $(2^R, \Delta)$ does not satisfy *CCC*.

I also call your attention to Lipecki's paper [18], where he for the first time mention the applications of abstract formulations for measures in a topological semigroup.

Corollary 1 includes results [3, Theorem 47.3], and [20, Lemma] as special cases, Corollary 2 involves results dealing with exhaustion [16, Theorem 3.1], [13, excercise 17.3]. For a further generalization of the last two mentioned results see Neubrunn's principle of the exhaustion in [2].

As to the applications of our Theorem 1 (or further results in this paper) I call your attention to the fact that if μ is a measure with values in a metrizable topological group, then μ satisfies *CCC*. (see Kluvánek [16, Section 3] and also more generally Lipecki's proposition in [17])

6. Applications to decompositions into totally atomic and atomless parts

Definition 12. *Let ν be a set function defined on $\mathscr{S}$ and $\mathscr{N}$ be its null system. Then we say that A is an atom (more exactly a ν-atom) if $A \in \mathscr{A}(\mathscr{N})$. The set of all ν-atoms will be denoted by $\mathscr{A}(\nu)$.*

Definition 13. *Let v be a set function defined on a σ-ring $\mathscr{S}$ and let $\mathscr{N}$ be its null system. Then we say that:*

(1) *v is atomless (atomic) if $\mathscr{A}(v)=\emptyset$ ($\mathscr{A}(v)\neq\emptyset$)*

(2) *v is totally atomic (resp uniformly totally atomic) if $\mathscr{N}$ is totally atomic (resp. uniformly totally atomic).*

Corollary 4. *Let v be an atomic additive set function on $\mathscr{S}$ such that its null system $\mathscr{N}$ satisfies $(\mathscr{A}(\mathscr{N})^*-\mathscr{N})C$. Then there exists $A\in\mathscr{A}(\mathscr{N})^*$ such that $v_{F^\perp}$ is atomless and v_F is uniformly totally atomic.*

If $\mathscr{N}$ is a σ ideal then the condition $(\mathscr{A}(\mathscr{N})^*-\mathscr{N})C$ holds iff $\mathscr{A}(\mathscr{N})C$.

The proof s evident from Theorem 2 and Lemma 9.

Corollary 5. *Let v_k be set functions defined on $\mathscr{S}$ and $\mathscr{N}_k$ be their null systems for $k=1, 2, \dots$ Let v be a set function the null class of which is a σ-ideal such that $\mathscr{N}\supset\bigcap_{k=1}^{\infty}\mathscr{N}_k$ and let $(\mathscr{N}\ \ \mathscr{N}_k)C$ be satisfied for all $k\in N$. Then there exists $F\in\mathscr{N}$ such that $\{A-F\colon A\in\mathscr{A}(v)\}\subset\bigcup_{k=1}^{\infty}\mathscr{A}(v_k)$*

The proof of the corollary is evident from Theorem 3. We point out that in the particular case if we put $v=\sum_{k=1}^{\infty}v_k$, then the condition $\bigcap_{k=1}^{\infty}\mathscr{N}_k\subset\mathscr{N}$ in Corollary 5 is fulfilled

Thus if we suppose in Corollary 5 $v_3=v_4=\dots=0$ and $v=v_1+v_2$ and $\mathscr{A}(\mathscr{N}_1)=\mathscr{A}(\mathscr{N}_2)=\emptyset$, and if moreover we assume $(\mathscr{N}-\mathscr{N}_1)C$ and $(\mathscr{N}-N_2)C$ instead of $(\mathscr{S}-\mathscr{N}_1)C$ and $(\mathscr{S}-\mathscr{N}_2)C$, we get the first part of the Hoffmann—Jørgensen's Theorem 4 in [15] and also Statement 1 in [19] and Corollary 3 in [18].

7. Remarks on the Hahn decomposition theorem

Let v be a singed measure defined on $\mathscr{S}$, which does not attain the value $-\infty$. If we put $\mathscr{N}=\{E\in\mathscr{S}\colon v(E)\leqq 0\}$, $\mathscr{P}=\{E\in\mathscr{S}\colon v(E)\geqq 0\}$ then $\mathscr{P}$, $\mathscr{N}$ are g-σ-ideals such that $\mathscr{P}\cup\mathscr{N}=\mathscr{S}$. It is well-known e.g. from [13] that $\mathscr{P}$, $\mathscr{N}$ satisfy the property (ix). See also [7].

Theorem 3 is thus a generalization of the classical Hahn decomposition theorem.

An abstract formulation of the Hahn theorem which generalized the classical version was introduced for the first time by Červeňanský and Dravecký in [7]. Their proposition will be denoted (CD). They consider systems $\mathscr{P}$, $\mathscr{N}$, $\mathscr{Z}$, which are analogues of the systems $\{E\in\mathscr{S}\colon v(E)\geqq 0\}$, $\{E\in\mathscr{S}\colon v(E)\leqq 0\}$, $\{\in E\mathscr{S}\colon v(E)=0\}$.

Among the hypotheses of the (CD) proposition there are the following assumptions:

a$_1$) $\emptyset \in \mathscr{X} \subset \mathscr{P} \cap \mathscr{N}$,
a$_2$) $(\mathscr{P} - \mathscr{X})$ satisfies (vi),
a$_3$) $(\mathscr{P} - \mathscr{X})C$.

It is possible to show that if $\mathscr{P}$, $\mathscr{N}$, $\mathscr{X}$ satisfy the assumptions in (CD), then $\mathscr{P}$, $\mathscr{N}$ are σ-ideals satisfying (viii), (ix) and thus Theorem 3 includes the (CD) proposition. The following example shows that the (CD) proposition does not imply our Theorem 3.

Example 4. Let $\mathscr{S} = \{A: A \subset R,\ A \text{ is at most countable}\}$ and $I = R - R_0$. Evidently $\mathscr{S}$ is a σ-ring. Let us define the systems $\mathscr{P}$, $\mathscr{N}$ as follows: $\mathscr{P} = \{A \in \mathscr{S}: A \subset I\}$, $\mathscr{N} = \{A \in \mathscr{S}: A \text{ is infinite}\} \cup \{A \in \mathscr{S}: \text{A is finite}, A \cap R_0 \neq \emptyset\} \cup \{\emptyset\}$. It is easily seen that $\mathscr{P}$, $\mathscr{N}$ are g-σ-ideals and that $\mathscr{P} \cup \mathscr{N} = \mathscr{S}$. Further $\mathscr{P} - \mathscr{N} = \{A \in S: A \text{ is finite}, A \subset I\} - \{\emptyset\}$, $\mathscr{N} - \mathscr{P} = \{A \in \mathscr{S}: A \cap R_0 \neq \emptyset\}$. As each set from $(\mathscr{N} - \mathscr{P})$ contains at least one rational number we have $(\mathscr{N} - \mathscr{P})C$. Moreover as each set $N \in \mathscr{S}$ is at most countable for all $N \in (\mathscr{N} - \mathscr{P})$ the condition $[N \,|\, (\mathscr{P} - \mathscr{N})]C$ is satisfied. Thus we have shown that $\mathscr{P}$, $\mathscr{N}$ satisfy the axioms (vi), (vii), (viii), (ix) and thus Theorem 4 can be successfully used.

We shall show that no system $\mathscr{X}$ exists for which the triple $(\mathscr{P}, \mathscr{N}, \mathscr{X})$ or $(\mathscr{N}, \mathscr{P}, \mathscr{X})$ satisfies a$_1$), a$_2$), a$_3$). Suppose the contrary. As $\mathscr{X} \subset \mathscr{P} \cap \mathscr{N}$ then $\mathscr{P} - \mathscr{X} = \mathscr{P} - \mathscr{P} \cap \mathscr{N} = \mathscr{P} - \mathscr{N} = \{A \in \mathscr{S}: A \subset I, A \text{ is finite}\} - \{\emptyset\}$. Due to a$_2$) $\mathscr{P} - \mathscr{X} \supset \{A \in \mathscr{S}: A \subset I\} - \{\emptyset\}$. Hence $\mathscr{X} = \mathscr{P} - (\mathscr{P} - \mathscr{X}) \subset \{\emptyset\}$ and thus $\mathscr{X} = \{\emptyset\}$.
Then $\mathscr{N} - \mathscr{X} = \{A \in \mathscr{S}: A \text{ is finite}, A \cap R_0 \emptyset\} \cup \{A \in \mathscr{S}: A \text{ is infinite}\}$. It is easily seen that neither the condition $(\mathscr{P} - \mathscr{X})C$ nor $(\mathscr{N} - \mathscr{X})C$ holds true and thus the triples $(\mathscr{P}, \mathscr{N}, \mathscr{X})$, $(\mathscr{N}, \mathscr{P}, \mathscr{X})$ do not satisfy the assumption a$_3$) in (CD).

8. Applications of the Hahn—Herer theorem

Definition 14. *Let μ be a set function. Then the system $\mathscr{M} = \{E \in \mathscr{S}: \mu(E) = 0\}$ will be called a null-valued system (more exactly a μ-null-valued system).*

Let us note that if μ is a measure (i.e. a σ-additive set function) with values in a topological semigroup, then $\mathscr{M}$ is a g-σ-ideal.

Corollary 6. *Let $\mathscr{S}$ be a σ-ring and μ be a set function defined on $\mathscr{S}$. Let the null-valued system of μ be a nonpathological g-σ-ideal.*

Then there exists a system $(A_t)_{t \in T}$ of pairwise disjoint members of $(\mathscr{S} - \mathscr{M})$ such that if some set A is from $A_t \,|\, \mathscr{M}$ for some $t \in T$, then A is μ-null and each set from $\left(\bigcup_{t \in T} A_t\right)^{\perp} \,|\, \mathscr{S}$ is μ-null.

The proof of the corollary is evident from Theorem 5. Let us also note that according to Lemma 15 a g-σ-ideal satisfies $(\mathscr{S}-\mathscr{M})C$, then $\mathscr{M}$ is not pathological.

Remark 4. Theorem 5 and Corollary 6 generalize Herer's result in [14] and also generalize Theorem 2 from Lipecki's paper ([18]).

In [14] Herer introduces Example 3 to which it is not possible to apply successfully the results mentioned in [18] and in [14]. But our Theorem 5 and Corollary 6 can be used also for this example to prove the existence of Hahn—Herer decomposition.

9. The Hahn decomposition theorem for measures with values in a partially ordered semigroup

In this section we shall introduce the Hahn theorem for measures with values in a partially ordered topological semigroup G. According to our agreement at the beginning the second chapter, G is an Abelian semigroup with the neutral element 0.

Let us recall some definitions. A partially ordered semigroup is a triple $(G, +, \leqq)$ where $(G, +)$ is a semigroup, $(G, \leqq)$ is a partially ordered set and moreover the partial ordering and the operation of addition are related by the so called monotonicity rule, which is in case of an Abelian semigroup as follows:

$$\text{(M)} \qquad a, b \in G, a \leqq b \Rightarrow a+c=b+c \text{ for all } c \in G.$$

If G is a partially ordered semigroup, then the system $\boldsymbol{P}=\{a \in G: a \geqq 0\}$ $(\boldsymbol{P}^{-}=\{a \in G: a \leqq 0\})$ is called a positive cone (negative cone).

Lemma 16. *Let G be a partially ordered semigroup. Then the positive and the negative cone satisfy properties*

(α) $\boldsymbol{P} \cap \boldsymbol{P}^{-}=\{0\}$

(β) $\boldsymbol{P}+\boldsymbol{P} \subset \boldsymbol{P}$ *and* $\boldsymbol{P}^{-}+\boldsymbol{P}^{-} \subset \boldsymbol{P}^{-}$

The proof of the property (α) follows from the antisymmetry of the partial ordering. (β) follows from (M) and from the transitivity of the partial ordering.

Definition 15. *Let G be a partially ordered semigroup with a positive (negative) cone $\boldsymbol{P}$ $(\boldsymbol{P}^{-})$. Then we say that A is purely positive (more exactly purely $(\mu, \boldsymbol{P})$-positive) if $\mu(E) \in \boldsymbol{P}$ for all $E \in A \mid \mathscr{S}$. We say that A is purely negative (more exactly $(\mu, \boldsymbol{P}^{-})$-negative) if $\mu(E) \in \boldsymbol{P}^{-}$ for all $E \in A \mid \mathscr{S}$.*

Lemma 17. *Every Hahn decomposition of an additive set function μ with values in a partially ordered semigroup G is the two-element Hahn—Herer decomposition.*

Proof. Let $\{A, A^{\perp}\}$ be Hahn's decomposition of an additive set function μ. Thus A is $(\mu, \boldsymbol{P}^{-})$-negative and $A^{\perp}$ is $(\mu, \boldsymbol{P})$-positive where $\boldsymbol{P}$, $\boldsymbol{P}^{-}$ are positive and negative cones of the partially ordered semigroup G. Let us put $\mathcal{P}=\{E\in\mathcal{S}\colon \mu(E)\in\boldsymbol{P}\}$ and $\mathcal{N}=\{E\in\mathcal{S}\colon \mu(E)\in\boldsymbol{P}^-\}$. We prove first that the following property is satisfied:

$$\text{(x)}\qquad E\in\mathcal{P}\cap\mathcal{N},\ F_1\in E\,|\,\mathcal{P},\ F_2\in E\,|\,\mathcal{N}\Rightarrow E-F_1\in\mathcal{N},\ E-F_2\in\mathcal{P}.$$

Indeed let $E\in\mathcal{P}\cap\mathcal{N}$ and $F_1\in E\,|\,\mathcal{P}$. Then $\mu(F_1)+\mu(E-F_1)=\mu(E)=0$. Then from the inequality $\mu(F_1)=0$ and using (M) we get $\mu(F_1)+\mu(E-F_1)\geqq\mu(E-F_1)$ and thus $0\geqq\mu(E-F_1)$. Hence $E-F_1\in\mathcal{N}$. Similarly we can prove that $E-F_2\in\mathcal{P}$.

We denote $\mathcal{M}=\{E\in\mathcal{S}\colon \mu(E)=0\}$. According to ($\alpha$) we have $\mathcal{M}=\mathcal{P}\cap\mathcal{N}$. To prove that $\{A, A^{\perp}\}$ form a Hahn—Herer's decomposition it is sufficient to prove that $A\,|\,\mathcal{M}\subset\mathcal{M}_0$ and $A^{\perp}\,|\,\mathcal{M}\subset\mathcal{M}_0$. Let $B\in A\,|\,\mathcal{M}$ and $C\in B\,|\,\mathcal{S}$. As $C\in A\,|\,\mathcal{S}$ and $A\,|\,\mathcal{S}\subset\mathcal{N}$ we have $C\in\mathcal{N}$. For the same reason also $B-C\in\mathcal{N}$. Since $B\in\mathcal{M}$ and $B-C\in\mathcal{N}$, by (x) we obtain $C\in\mathcal{P}$ and thus $C\in\mathcal{M}$. We have proved that $B\,|\,\mathcal{S}\subset\mathcal{M}$ and thus $A\,|\,\mathcal{M}\subset\mathcal{M}_0$. The inclusion $A^{\perp}\,|\,\mathcal{M}\subset\mathcal{M}_0$ can be proved analogously.

Theorem 6. *Let G be a partially ordered Hausdorff topological semigroup and μ be a measure with values in G. Let positive and negative cones of G satisfy:*

(γ) *$x_n\in\boldsymbol{P}$ (or $x_n\in\boldsymbol{P}^-$) for all $n\in N$ and $\sum_{n=1}^{\infty}x_n\in(\boldsymbol{P}\cup\boldsymbol{P}^-)$, then $\sum_{n=1}^{\infty}x_n\in\boldsymbol{P}$ (or $\sum_{n=1}^{\infty}x_n\in\boldsymbol{P}^-$)*

(δ) $\{\mu(E)\colon E\in\mathcal{S}\}\subset\boldsymbol{P}\cup\boldsymbol{P}^-$

(ε) $\{A\in\mathcal{S}\colon \mu(A)\in(\boldsymbol{P}^- -\boldsymbol{P})\}C$

(ζ) *if $\mu(E)\in(\boldsymbol{P}^- -\boldsymbol{P})$, then $\{A\in E\,|\,\mathcal{S}\colon \mu(A)\in(\boldsymbol{P}-\boldsymbol{P}^-)\}C$.*

Then there exists $A\in\mathcal{S}$ such that A is purely $(\mu,\boldsymbol{P}^-)$-negative and $A^{\perp}$ is purely $(\mu,\boldsymbol{P})$-positive.

Moreover $\{A, A^{\perp}\}$ is a two-element Hahn—Herer decomposition.

Proof. Let us put $\mathcal{P}=\{E\in\mathcal{S}\colon \mu(E)\in\boldsymbol{P}\}$ and $\mathcal{N}=\{E\in\mathcal{S}\colon \mu(E)\in\boldsymbol{P}^-\}$. According to ($\alpha$) $\emptyset\in\mathcal{P}\cap\mathcal{N}$. Let $E_n\in\mathcal{P}$ for all $n\in\boldsymbol{N}$. Then $\mu(E_n)\in\boldsymbol{P}$ and according to (δ) $\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\in\boldsymbol{P}\cup\boldsymbol{P}^-$. Hence according to ($\gamma$) we get that $\sum_{n=1}^{\infty}\mu(E_n)=\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\in\boldsymbol{P}$ and thus $\bigcup_{n=1}^{\infty}E_n\in\boldsymbol{P}$. Analogically we could show that $\mathcal{N}$ satisfies (iii) and thus we have proved that $\mathcal{P}$, $\mathcal{N}$ are g-σ-ideals.

It is easily seen that $\mathcal{P}-\mathcal{N}=\{E\in\mathcal{S}\colon \mu(E)\in(\boldsymbol{P}-\boldsymbol{P}^-)\}$ and $\mathcal{N}-\mathcal{P}=\{E\in\mathcal{S}\colon \mu(E)\in(\boldsymbol{P}^- -\boldsymbol{P})\}$ and condition (ζ) means that for all $E\in(\mathcal{N}-\mathcal{P})$ one has $[E\,|\,(\mathcal{P}-\mathcal{N})]C$ and thus $\mathcal{P}$, $\mathcal{N}$ satisfy (ix). From (δ) we get that $\mathcal{P}$, $\mathcal{N}$ satisfy (viii).

According to Theorem 2 there exists $A\in\mathcal{S}$ such that $A\,|\,\mathcal{S}\subset\mathcal{N}$ and $A^{\perp}\,|\,\mathcal{S}\subset\mathcal{P}$ and thus A is purely $(\mu,\boldsymbol{P}^-)$-negative and $A^{\perp}$ is purely $(\mu,\boldsymbol{P})$-positive.

The following example shows that the condition (γ) is not fulfilled even in the case when G is a partially ordered group.

Example 5. Let G be the additive group of real numbers and let us put $P = \{r - z \cdot e: r \in R_0, r \geqq 0, z \geqq 0, z \text{ is an integer}\}$ where e is the base of the natural logarithm. Since G is a group, then $P^- = \{x \in G: -x \in P\}$. It is easy to verify that the conditions $P \cap P^- = \{0\}$ and $P + P \subset P$ are fulfilled and thus according to Theorem II.2. in [11], P defines a partial ordering.

But the condition (γ) is not fulfilled because if we define $x_n = \frac{1}{(n-1)!}$ for all $n \in N$, then $x_n \in P$ for all $n \in N$, but $\sum_{n=1}^{\infty} x_n \in P$.

REFERENCES

[1] NEUBRUNN, T.: Замечание об абсолютной непрерывности мер. Rus ian Mat -fy Čas 16 1966, 21—30.

[2] NEUBRUNN T.: On the equivalence of an exhaustion principle and the ax om of ch i e Fun Math 60, 1967, 209—211.

[3] BERBERIAN, D. K.: Measure and integration, New York 1960.

[4] BROOKS, S. K.: The Lebesgue decomposition theorem for measures. Th Am r Mat . M on h 78, N. 6, 1971, 660—662.

[ ] CAPEK, P.: Theorèmes de decomposition en theorie de la mesure I, II. Pub tions du Se n d'Analyse de Brest, juin 1976.

[6] CAPEK, P.: Théorèmes de décomposition en theorie de la measure, R.A.S P. t 2 5, 19 7 Serie A-537

[7] ČERVEŇANSKY, J., DRAVECKY, J.: A note on Hahn decomposition. Acta F R N U iv. Cc men. — Mathematica XXV — 1971, 27—29.

[ ] FICKER, V.: Dominated classes and related questions Acta F.R N Un v Comen — Mathematica X, 7, 1966, 3—18.

[9] FICKER, V.: An abstract formulation of the Lebesgue decomposition theorem. Aust. M h Soc 12, 1971, 101—105.

[10] FICKER, V.: On the equivalence of a countable disjoint class of sets of positive me r nd a weaker condition than total σ-finiteness of measures. Bull. Austral Math. So 1 19 9 237—243.

[11] FUCHS, L.: Partially ordered algebraic system. Pergamon Press 1963.

[12] HAHN, H., ROSENTHAL, A.: Set functions. The University of New Mexico Press 1948.

[13] HALMOS, P. R.: Measure theory. Van Nonstrand—New York 1950.

[14] HERER. W.: Decomposition of measures with values in a topological group. Bull. de l'Acad Polon. des Sciences Série Math. Astron. et Phys. 20, 1972, 203—205.

[15] HOFFMANN—JØRGENSEN, J.: Vector measures. Math. Scand. 28, 1971, 5—32.

[16] KLUVÁNEK, I.: К теории векторныцх мер. Mat.-fyz. Čas. 11, 3, 1961.

[17] LIPECKI, Z.: A characterization of group-valued measures satisfying the countable chain condition. Coll. Math. 31, 1974, 231—234.

[18] LIPECKI, Z.: Decomposition theorems for Boolean rings, with applications to semigroup-valued measures. Commentationes Mathematicae. XX, 1978, 397—403.

[19] MUSIAŁ, K.: Absolute continuity of vector measures. Coll. Math. 27, 1973, 319—321.
[20] MUSIAŁ, K.: Absolute continuity and the range of group valued Vmeasures. Bull. Acad. Polon. Sci. Série Sci., Math., Astr. et Phys. 21, 1973, 105—113.
[21] TRAYNOR, T.: Decomposition of group-valued additive set functions. Ann. Inst. Fourier, Grenoble, 22, 3, 1972, 131—140.
[22] YAM TING WOO, J.: An elementary proof of the Lebesgue decomposition theorem. The Amer. Math. Monthly 78, 7, 1971, p. 783.
[23] DOBRAKOV, I.: On submeasures I. Dissertationes Mathemat. CXII, 1974.
[24] M^me GODET—THOBIE, C.: Multimesures et Multimesures de Transitions. Thèse. Montpellier. 1975.

Received August 4, 1978

Katedra matematickej analýzy
Matematicko fyzikálnej fakulty UK
Mlynská dolina
816 31 Bratislava

ТЕОРЕМЫ О РАЗЛОЖЕНИИ В ТЕОРИИ МЕРЫ

Петер Цапек

Резюме

В первой главе работы были получены абстрактные формулировки следующих теорем о разложении в теории меры: теоремы Лебега, теоремы о разложении меры на неатомическую а вполне атомическую части, теоремы Хана, теоремы Хана—Герера. Здесь также приводится абстрактная формулировка и обобщение теоремы Хоффманн—Йогенсена о сумме двух безатомических мер. Во второй главе приведено применение абстрактных формулировок и сранение их с известными родственными результатами. В девятом разделе доказана теоремы Хана для мер со значениями в частично упорядноченной топологической полугруппе, из которой следует как частичный случай класическая теорема Хана.