Rastislav Potocký

On random variables having values in a vector lattice

*Mathematica Slovaca*, Vol. 27 (1977), No. 3, 267--276

Persistent URL: http://dml.cz/dmlcz/131648

Math. Slovaca 27, 1977, No. 3, 267—276

# ON RANDOM VARIABLES HAVING VALUES IN A VECTOR LATTICE

RASTISLAV POTOCKY

In this paper an integration theory of the Daniel type for functions with values in a vector lattice is presented. The integral is defined on a simple family of functions first, then a method of extension is used. We begin with elementary functions. (For terminology see [1]).

**Definition 1.1.** *Let $(\Omega, S, P)$ be a probability space, $X$ be any Dedekind $\sigma$-complete vector lattice. A function $f$: $\Omega \rightarrow X$ is said to be an elementary random variable if there are a sequence of mutual disjoint sets $E_i$, $E_i \in S$, $\cup E_i = \Omega$ and a sequence $\{x_i\}$ of elements of $X$ such that $f(\omega) = x_i$ for every $\omega \in E_i$. An elementary random variable is said to be integrable if the series $\Sigma x_i P(E_i)$ is absolutely o-convergent. The integral of $f$ is then defined as follows*

$$\int f \, \mathrm{d}P = \Sigma x_i P(E_i).$$

The set of all elementary random variables will be denoted by $E$. The order relation in $E$ is defined in the usual manner, i.e. $f \leq g$ iff $f(\omega) \leq g(\omega)$ for every $\omega \in \Omega$. The order convergence in $X$ and the order relation in $E$ then imply an order convergence in $E$. This will be denoted by $f_n \rightarrow f$ or $f_n \uparrow f$ (resp. $f_n \downarrow f$) if $f_n$ is increasing (resp. decreasing).

The integral as defined in 1.1. is a non-negative linear operator on $E$. In order to prove that it is continuous under monotone limits, i.e. that $f_n \downarrow 0$ implies $\int f_n \, \mathrm{d}P \downarrow 0$, we need the following two lemmas.

**Lemma 1.1.** *Let $T$ be an o-continuous linear functional defined on $X$. Then $T^+(T^-)$, $T^+(x) = \sup T\langle 0, x\rangle$, $x \geq 0$ $(T^-(x) = \sup T\langle -x, O\rangle$, $x \geq 0)$ is* o-continuous.

Proof. Given a sequence $\{x_n\}$ in $X$ such that $x_n \downarrow 0$ we have to prove that $T^+(x_n) \downarrow 0$. If not, there exists a positive number $c$ such that $T^+(x_n) \downarrow c$. Hence we have $T^+(x_n) \geq c$ for every $n$. It follows from the definition of $T^+$ that for every $n$ there exists a $z_n \in \langle 0, x_n\rangle$ such that $c - \frac{c}{2^n} \leq T(z_n)$. It is evident that $z_n \rightarrow 0$ and consequently $T(z_n) \rightarrow 0$. On the other hand $c \leq \lim T(z_n)$, a contradiction.

**Lemma 1.2.** *If $f_n \in E(R)$, i.e. elementary random variables with values in $R$ (the field of real numbers), $f_n$ integrable, then $f_n \downarrow 0$ implies $\int f_n \, \mathrm{d}P \downarrow 0$.*

Proof. See [3], page 126.

**Theorem 1.1.** *Let $(\Omega, S, P)$ be a probability space, $X$ be a locally convex space with an ordering given by a closed cone $K$. Let $x_n \xrightarrow{o} x$ imply $T(x_n) \to T(x)$ for every $T \in X^*$. Then $f_n \in E$, $f_n$ integrable and $f_n \downarrow 0$ imply $\int f_n \, \mathrm{d}P \downarrow 0$.*

Proof. Given any $z > 0$, $z \in X$ we have to prove the existence of a natural number $n$ such that $\int f_n \, \mathrm{d}P \notin \langle z, \infty)$. Since $z > 0$ and $K$ is a closed cone, there exists a continuous linear functional $T$ on $X$ such that $0 < b = \inf T\langle z, \infty)$. Since $T$ is o-continuous, by hypothesis, and since every o-continuous linear functional is o-bounded (i.e. maps o-bounded sets into bounded sets), if follows from the Riesz decomposition theorem that $T$ is the difference between two monotone linear functionals, namely $T^+$ and $T^-$. By lemma 1.1. $T^+$ and $T^-$ are o-continuous linear monotone functionals. It is clear that for such functionals $f_n \downarrow 0$ implies $T^+ f_n \downarrow 0$ ($T^- f_n \downarrow 0$), i.e. $T^+ f_n(\omega) \downarrow 0$ for every $\omega$($T^- f_n(\omega) \downarrow 0$ for every $\omega$). It follows by lemma 1.2. that $\int T^+ f_n \, \mathrm{d}P \downarrow 0$ ($\int T^- f_n \, \mathrm{d}P \downarrow 0$). Moreover we have $T^+ \int f_n = T^+$ ($o$-$\lim \Sigma x_i P(E_i) = o$-$\lim T^+(\Sigma x_i P(E_i) = o$-$\lim \quad \Sigma T^+(x_i) P(E_i) = \Sigma T^+(x_i) P(E_i) =$ $\int T^+ f_n \, \mathrm{d}P$. (Similarly for $T^-$). Thus there exists a natural number $m$ such that for every $n \geqslant m$, $T^+ \int f_n \, \mathrm{d}P \leqslant \frac{b}{4}$, $T^- \int f_n \, \mathrm{d}P \leqslant \frac{b}{4}$. Since for every $x \geqslant 0$ $|Tx| \leqslant T^+ x + T^- x$, it follows that $|T \int f_n \, \mathrm{d}P| \leqslant T^+ \int f_n \, \mathrm{d}P + T^- \int f_n \, \mathrm{d}P \leqslant \frac{b}{2}$.

It is not difficult to present the examples of spaces in which the conditions of theorem 1.1. are fulfilled. We know that in some spaces (e.g. in complete metrizable topological linear spaces ordered by a closed cone) the last condition in theorem 1.1. implies the so-called normality of cone, which consequently implies that $x_n \downarrow x$ in the topology of the space $X$, whenever $x_n \downarrow x$ in order (see [2], [4] and [5] for detailed discussion). It is worth mentioning that this is not always the case.

Example 1. Let $F$ be the space of all real sequences having only a finite number of non-zero terms ordered by the partial sum cone $P_s$, i.e. by the set of all sequences in $F$ having all partial sums non-negative. Let the supremum norm be given on $F$, i.e. $\|x\| = \sup_i |x_i|$.

We recall that a cone $K$ gives a normal ordering of a normed linear space iff there exists $a > 0$ such that for $x$, $y$ in $K \|x + y\| \geqslant a\|x\|$. Jameson has shown (see [4], page 94) that $P_s$ is not normal in $F$ with respect to the supremum norm.

It follows that $P_s$ is closed in this topology. If this is not the case, there exists a sequence $\{x_n\}$ of elements of $P_s$ converging in the topology to a $x$, which does not belong to $P_s$. From it there follows the existence of a natural number $k$ such that

$s_k = \sum_1^k x_i = c < 0$. Put $d = -c$ and consider $\frac{d}{2k}$. There exists a natural number $n_0$ such that for every $n \geqslant n_0$ $\|x_n - x\| < \frac{d}{2k}$, i.e. $|x_{ni} - x_i| < \frac{d}{2k}$ for every $i$. We have

$$\sum_1^k x_{ni} = \sum_1^k (x_{ni} - x_i) + \sum_1^k x_i < \frac{d}{2} + c = \frac{c}{2} < 0.$$

It is well known that every continuous linear functional on $F$ may be written in the form $f(x) = \Sigma x_i u_i$, $\{u_i\}$ a sequence in $l$. All we need to prove is the o-continuity of such functionals. Let a sequence $x_n$ of elements of $F$ such that $x_n \downarrow 0$ be given. It follows that $\inf_n s_k^n = \inf_n \Sigma\ x_i^n = 0$ for every $k$. (Indeed, the existence of a natural number $k$ such that $\inf_n s_k^n \geqslant c > 0$, i.e. $s_k^n \geqslant c$ for every $n$ leads to a contradiction, since the element of $F$ having $c$ in the place $k$, $-c$ in the place $k+1$ and 0 elsewhere precedes all $x_n$, but does not precede 0).

Denote $\max_i s_i^1$ by $L$ (it exists, since all $s_i^1$ are the same from an index $j$). It follows that $|s_i^n| \leqslant L$ for every $(i, n)$, since $x_n$ is decreasing. Given $b > 0$ there exists $N$ such that $\sum_{N+1} |u_i| < \frac{b}{6L}$. Put $\max_{1 \leqslant i \leqslant N} |u_i| = M$. Then a natural number $N_0$ exists such that for every $n \geqslant N_0$ and for every $i \in \{1, \ldots, N\}$ $|s_i^n| < \frac{b}{6NM}$. We have, consequently,

$$|f(x^n)| \leqslant \Sigma |x_i^n|\ |u_i| \leqslant \Sigma |s_i^n|\ |u_i| + \Sigma |s_{i-1}^n|\ |u_i| =$$
$$= \sum_1^N |s_i^n|\ |u_i| + \sum_1^N |s_{i-1}^n|\ |u_i| + \sum_{N+1}^\infty |s_i^n|\ |u_i| +$$
$$+ \sum_{N+1}^\infty |s_{i-1}^n|\ |u_i| \leqslant 2 \sum_1^N \frac{b}{6NM} |u_i| + 2L \sum_{N+1}^\infty |u_i| < b.$$

In the general case, $z_n \to 0$ implies the existence of a sequence $x_n \downarrow 0$ such that $|z_n| \leqslant x_n \downarrow 0$. Hence we have

$$|f(z_n)| = |\Sigma z_i^n u_i| \leqslant \Sigma |z_i^n|\ |u_i| = \Sigma |S_i^n - S_{i-1}^n|\ |u_i| \leqslant$$
$$\leqslant \Sigma |s_i^n|\ |u_i| + \Sigma |s_{i-1}^n|\ |u_i|,$$

$S_i^n$, $s_i^n$ denoting the partial sums of $z_n$ and $x_n$, respectively.

We present now sufficient conditions for the assumptions of theorem 1.1. to be fulfilled. The proofs are known and therefore may be omitted.

**Proposition 1.2.** *Let $X$ be a semireflexive locally convex space ordered by a closed cone and let every continuous linear functional be absolutely majorized (i.e. $|f(x)| \leq g(x)$, $x \geq 0$, $g(x)$ a monotone functional). Then $x_n \xrightarrow{o} x$ implies $T(x_n) \to T(x)$ for every $T \in X^*$.*

Proof. See [4].

**Proposition 1.3.** *Let $X$ be a complete topological linear space, ordered by a closed cone with a bounded base $B$ such that $0 \notin \bar{B}$. Then $x_n \xrightarrow{o} x$ implies $T(x_n) \to T(x)$ for every $T \in X^*$.*

Proof. See [2].

It is interesting that in some cases we can do without topology, i.e. we can state and prove theorem 1.1. replacing continuous linear functionals by $o$-bounded ones. From now on $X^+$ denotes $o$-dual of $X$, i.e. the set of all $o$-bounded linear functionals on $X$. We recall that $X$ is said to be regularly ordered by a cone, if $X^+$ separates points of $X$.

**Theorem 1.4.** *Let $(\Omega, S, P)$ be a probability space, $X$ be a vector lattice regularly ordered by a cone. If every o-bounded linear functional is o-continuous, then $f_n \downarrow 0$, $f_n$ elementary integrable random variables, implies $\int f_n \, dP \downarrow 0$.*

Proof. Analogous to that of theorem 1.1.

**Corollary 1.5.** *Let $X$ be a linear space ordered by a cone with the order topology, i.e. the largest locally convex topology making all o-intervals bounded. If every o-bounded linear functional is o-continuous, then $f_n \downarrow 0$, $f_n$ elementary integrable random variables implies $\int f_n \, dP \downarrow 0$.*

Proof. In such space $X^+$ separates points of $X$.

There is another interesting application of theorem 1.4. We recall that a vector lattice is called a regular vector lattice if it has the diagonal property for $o$-convergence (see [1]). For such spaces the following theorem holds, which should be compared with [6], lemma 2.

**Theorem 1.6.** *Let $X$ be a Dedekind $\sigma$-complete regular vector lattice. If $X^+$ separates points of $X$, then $f_n \downarrow 0$ implies $\int f_n \, dP \downarrow 0$.*

Proof. If a vector lattice has the diagonal property, then every o-bounded linear functional is o-continuous (see [4]).

## II.

**Definition 2.1.** *A function $f$ defined on $\Omega$ and having values in $X$ is said to be a random variable if there exist an increasing sequence $\{f_n\}$ of elementary random* *variables such that $f_n \uparrow f$ and a decreasing sequence $\{g_n\}$ of elementary random variables such that $g_n \downarrow f$.*

Remark. Let $X$ be a metric space. A function $f\colon \Omega \to X$ is called a *random element*, iff $f^{-1}(B) \in S$ for every Borel set $B$ of $X$.

We recall that a subset $H$ of a cone $K$ in an ordered vector space exhausts $K$ if for every $x \in K$ there exists $h \in H$ and a natural number $n$ such that $x \leqslant nh$ (see [4]).

**Proposition 2.1.** *Let $X$ be a complete separable linear metric space ordered by a closed cone $K$ such that a countable set exhausts $K$ and the mapping $x \to x^+$ is continuous at 0. Then every random element is a random variable.*

Proof. Let $D$ be a countable dense set in $X$. For every $x \in X$ there exists a sequence $\{x_n\} \subset D$, converging to $x$, i.e. $r(x_n - x, 0) \to 0$. Hence, by hypothesis $r(|x_n - x|, 0) \to 0$. Therefore we can choose a subsequence $x_{n_k}$ such that $r(|x_{n_k} - x|, 0) < \frac{1}{k^3}$. It follows that the series $\Sigma k|x_{n_k} - x|$ metric-converges. Denoting its sum by $z$, we have $|x_{n_k} - x| \leqslant \frac{1}{k} z \leqslant \frac{1}{k} h$, $h \in H$, since $K$ is closed and denoting $x_{n_k} - \frac{1}{k} h$ by $z_{n_k}$, we have that $|z_{n_k} - x| \leqslant \frac{2}{k} h$, $z_{n_k} \leqslant x$.

Thus there exist sequences $z_n$ and $u_n$ such that the following set equality holds:

$$X = \bigcup_{u_n} \bigcap_{k} \bigcup_{x_k} \left\{x;\ |x - x_k| \leqslant \frac{1}{k} u_n\right\} - \\ - \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k-1} \left\{x;\ |x - x_i| \leqslant \frac{1}{k} u_m\right\}.$$

We define the countably-valued functions $T_k$ as follows

$$T_k(x) = x_1 \quad \text{if} \quad x \in \bigcup_{u_n} \left\{x;\ |x - x_1| \leqslant \frac{1}{k} u_n\right\} \quad \text{and}$$

$$T_k(x) = x_s \quad \text{if} \quad x \in \bigcup_{u_n} \left\{x;\ |x - x_s| \leqslant \frac{1}{k} u_n\right\} - \\ - \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{s-1} \left\{x;\ |x - x_i| \leqslant \frac{1}{k} u_m\right\}.$$

Then we put

$$T_1^*(x) = T_1(x), \quad T_k^*(x) = T_{k-1}^*(x) \vee T_k(x).$$

The next example shows that the continuity of the mapping $x \to x^+$ at 0 does not imply, by itself, the normality of the corresponding cone.

Example 2. Consider the space $l$ ordered by the partial sum cone $P_s$ with the norm $\|x\| = \Sigma |x_n| \cdot P_s$ is not normal with respect to this norm, since

$$x_n = e_1 - \frac{1}{2} e_2 + \frac{1}{3} e_3 - \ldots \pm \frac{1}{n} e_n$$

and

$$y_n = \frac{1}{2} e_2 - \frac{1}{3} e_3 + \ldots \mp \frac{1}{n} e_n$$

belong to $P_s$ and we have $\|x_n + y_n\| = \|e_1\| = 1$, but

$$\|x_n\| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \quad \|y_n\| = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

We show now that $x \to x^+$ is continuous for all $x$. Given $c > 0$, there exists $N$ such that $|x_r| < c$ for $r > N$. Consider $y \in l$ such that $\|y - x\| \leq \frac{c}{N}$. Define $X_r = x_1 + \ldots + x_r$, $X_0 = 0$ and $Y_r = y_1 + \ldots + y_r$, $Y_0 = 0$. We have $x^+ = \{a_r\}$, $y^+ = \{b_r\}$, where $a_r = X_r^+ - X_{r-1}^+$, $b_r = Y_r^+ - Y_{r-1}^+$. It follows that

$$\|y^+ - x^+\| = \sum |b_r - a_r| \leq 2 \sum_{r=1}^{N-1} |Y_r - X_r| + \sum_N^{\infty} |b_r - a_r|.$$

From this we obtain

$$\sum_1^{N-1} |Y_r - X_r| \leq \sum_1^{N-1} \sum_1^{r} |y_i - x_i| \leq c$$

and

$$\sum_N^{\infty} |b_r - a_r| \leq \sum_N^{\infty} |b_r| + \sum_N^{\infty} |a_r| \leq \sum_N^{\infty} |y_r| + \sum_N^{\infty} |x_r|$$

$$\sum_N^{\infty} |y_r - x_r| + 2 \sum_N^{\infty} |x_r| \leq 3c, \text{ i.e. } \|y^+ - x^+\| \leq 5c.$$

In the converse direction we can prove the following

**Proposition 2.2.** *Let $X$ be a separable locally convex metrizable linear space with a closed ordering. Let a countable set exhaust the cone. Let $x_n \overset{o}{\to} x$ imply $T(x_n) \to T(x)$ for every $T \in X^*$. Then the uniform o-limit of every sequence of random elements is a random element.*

Proof. Since every closed set $C$ in $X$ can be written in the form $C = \bigcap_p \bigcup_{n=1} \left\{x; |x - x_n|_p \leq \frac{1}{p}\right\}$, where $\{p\}$ is an increasing sequence of seminorms in $X$, we may restrict our attention to convex closed sets. For such a set $C$ we have

$$V^{-1}(C) = \bigcup_{\{o_n\}\in B} \bigcap_{n} \bigcup_{x_n \in C} V_n^{-1}\{x\,;\,|x - x_n| \leq o_n\}$$

both $B$ and $\{x_n\}$ countable subsets of the set of all sequences o-converging to 0 and $C$ respectively.

Indeed the right-hand side of the equality implies that a sequence $x_n \in C$ w-converges to $V(\omega)$, i.e. that $V(\omega)$ belongs to the weak closure of $C$ and hence to the closure of $C$.

**Proposition 2.3.** *Let $X$ be a separable locally convex metrizable linear space with a cone $K$ such that $X^*$ has a countable basis. Let $x_n \xrightarrow{o} x$ imply $T(x_n) \rightarrow T(x)$ for every $T \in X^*$. If $V_n(\omega) \rightarrow V(\omega)$ for every $\omega \in \Omega$, $V_n$ random elements, then $V$ is a random element.*

Proof.

$$V^{-1}(C) = \bigcap_{n} \bigcup_{x_n \in C} \bigcap_{T \in B} \bigcup_{\{a_n\} \in A} V_n^{-1}\{x\,;\,|T(x) - T(x_n)| \leq a_n\},$$

$\{x_n\}$, $B$, $A$ countable subsets of $C$, $X^*$ and the set of all null sequences, respectively.

**Proposition 2.4.** *Let $X$ be a separable locally bounded locally convex metrizable linear space, $X^*$ be separable in the strong topology. Let $x_n \xrightarrow{o} x$ imply $T(x_n) \rightarrow T(x)$ for every $T \in X^*$. If $V_n(\omega) \rightarrow V(\omega)$ for every $\omega \in \Omega$, $\{V_n\}$ a sequence of random elements, then $V$ is a random element.*

Proof.

$$V^{-1}(C) = \bigcup_{\substack{O \subset C \\ O \in B}} \bigcap_{n} \bigcup_{x_n \in A} \bigcap_{T \in D} \bigcup_{\{a_n\} \in E} V_n^{-1}\{x\,;\,|T(x) - T(x_n)| \leq a_n\},$$

$B$ a countable set of bounded subsets of $C$; $A$, $D$, $E$ countable subsets of $C$, $X$ and the set of all sequences converging to 0, respectively.

## III.

Consider now spaces with the following properties (which we refer to as $(A)$ and $(B)$, respectively):

$(A)$ *$X$ is a locally convex space with an ordering given by a closed cone such that $x_n \xrightarrow{o} x$ implies $Tx_n \rightarrow Tx$ for every $T \in X^*$.*

$(B)$ *$X$ is a vector lattice regularly ordered by a cone such that every o-bounded linear functional is o-continuous.*

**Definition 3.1.** *Let $(\Omega, S, P)$ be a probability space, $X$ be a Dedekind $\sigma$-complete vector lattice ordered by a cone such that either $(A)$ or $(B)$ holds. A random variable $f\colon \Omega \to X$ is called integrable if there exist an increasing sequence $\{f_n\}$ of elementary integrable random variables, such that $f_n \uparrow f$ and a decreasing sequence $\{g_n\}$ of elementary integrable random variables such that $g_n \downarrow f$, both with uniformly bounded integrals. The integral of $f$ is defined by $\int f \,\mathrm{d}P = \lim_n \int f_n \,\mathrm{d}P$.*

This definition is justified, since one can show that the value of integral does not depend on the choice of the sequence $\{f_n\}$ and the sequence $\{g_n\}$.

It is easy to show that the integral just defined is a monotone linear operator on the set of all integrable functions. It also has the property that the absolute value of an integrable function as well as its positive and negative parts are integrable.

**Theorem 3.1.** *Let $(\Omega, S, P)$ be a probability space, $X$ be a Dedekind $\sigma$-complete vector lattice ordered by a cone such that either $(A)$ or $(B)$ holds. If $\{f_n\}$ is an increasing sequence of integrable random variables which converges to a function $f$ bounded from above by an integrable random variable and such that* $\lim \int f_n \,\mathrm{d}P$ *exists, then $f$ is integrable and $\int f \,\mathrm{d}P = \lim_n \int f_n \,\mathrm{d}P$.*

Proof. For each $n$ there is an increasing sequence $\{u_{ni}\}$ of elementary integrable random variables such that $|u_{ni} - f_n| \leq z_{ni}$, $\inf_i z_{ni} = 0$. Define $u_n = \sup_{k \leq n} u_{kn}$. We hawe

$$0 \leq f - u_n \leq \inf_{k \leq n} |f - u_{kn}| \leq \inf_{k \leq n} \{|f - f_k| + |f_k - u_{kn}|\} \leq$$

$$\leq \inf_{k \leq n} \{z_{kn} + o_k\} = w_n$$

and it is immediately that $\inf_n w_n = 0$. The rest of the proof follows easily.

**Theorem 3.2.** *If $f_n$ is a sequence of integrable functions which converges to a function $f$ such that* $\lim \int f_n$ *exists and if $g$ is an integrable function such that $|f_n| \leq g$ for every $n$, then $f$ is integrable and $\int f \,\mathrm{d}P = \lim_n \int f_n \,\mathrm{d}P$.*

**Theorem 3.3.** *Let $g$ and $h$ be integrable functions and let $f_n$ be a sequence of integrable functions such that $f_n \geq g$ resp. $f_n \leq h$. Then, if* $\liminf \int f_n \,\mathrm{d}P < \infty$, *the function* $\liminf f_n$ *is integrable and* $\int \liminf f_n \,\mathrm{d}P \leq \liminf \int f_n \,\mathrm{d}P$, *resp. if* $\limsup \int f_n \,\mathrm{d}P > -\infty$, *the function* $\limsup f_n$ *is integrable and* $\int$ *lim sup* $f_n \,\mathrm{d}P \geq \limsup \int f_n \,\mathrm{d}P$.

At the end of this paper we give the outline of two other possible approaches to the problem we are dealing with.

**Definition 3.2.** *Let the assumptions be as above; a random variable f is said to be integrable if there exist a sequence* $\{f_n\}$ *of elementary integrable random variables and a sequence* $\{g_n\}$ *of elementary integrable random variables such that* $g_n \downarrow 0$ *and* $|f_n - f| \leq g_n$. *The integral of f is defined by* $\int f \, \mathrm{d}P = \lim_n \int f_n \, \mathrm{d}P$.

This definition is justified, since a sequence $x_n$ in a Dedekind $\sigma$-complete vector lattice o-converges iff the double sequence $|x_m - x_n|$ $o$-converges to 0.

We present now an analogue of theorem 3.1.

**Theorem 3.4.** *Let* $(\Omega, S, P)$ *be a probability space, X be a Dedekind σ-complete vector lattice such that either (A) or (B) holds. If a sequence* $\{f_n\}$ *of integrable random variables converges to a random variable f, i.e.* $|f_n - f| \leq o_n$, $\{o_n\}$ *a sequence of elementary integrable random variables which converges to 0, then f is integrable and* $\int f \, \mathrm{d}P = \lim_n \int f_n \, \mathrm{d}P$.

There is another way how to define the integral of a random variable. Following an idea of Bochner, we introduce the next

**Definition 3.3.** *Let the assumptions be as above, a random variable f is called integrable if a sequence* $\{f_n\}$ *of elementary integrable random variables such that* $\lim \int f_n \, \mathrm{d}P$ *exists, tends to f in order and if, moreover,* $T|f_n| \leq T|f|$, *for every linear functional mentioned in (A) and (B), respectively. Then we define* $\int f \, \mathrm{d}P = \lim_n \int f_n \, \mathrm{d}P$.

One can easily show, by using the method of theorems 1.1. and 1.4. that this definition is justified. Similarly we can prove the analogues of theorems 3.1., 3.2. and 3.3.

Remark. It should be emphasized that there are several papers discuissing similar problems, however, from other points of view. Among them the most interesting seem to be papers [7], [8], [9].

REFERENCES

[1] LUXEMBURG, W. A., ZAANEN, A. C.: Riesz spaces 1. Amsterdam 1971.
[2] JAMESON, G.: Ordered linear spaces. Berlin 1970.
[3] KAPPOS, D. A.: Probability algebras and stochastic spaces. New York 1970.
[4] PERESSINI, A. L.: Ordered topological linear spaces. New York 1967.
[5] SCHAEFER, H. H.: Topological vector spaces. New York 1966.
[6] POTOCKÝ, R.: On the integration of functions with values in complete vector lattices. Acta F.R.N. Univ. Comen. — Mathematica XXVI, 1972, 83—91.
[7] WRIGHT, J. D. MAITLAND: Stone — algebra — valued measures and integrals. Proc. London math. soc. 19, 1969, 107—122.
[8] RIEČAN, B.: О продолжении операторов с значениями в линейных полуупорядоченных пространствах. Čas. pest. mat. 93, 1968, 459—471.

[9] VRACIU, G.: V-integrala in spatii liniare ordonate. Studii si cerc. mat. 26, 1974, No 7, 1051—1055.

Received December 10, 1975

*Katedra numerickej matematiky
a matematickej štatistiky
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava*

## О СЛУЧАЙНЫХ ВЕЛИЧИНАХ С ЗНАЧЕНИЯМИ В НЕКОТОРОЙ ВЕКТОРНОЙ РЕШЕТКЕ

Растислав Потоцки

Резюме

Функция $f$, определенная на вероятностном пространстве ($\psi$, $S$, $P$) со значениями в некоторой векторной решетке называется элементарной случайной величиной, если существует полная счетная система попарно непересекающихся событий такая, что $f$ постоянна на всяком элементе системы. Элементарная случайная величина называется интегрируемой, если естественным способом определенный ряд абсолютно сходится по упорядочению. Сумма ряда называется интегралом от $f$.

В первой части работы приводятся достаточные условия, при которых выше определенный интеграл является непрерывным линейным оператором. Во второй части вводится понятие случайной величины как порядкового предела двух последовательностей – возрастающей и убывающей – элементарных случайных величин. В последней части излагается теория интеграла и доказываются некоторые классические результаты.