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BARRELLEDNESS OF THE SPACE OF DOBRAKOV INTEGRABLE FUNCTIONS

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ABSTRACT. We use a continuous version of the Gliding Hump Principle to show that the space of Dobrakov integrable functions is barrelled under the assumption that the measure is countably additive in the uniform operator topology.

In [DFP1-2], Drewnowski, Florencio and Paúl established abstract Uniform Boundedness results and then used them to establish the barrelledness of the space of Pettis integrable functions. In [Sw2] we established a somewhat similar uniform boundedness result and also used the result to establish the barrelledness of the space of Pettis integrable functions. In [D1] (and subsequent papers) I. Dobrakov developed a theory for integrating vector-valued functions with respect to operator-valued measures which gave an extension of the Pettis integral when the measure was scalar valued. In this note we show that the uniform boundedness result of [Sw2] can also be used to establish the barrelledness of the space of Dobrakov integrable functions.

We begin by fixing the notation which will be employed in the sequel; otherwise, our terminology is standard and basically follows [DS]. Let X, Y be (real) Banach spaces and $L(X,Y)$ the space of continuous linear operators from X into Y. Let Σ be a σ-algebra of subsets of a set S and let $m\colon \Sigma \to L(X,Y)$ be countably additive with respect to the strong operator topology of $L(X,Y)$. A function $f\colon S \to X$ is (*strongly*) *measurable* if there exists a sequence of X-valued, Σ-simple functions $\{f_k\}$ converging to f pointwise on S. A measurable function $f\colon S \to X$ is m*-integrable* or *integrable with respect to* m if there exists a sequence of X-valued, m-integrable simple functions $\{f_k\}$ such that $f_k \to f$ m-a.e. and $\lim \int_E f_k \, dm = \gamma(E)$ exists for every $E \in \Sigma$; the *integral* of f over E is defined to be $\gamma(E)$ and is denoted by $\int_E f \, dm = \gamma(E)$ ([D1]). Moreover, $\int f \, dm$ is a countably additive Y-valued measure and the

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limit above is uniform for $E \in \Sigma$ ([D1; Theorem 7]). Let $L^1(m)$ be the space of all m-integrable functions. We define an L^1-seminorm on $L^1(m)$ by $\|f\|_1 = \sup\left\{\left\|\int_E f\ dm\right\| : E \in \Sigma\right\}$; since $\int f\ dm$ is countably additive, $\|f\|_1 < \infty$ ([D1; Theorem 2]).

EXAMPLE 1. Let $\mu : \Sigma \to \mathbb{R}$ be a countably additive, positive measure. Define an $L(X,X)$-valued measure $\hat{\mu}$ by $\hat{\mu}(E)x = \mu(E)x$. Then f is $\hat{\mu}$-integrable if and only if f is measurable and Pettis μ-integrable and $\int_E f\ d\hat{\mu} = \int_E f\ d\mu$ ([D1; Example 3.2]). Moreover, if f is $\hat{\mu}$-integrable, $2\|f\|_1 \geq \sup\left\{\int_S |x'f|\ d\mu : \|x'\| \leq 1\right\} = \|f\|_1' \geq \|f\|_1$ and $\|\ \|_1'$ is the usual semi-norm defined on the space of Pettis integrable functions ([Pe]).

It is well-known that the space of Pettis integrable functions is not, in general, complete under $\|\ \|_1$ ([Pe], [Th]) so it follows from Example 1 that $L^1(m)$ is, in general, not complete under $\|\ \|_1$. As is the case for the space of Pettis integrable functions, we show that $L^1(m)$ is a barrelled space under $\|\ \|_1$ when m is countably additive in the norm topology of $L(X,Y)$.

Remark 2. Dobrakov defines a semi-norm p_1 on a subspace $\mathcal{L}^1(m)$ of $L^1(m)$ by setting $p_1(f) = \left\{\int_S \|f(\cdot)\|\ dv(y'm) : \|y'\| \leq 1\right\}$, where $v(y'm)$ is the variation of the measure $y'm(E) = \langle y', m(E)\rangle$ ([D2; Theorem 4]), and defining $\mathcal{L}^1(m)$ to be all functions f with $p_1(f) < \infty$. He then shows that $\mathcal{L}^1(m)$ is complete under p_1; however, $\mathcal{L}^1(m)$ is generally a proper subspace of $L^1(m)$. Indeed, in the case of scalar valued measures as in Example 1, $p_1(f) = \int_S \|f(\cdot)\|\ d\mu$ so $\mathcal{L}^1(\hat{\mu})$ coincides with the subspace of $L^1(\hat{\mu})$ consisting of the Bochner integrable functions which is generally a proper subspace of the space of Pettis integrable functions ([D2; p. 688]).

We next describe the uniform boundedness result which will be employed in our proof of the barrelledness of $L^1(m)$.

Let E be a Hausdorff locally convex space. If E' is the dual of E, we denote the weak* topology of E' (strong topology of E) by $\sigma(E',E)$ ($\beta(E,E')$). Suppose $P : \Sigma \to L(E,E)$ and write $P_A = P(A)$ for $A \in \Sigma$. Let P satisfy.

(i) $P_\phi = 0$ and $P_S = I$, where I is the identity operator on E,
(ii) P is finitely additive with $P_{A\cap B} = P_A P_B$.

We impose 2 further conditions on P in order to establish a uniform boundedness result for E. First, we have a gliding hump condition.

We say that P *satisfies the strong Gliding Hump Property* (*strong* (*GHP*)) if:

> whenever $\{x_j\}$ is a null sequence in E and $\{A_j\}$ is a pairwise disjoint sequence from Σ, there is a subsequence $\{n_j\}$ such that the series $\sum_{j=1}^{\infty} P_{A_{n_j}} x_{n_j}$ is convergent to an element in E.

We also consider a decomposition property for P. If $y' \in E$ and $x \in E$, we write $y'Px$ for the scalar valued set function $A \to \langle y', P_A x\rangle$, $A \in \Sigma$, and we let $|y'Px|$ denote the variation of $y'Px$. We say that P *satisfies property* (D) if:

(D) For every $\varepsilon > 0$, $y' \in E'$, $x \in E$, there exist $A_1, \dots, A_k \in \Sigma$ pairwise disjoint such that $S = \bigcup_{i=1}^{k} A_i$ and $|y'Px|(A_i) < \varepsilon$ for $i = 1, \dots, k$.

Remark 3. If $y'Px$ is non-atomic for every $y' \in E'$, $x \in E$, then (D) is satisfied ([RR; 5.1.6]). In particular, if λ is a finite, positive, non-atomic measure on Σ, E is a normed space and $\lim_{\lambda(A)\to 0} \|P_A x\| = 0$ for every $x \in E$, then this condition is satisfied.

We employ the following uniform boundedness result which was established in [Sw2] (this result requires the assumption that $P_{A\cap B} = P_A P_B$).

THEOREM 4. *Suppose P satisfies conditions* (D) *and strong* (*GHP*). *If $B \subset E'$ is $\sigma(E', E)$ bounded and $A \subset E$ is bounded, then*

$$\sup\{|\langle x', x\rangle| : x' \in B,\ x \in A\} < \infty .$$

That is, if $A \subset E$ is bounded, then A is strongly bounded so that E is a Banach-Mackey space ([Wi; §10.4]).

Remark 5. A stronger result involving a weaker gliding hump property was established in [Sw2], but Theorem 4 will suffice for the application to $L^1(m)$ that we require.

We first consider the strong (GHP) for $L^1(m)$. In this case the mapping P is given by $P_A f = C_A f$ for $A \in \Sigma$, $f \in L^1(m)$.

THEOREM 6. *$L^1(m)$ has strong* (*GHP*).

Proof. Let $f_k \to 0$ in $L^1(m)$ and $\{E_k\} \subset \Sigma$ pairwise disjoint. Pick a subsequence satisfying $\|f_{n_k}\|_1 < 1/2^k$. Let f be the pointwise limit of $\sum_{k=1}^{\infty} C_{E_{n_k}} f_{n_k}$. For each $E \in \Sigma$, if $q > p$, $\left\| \sum_{k=p}^{q} \int_E C_{E_{n_k}} f_{n_k}\, dm \right\|_1 \le \sum_{k=p}^{q} 1/2^k$ so $\lim_p \sum_{k=1}^{p} \int_E C_{E_{n_k}} f_{n_k}\, dm = \nu(E)$ exists. By [D1; Theorem 16] f is m-integrable and

$$\int_E f \, dm = \sum_{k=1}^{\infty} \int_E C_{E_{n_k}} f_{n_k} \, dm \qquad \text{uniformly for} \quad E \in \Sigma. \tag{1}$$

Now (1) implies that $\sum_{k=1}^{\infty} C_{E_{n_k}} f_{n_k} = f$ in $\| \|_1$. □

We next consider property (D) and the barrelledness of $L^1(m)$ when the measure m is non-atomic.

THEOREM 7. *Assume that m is non-atomic and countably additive with respect to the norm topology of $L(X,Y)$. Then $L^p(m)$ has* (D).

P r o o f. By [DU; I.2.6] or [DS; IV.10.5] there exists a control measure λ for m, i.e., $\lim_{\lambda(E)\to 0} m(E) = 0$ and m and λ have the same null sets. Since m is non-atomic, λ is also non-atomic. Let $f \in L^1(m)$. By [D1; Theorem 3] the set function $E \mapsto \int_E f \, dm$ is continuous with respect to the scalar semi-variation $\|m\|(\cdot)$ and, therefore, is continuous with respect to the control measure λ. Hence, $\lim_{\lambda(E)\to 0} \left\| \int_E f \, dm \right\| = 0$ and $\lim_{\lambda(E)\to 0} \|C_E f\|_1 = 0$. The result now follows from Remark 3. □

COROLLARY 8. *Let m satisfy the assumptions in Theorem 7. Then $L^1(m)$ is barrelled.*

P r o o f. By Theorems 4, 6 and 7, $L^1(m)$ is a Banach-Mackey space and, hence, is barrelled ([Wi; 10.4.12]). □

We next consider the case when m may have atoms. We assume henceforth that m is countably additive with respect to the norm topology of $L(X,Y)$ and λ is a control measure for m as in the proof of Theorem 7. Then there exist pairwise disjoint $\{A_k : k \geq 0\} \subset \Sigma$ such that λ restricted to $A_0 \cap \Sigma$ is non-atomic and each A_k for $k \geq 1$ is an atom for λ ([RR; 5.2.13]). In the computations below we tacitly assume that $\{A_k : k \geq 1\}$ is infinite; the case when the set is finite is much simpler. Let $E_0 = A_0$, $E_1 = \bigcup_{k\geq 1} A_k$ and let m_0 (λ_0) be m (λ) restricted to $E_0 \cap \Sigma$ and m_1 (λ_1) be m (λ) restricted to $E_1 \cap \Sigma$. Now m_0 is non-atomic, m_1 is purely atomic and $m_1 = m_0 + m_1$. Hence, we have $L^1(m) = L^1(m_0) \oplus L^1(m_1)$. Since $L^1(m_0)$ is barrelled by Corollary 8, to show that $L^1(m)$ is barrelled it suffices to show that $L^1(m_1)$ is barrelled when m_1 is purely atomic. Since each A_k is a λ-atom and m and λ have the same null sets, m takes on only 2 values, $m(A_k)$ and 0, when restricted to $A_k \cap \Sigma$, i.e., A_k is an atom for m.

We consider the integrability of a function f over E_1. For this we first require a lemma.

LEMMA 9. *Let $f\colon S \to X$ be λ-measurable. Then $f =$ constant λ-a.e. in each A_k for $k > 1$.*

Proof. First suppose f is simple with $f = \sum_{k=1}^{n} C_{B_k} x_k$, where $\{B_i\} \subset \Sigma$ are pairwise disjoint and $x_k \in X$. Since A_k is an atom for λ, $f =$ constant λ-a.e. in A_k. In general, if f is λ-measurable, there exist simple functions $\{f_j\}$ such that $f_j \to f$ λ-a.e. If $f_j = x_j$ λ-a.e. in A_k, then there exists $x \in X$ such that $x_j \to x$ and $f = x$ λ-a.e. in A_k. □

PROPOSITION 10. *Let $f\colon S \to X$ be λ-measurable and suppose $f = x_k$ λ-a.e. in A_k (Lemma 9) for $k > 1$. Then f is m-integrable over E_1 if and only if $\sum_k m(A_k)x_k$ is subseries convergent. In this case,*

$$\int_{E_1} f \, dm = \sum_k m(A_k)x_k .$$

Proof. If f is m-integrable over E_1, then $\int f \, dm$ is countably additive so $\int_{E_1} f \, dm = \sum_k \int_{A_k} f \, dm = \sum_k m(A_k)x_k$ ([D1; Theorem 3]).

For the converse, set $f_n = \sum_{k=1}^{n} C_{A_k} x_k$ so $f_n \to f$ λ-a.e. on E_1 and for every $B \in \Sigma$, $B \subset E_1$,

$$\int_B f_n \, dm = \sum_{k=1}^{n} m(A_k \cap B)x_k \to \sum_k m(A_k \cap B)x_k$$

by the subseries convergence of the series ($m(A_k \cap B)$ is either 0 or $m(A_k)$ since A_k is an atom for m). By [D1; Theorem 16], f is m-integrable over E_1 with $\int_{E_1} f \, dm = \sum_k m(A_k)x_k$.

We now define a vector-valued sequence space which is isometric to $L^1(m_1)$. Let $\ell^1(m_1) = \left\{x = \{x_k\} : x_k \in X \text{ and } \sum_k m(A_k)x_k \text{ is subseries convergent}\right\}$ and define a semi-norm on $\ell^1(m_1)$ by $q_1(x) = \sup\left\{\left\|\sum_{k\in\sigma} m(A_k)x_k\right\| : \sigma \subset \mathbb{N}\right\}$. Then $L^1(m_1)$ and $\ell^1(m_1)$ are linearly isometric under the correspondence $x = \{x_k\} \leftrightarrow f = \sum_k C_{A_k} x_k$ given by Proposition 10. □

We show that $\ell^1(m_1)$ (and, hence, $L^1(m_1)$) is barrelled by using results for vector sequence spaces established in [LS] and [Sw1]. Let e^k be the scalar sequence with a 1 in the kth coordinate and 0 elsewhere; if $z \in X$, let $e^k \otimes z$ be the X-valued sequence with z in the kth coordinate and 0 elsewhere.

PROPOSITION 11. *If* $x = \{x_k\} \in \ell^1(m_1)$, *then* $x = \sum_k e^k \otimes x_k$ (*i.e.*, $\ell^1(m_1)$ *is an AK-space* ([Sw3])).

Proof. Set $x^n = \sum_{k=1}^{n} e^k \otimes x_k$. Then

$$q_1(x - x^n) = \sup\Big\{ \Big\| \sum_{k\in\sigma} m(A_k)x_k \Big\| : \sigma \subset \{n+1, \dots\} \Big\} \to 0$$

by the subseries convergence of the series $\Sigma m(A_k)x_k$. □

Let $c_0(X')$ be all X'-valued sequences which are norm convergent to 0.

PROPOSITION 12. *Let* $f \in \ell^1(m_1)'$. *Then there exists* $x' = \{x'_k\} \in c_0(X')$ *such that* $\langle f, x\rangle = \sum_k \langle x'_k, x_k\rangle$ *for every* $x = \{x_k\} \in \ell^1(m_1)$.

Proof. Define $x'_k \colon X \to \mathbb{R}$ by $\langle x'_k, z\rangle = \langle f, e^k \otimes z\rangle$. Then $x'_k \in X'$ and $\|x'_k\| \le \|f\|\,\|m(A_k)\|$ since

$$|\langle x'_k, z\rangle| \le \|f\|\,\|e^k \otimes z\| = \|f\|\,\|m(A_k)z\| \le \|f\|\|m(A_k)\|\|z\|\,.$$

Since $m(A_k) \to 0$, $x' = \{x'_k\} \in c_0(X')$, and by Proposition 11

$$\langle f', x\rangle = \sum_k \langle f, e^k \otimes x_k\rangle = \sum_k \langle x'_k, x_k\rangle\,.$$

The β-dual of an X-valued sequence space E is defined to be $E^\beta = \Big\{x' = \{x'_k\} : x'_k \in X',\ \sum_k \langle x'_k, x_k\rangle$ converges for all $x = \{x_k\} \in E\Big\}$ ([LS]). Thus, Proposition 12 asserts that $\ell^1(m_1)' \subset \ell^1(m_1)^\beta$. We next want to show that $\ell^1(m_1)' = \ell^1(m_1)^\beta$. For this we employ a result of [LS]. Let E be an X-valued sequence space which has a vector topology. If $x = \{x_k\} \in E$ and $\sigma \subset \mathbb{N}$, let $C_\sigma x$ be the coordinatewise product of x and the characteristic function C_σ. By an interval in $\mathbb{N}$ we mean a set of the form $[m, n] = \{j \in \mathbb{N} : m \le j \le n\}$, where $m, n \in \mathbb{N}$; a sequence of intervals $\{I_j\}$ in $\mathbb{N}$ is increasing if $\max I_j < \min I_{j+1}$. The space E has the zero gliding hump property (0-GHP) if whenever $x^k \to 0$ in E and $\{I_k\}$ is an increasing sequence of intervals, there exists a subsequence $\{n_k\}$ such that $x = \sum_{k=1}^{\infty} C_{I_{n_k}} x^{n_k} \in E$, where the series converges coordinate-wise ([LS]). □

PROPOSITION 13. $\ell^1(m_1)$ *has* 0-*GHP.*

Proof. Let $x^k = \{x^k_j\} \to 0$ in $\ell^1(m_1)$ and let $\{I_k\}$ be an increasing sequence of intervals. Pick an increasing sequence $\{n_k\}$ such that $q_1(x^{n_k}) < 1/2^k$

and let $x = \sum_{k=1}^{\infty} C_{I_{n_k}} x^{n_k}$ (coordinate-wise sum). We show $x \in \ell^1(m_1)$. Let $\sigma \subset \mathbb{N}$ be infinite and set $z_j = \sum_{i \in I_{n_j} \cap \sigma} m(A_i) x_i^{n_j}$. Since $\|z_j\| < 1/2^j$, the series $\sum_{k \in \sigma} m(A_k) x_k = \sum_{j=1}^{\infty} z_j$ converges, i.e., the series $\sum m(A_k) x_k$ is subseries convergent. □

From [LS; Corollary 4] and Proposition 12, we obtain:

PROPOSITION 14. $\ell^1(m_1)^\beta = \ell^1(m_1)'$.

An X-valued sequence space E is said to be monotone if $C_\sigma x \in E$ whenever $x \in E$ and $\sigma \subset \mathbb{N}$. $\ell^1(m_1)$ is obviously monotone. Using the results above and results from [Sw1], we obtain:

COROLLARY 15. *$\sigma(\ell^1(m_1), \ell^1(m_1)')$ is sequentially complete and $\ell^1(m_1)$ is barrelled.*

P r o o f. Since $\ell^1(m_1)$ is monotone, $\sigma(\ell^1(m_1), \ell^1(m_1)^\beta)$ is sequentially complete by [Sw1; Theorem 8]. The first statement now follows from Corollary 14.

Any weakly sequentially complete space is a Banach-Mackey space ([Wi; 10.4.8]) and since $\ell^1(m_1)$ is normed, $\ell^1(m_1)$ is barrelled ([Wi; 10.4.12]). □

From Corollaries 8 and 15, we have:

THEOREM 16. *If m is countably additive with respect to the norm topology of $L(X, Y)$, then $L^1(m)$ is barrelled.*

PROBLEM. Does the conclusion of Theorem 16 hold if m is countably additive in the strong operator topology?

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