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KALMAN FILTER WITH VARIANCE COMPONENTS[1]

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ABSTRACT. Predicting and interpolating in the Kalman filter depend on the knowledge of stochastical properties both of the process and measurement noise. The case when one unknown parameter $\sigma^2_{\mathbf{Q}}$ occurs in the process noise and one unknown parameter $\sigma^2_{\mathbf{R}}$ occurs in the measurement noise whose ratio $\sigma^2_{\mathbf{Q}}/\sigma^2_{\mathbf{R}}$ is also unknown is investigated in the paper.

Introduction

The main feature of algorithms of Kalman filters is their iterative character and the possibility to realize all calculations on line. The necessity of this is quite obvious, e.g., in the case of tracking positions of a moving satellite. The same approach must be respected in the case of estimating parameters of covariance matrices. Thus, an attempt is made in the paper to construct the MINQUE procedure which can be realized on line.

In what follows, $\boldsymbol{Y}$ denotes an n-dimensional random (observation) vector; the notations $E(\boldsymbol{Y} \mid \boldsymbol{\beta})$ and $\operatorname{Var}(\boldsymbol{Y} \mid \boldsymbol{\vartheta})$ are used for its mean value and covariance matrix, respectively. Here $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$ are parameters of the distribution function of the vector $\boldsymbol{Y}$ (the general notations $E(\boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta})$ and $\operatorname{Var}(\mathbf{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta})$ are not used here as a consequence of the assumption that the mean value and the covariance matrix are independent of the parameter $\boldsymbol{\vartheta}$ and $\boldsymbol{\beta}$, respectively).

1. Definitions and auxiliary statements

DEFINITION 1.1. Let $\boldsymbol{x}_0 \in \mathbb{R}^k$ (k-dimensional Euclidean space) be an unknown vector and

$$\boldsymbol{x}_j = \mathbf{A}_{j-1}\boldsymbol{x}_{j-1} + \boldsymbol{\Gamma}_{j-1}\boldsymbol{\xi}_{j-1}\,, \qquad j = 1, 2, \dots\,,$$

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where $\mathbf{A}_0, \mathbf{A}_1, \dots$ are given $k \times k$ regular matrices, $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \dots$ given matrices and $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots$ is a sequence of random vectors (process noise) which are stochastically independent and such that $E(\boldsymbol{\xi}_j) = \boldsymbol{O}$, $\operatorname{Var}(\boldsymbol{\xi}_j) = \overline{\mathbf{Q}}_j = \sigma^2_{\mathbf{Q}} \mathbf{Q}_j$, $j = 0, 1, \dots$.

The position of the jth point $\boldsymbol{x}_j$ of the orbit $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ is indirectly measured by an observation vector

$$\boldsymbol{v}_j = \mathbf{C}_j \boldsymbol{x}_j + \boldsymbol{\eta}_j \,, \qquad j = 0, 1, \dots \,,$$

where $\mathbf{C}_j$ is a given $n \times k$ matrix whose rank $r(\mathbf{C}_j) = k \le n$ (in what follows, usually $k < n$ is assumed), and $\boldsymbol{\eta}_j$ is an n-dimensional random vector (measurement noise) such that $E(\boldsymbol{\eta}_j) = \boldsymbol{O}$ and $\operatorname{Var}(\boldsymbol{\eta}_j) = \overline{\mathbf{R}}_j = \sigma^2_{\mathbf{R}} \mathbf{R}_j$, $j = 0, 1, \dots$. All the vectors $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\eta}_0, \boldsymbol{\eta}_1, \dots$ are stochastically independent and all the matrices $\overline{\mathbf{Q}}_0, \overline{\mathbf{Q}}_1, \dots, \overline{\mathbf{R}}_0, \overline{\mathbf{R}}_1, \dots$ are positively definite (p.d.).

In the following the matrices $\mathbf{Q}_0, \mathbf{Q}_1, \dots$ and $\mathbf{R}_0, \mathbf{R}_1, \dots$ will be considered to be known while the factors $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ will be considered to be unknown.

The algorithm, which gives the best prediction (in the mean square error sense) of $\boldsymbol{x}_k$ on the basis of $(\boldsymbol{v}'_0, \dots, \boldsymbol{v}'_j)'$ ($'$ denotes the transposition) and its correction caused by an adding of the observation $\boldsymbol{v}_{j+1}$ is called the *Kalman filter* (*KF*). Sometimes it is called the *discrete KF* as the continuous version can also be considered (in more detail, see, e.g., [1] and [6]).

LEMMA 1.2. *Let $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ be a priori known, i.e., $\overline{\mathbf{Q}}_0, \overline{\mathbf{Q}}_1, \dots$ and $\overline{\mathbf{R}}_0, \overline{\mathbf{R}}_1, \dots$ are known, and let $\hat{\boldsymbol{x}}_{k|j}$ be the best prediction of $\boldsymbol{x}_k$ on the basis of the vector $(\boldsymbol{v}'_0, \dots, \boldsymbol{v}'_j)'$,*

$$\mathbf{\Phi}_{j,k} = \mathbf{A}_j^{-1} \mathbf{A}_{j+1}^{-1} \dots \mathbf{A}_{k-1}^{-1} \,, \quad j < k \,, \qquad \mathbf{\Phi}_{k,k} = \mathbf{I} \quad (\textit{identical matrix}),$$

$$\mathbf{H}_{k,j} = \begin{pmatrix} \mathbf{C}_0 \mathbf{\Phi}_{0,k} \\ \mathbf{C}_1 \mathbf{\Phi}_{1,k} \\ \vdots \\ \mathbf{C}_j \mathbf{\Phi}_{j,k} \end{pmatrix}, \qquad \varepsilon_{k,j} = \begin{pmatrix} -\mathbf{C}_0 \sum\limits_{i=1}^{k} \mathbf{\Phi}_{0,i} \mathbf{\Gamma}_{i-1} \boldsymbol{\xi}_{i-1} + \boldsymbol{\eta}_0 \\ -\mathbf{C}_1 \sum\limits_{i=2}^{k} \mathbf{\Phi}_{1,i} \mathbf{\Gamma}_{i-1} \boldsymbol{\xi}_{i-1} + \boldsymbol{\eta}_1 \\ \vdots \\ -\mathbf{C}_j \sum\limits_{i=j+1}^{k} \mathbf{\Phi}_{j,i} \mathbf{\Gamma}_{i-1} \boldsymbol{\xi}_{i-1} + \boldsymbol{\eta}_j \end{pmatrix},$$

$\operatorname{Var}(\varepsilon_{k,j}) = \mathbf{W}_{k,j}^{-1}$ *and* $\mathbf{P}_{k,j} = (\mathbf{H}'_{k,j} \mathbf{W}_{k,j} \mathbf{H}_{k,j})^{-1}$.

Then

$$\begin{aligned}
\hat{\mathbf{x}}_{k-1|k-1} &= \mathbf{P}_{k-1,k-1}\mathbf{H}'_{k-1,k-1}\mathbf{W}_{k-1,k-1}(\boldsymbol{v}'_0, \boldsymbol{v}'_1, \dots, \boldsymbol{v}'_{k-1})'\,, \\
\hat{\mathbf{x}}_{k|k-1} &= \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}\,, \\
\mathbf{P}_{k,k-1} &= \mathbf{A}_{k-1}\mathbf{P}_{k-1,k-1}\mathbf{A}'_{k-1} + \boldsymbol{\Gamma}_{k-1}\overline{\mathbf{Q}}_{k-1}\boldsymbol{\Gamma}'_{k-1}\,, \\
\mathbf{G}_k &= \mathbf{P}_{k,k-1}\mathbf{C}'_k\big(\overline{\mathbf{R}}_k + \mathbf{C}_k\mathbf{P}_{k,k-1}\mathbf{C}'_k\big)^{-1}\,, \\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{G}_k\big(\boldsymbol{v}_k - \mathbf{C}_k\hat{\mathbf{x}}_{k|k-1}\big)\,, \\
\mathbf{P}_{k|k} &= (\mathbf{I} - \mathbf{G}_k\mathbf{C}_k)\mathbf{P}_{k,k-1}\,, \\
\hat{\mathbf{x}}_{k+1|k} &= \mathbf{A}_k\hat{\mathbf{x}}_{k|k} \dots \quad \textit{etc.}
\end{aligned} \tag{1.1}$$

Proof see, e.g., in [1; p. 27]. □

Remark 1.3. Lemma 1.2 is one of results of the KF theory. It demonstrates the important feature of this theory, namely, its iterative character. However, the best predictors given in (1.1) can be calculated only if the factors $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ are known, which is not our case. As the KF theory is based on the "on line" approach, the estimation of $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ should respect this approach as well. One of possible algorithms respecting this approach is given in Section 2.

Definition 1.4. The model

$$\left(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right), \qquad \boldsymbol{\beta} \in \mathbb{R}^k, \ \ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} \subset \mathbb{R}^p\,, \tag{1.2}$$

is said to be a *linear model with variance-covariance components.* If $r(\mathbf{X}) = k < n$ and $\mathrm{Var}(\boldsymbol{Y} \mid \boldsymbol{\vartheta})$ is p.d., the model (1.2) is said to be *regular*; here $\boldsymbol{Y}_{n,1}$ is an n-dimensional random vector, $E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta}$, $\mathbf{X}$ is a known $(n \times k)$-dimensional design matrix, $\boldsymbol{\beta}$ a k-dimensional vector of unknown parameters, $\mathrm{Var}(\boldsymbol{Y} \mid \boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\boldsymbol{\vartheta}} \subset \mathbb{R}^p$ are unknown second order parameters – variance components (their definition domain is assumed to be an open set in $\mathbb{R}^p$), and $\mathbf{V}_1, \dots, \mathbf{V}_p$ are known symmetric matrices such that $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is p.d. (for more detail see [3]).

Let $\boldsymbol{Y}_1$ be a subvector of the vector $\boldsymbol{Y}$ and $\left(\boldsymbol{Y}_1, \mathbf{X}_1\boldsymbol{\beta}_1, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_{i1} = \boldsymbol{\Sigma}_{11}\right)$ the linear submodel of the model (1.2) corresponding to $\boldsymbol{Y}_1$; i.e., $\mathbf{X}_1$ consists of proper rows of the matrix $\mathbf{X}$ and $\mathbf{V}_{i1}$ of proper submatrices of the matrix $\mathbf{V}_i$. Let the mentioned submodel be also regular. If the covariance matrix $\boldsymbol{\Sigma}_{11}$ is known, the *BLUE* (*best linear unbiased estimator*) $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ of $\boldsymbol{\beta}$ based on $\boldsymbol{Y}_1$ is a statistic satisfying the following conditions:

(i) $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) = \mathbf{T}^* \boldsymbol{Y}_1$,
(ii) $E[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) \mid \boldsymbol{\beta}] = \boldsymbol{\beta} \quad \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}$,
(iii) $\forall\{\mathbf{T} \neq \mathbf{T}^*,\ \mathbf{T}$ fulfilling (ii)$\}$ $\quad \mathrm{Var}(\mathbf{T}\boldsymbol{Y}_1 \mid \boldsymbol{\Sigma}_{11}) >_L \mathrm{Var}(\mathbf{T}^*\boldsymbol{Y}_1 \mid \boldsymbol{\Sigma}_{11})$.

Here $>_L$ denotes the ordering of positive semidefinite matrices in the Loewner sense, i.e., $\mathbf{A} >_L \mathbf{B} \iff \mathbf{A} - \mathbf{B}$ is p.s.d.

If $\boldsymbol{\Sigma}_{11}$ is substituted by $\mathbf{I}$, the estimator of $\boldsymbol{\beta}$ is called the *OLS-estimator* (*ordinary least squares-estimator*).

LEMMA 1.5. *Let the covariance matrix* $\begin{pmatrix} \boldsymbol{\Sigma}_{11}, & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21}, & \boldsymbol{\Sigma}_{22} \end{pmatrix}$ *in a regular model*

$$\left[\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_{11}, & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21}, & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right], \qquad \boldsymbol{\beta} \in \mathbb{R}^k, \tag{1.3}$$

be known. Let $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ *and* $\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ *be the BLUEs of* $\boldsymbol{\beta}$ *based on* $\boldsymbol{Y}_1$ *and* $(\boldsymbol{Y}_1', \boldsymbol{Y}_2')'$, *respectively. Then*

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = {} & \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) + (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1})' \\ & \cdot \left[\boldsymbol{\Sigma}_{22.1} + (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'(\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'\right]^{-1} \\ & \cdot \left\{\boldsymbol{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}[\boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)]\right\}.\end{aligned}$$

Here $\boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{21}$.

P r o o f . The model

$$\left[\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Y}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1 \end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_{11}, & \mathbf{O} \\ \mathbf{O}, & \boldsymbol{\Sigma}_{22.1} \end{pmatrix}\right]$$

is equivalent to the model (1.3).

Thus

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = {} & \left[\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1 + (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'\boldsymbol{\Sigma}_{22.1}^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)\right]^{-1} \\ & \cdot \left[\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Y}_1 + (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'\boldsymbol{\Sigma}_{22.1}^{-1}(\boldsymbol{Y}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Y}_1)\right].\end{aligned}$$

Taking into account the equivalences

$$\begin{aligned}& \left[\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1 + (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'\boldsymbol{\Sigma}_{22.1}^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)\right]^{-1} \\ & = (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1} - (\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)' \\ & \cdot \left[\boldsymbol{\Sigma}_{22.1} + (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)(\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)'\right]^{-1} \\ & \cdot (\mathbf{X}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)(\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1},\end{aligned}$$

$$\begin{aligned}&\left[\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1 + (\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)'\Sigma_{22.1}^{-1}(\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)\right]^{-1}\\ &\cdot(\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)'\Sigma_{22.1}^{-1}\\ =\,&(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)'\\ &\cdot\left[\Sigma_{22.1} + (\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)(\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1)'\right]^{-1}\end{aligned}$$

and the expression $\hat{\beta}(\boldsymbol{Y}_1) = (\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\Sigma_{11}^{-1}\boldsymbol{Y}_1$ we easily finish the proof. □

REMARK 1.6. *Let $\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$, where $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$ are stochastically independent random vectors, and $\mathbf{X}$ a given $n \times k$ matrix such that $r(\mathbf{X}) = k < n$. Let the mean value $\overline{\boldsymbol{\beta}} = E[\boldsymbol{\beta}]$ be unknown while the covariance matrices $\mathrm{Var}(\boldsymbol{\beta}) = \Sigma_{\boldsymbol{\beta}}$ and $\mathrm{Var}(\boldsymbol{\varepsilon}) = \Sigma$ are known and, simultaneously, $r(\Sigma) = n$. Then the best (in the mean square error sense) linear prediction of the random vector $\boldsymbol{\beta}$ is*

$$\begin{aligned}\boldsymbol{\beta}^* &= \left[\mathbf{X}'(\Sigma + \mathbf{X}\Sigma_{\boldsymbol{\beta}}\mathbf{X}')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(\Sigma + \mathbf{X}\Sigma_{\boldsymbol{\beta}}\mathbf{X}')^{-1}\boldsymbol{Y} = \hat{\overline{\boldsymbol{\beta}}}\\ &= (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y}.\end{aligned}$$

Here, $\hat{\overline{\boldsymbol{\beta}}}$ is the BLUE of $\overline{\boldsymbol{\beta}}$ based on $\boldsymbol{Y}$, i.e., it equals the BLUE in the linear model $(\boldsymbol{Y}, \mathbf{X}\overline{\boldsymbol{\beta}}, \Sigma + \mathbf{X}\Sigma_{\boldsymbol{\beta}}\mathbf{X}')$.

P r o o f is easy, and therefore it is omitted. □

Thus it can be seen that predictions and least squares estimators in the considered model coincide in the given sense.

DEFINITION 1.7. Within the model $\left(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right)$ (cf. Definition 1.4), the $\boldsymbol{\vartheta}_0$-*MINQUE* (*minimum norm quadratic estimator*) of a linear function $g(\boldsymbol{\vartheta}) = \boldsymbol{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, is

$$\sum_{i=1}^{p}\lambda_i\boldsymbol{Y}'(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+\mathbf{V}_i(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+\boldsymbol{Y},$$

under the condition that the system of equations

$$\mathbf{S}_{(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+}\boldsymbol{\lambda} = \boldsymbol{g}$$

is consistent; here $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_p)'$ and

$$\left\{\mathbf{S}_{(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+}\right\}_{i,j} = \mathrm{Tr}\left[(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+\mathbf{V}_i(\mathbf{M_X}\Sigma_0\mathbf{M_X})^+\mathbf{V}_j\right], \qquad i,j = 1,\dots,p,$$

$\Sigma_0 = \sum_{i=1}^{p}\vartheta_{i,0}\mathbf{V}_i$, $\mathbf{M_X} = \mathbf{I} - \mathbf{X}\mathbf{X}^+$, and $^+$ denotes the Moore-Penrose g-inverse (in more detail, cf. [4]).

In the following, $\operatorname{vec}(\mathbf{A})$ means the vector that arises by arranging columns of the matrix $\mathbf{A}$ one below the other.

Let $\mathbf{C}_{m,n}$ be the $(mn) \times (mn)$ matrix with the property

$$\forall\{\mathbf{A} : \mathbf{A} \text{ is } m \times n \text{ matrix}\} \quad \mathbf{C}\operatorname{vec}(\mathbf{A}') = \operatorname{vec}(\mathbf{A})$$

(in more detail, cf. [5; p. 10]).

LEMMA 1.8. *For the observation vector $\boldsymbol{Y}$ in the model from Definition 1.4 it is valid*

$$E(\boldsymbol{Y} \otimes \boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}) = (\mathbf{X} \otimes \mathbf{X})\boldsymbol{\beta}^{2\otimes} + \tilde{\mathbf{V}}\boldsymbol{\vartheta},$$

where

$$\tilde{\mathbf{V}} = \left[\operatorname{vec}(\mathbf{V}_1), \dots, \operatorname{vec}(\mathbf{V}_p)\right].$$

If, in addition, the vector $\boldsymbol{Y}$ is normally distributed, i.e., $\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \Sigma)$, then

$$\begin{aligned}&\operatorname{Var}(\boldsymbol{Y} \otimes \boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta})\\ &= (\mathbf{I}_{n^2,n^2} + \mathbf{C}_{n,n})(\Sigma \otimes \Sigma) + \Sigma \otimes (\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}') + (\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}') \otimes \Sigma\\ &\qquad + \left[(\mathbf{X}\boldsymbol{\beta}) \otimes \mathbf{I}\right]\Sigma\left[\mathbf{I} \otimes (\boldsymbol{\beta}'\mathbf{X}')\right] + \left[\mathbf{I} \otimes (\mathbf{X}\boldsymbol{\beta})\right]\Sigma\left[(\boldsymbol{\beta}'\mathbf{X}') \otimes \mathbf{I}\right].\end{aligned}$$

Proof. See, e.g., [2]. □

Remark 1.9. Each invariant estimator of the parameter $\boldsymbol{\vartheta}$ (i.e., an estimator whose realizations – estimates – do not depend on the value of $\boldsymbol{\beta}$) is a function of the maximum invariant $\mathbf{M_X}\boldsymbol{Y}$ (in more detail, cf. [5]). By this reason, the model

$$\left(\mathbf{M_X}\boldsymbol{Y}, \boldsymbol{O}, \sum_{i=1}^{p} \vartheta_i \mathbf{M_X}\mathbf{V}_i\mathbf{M_X}\right)$$

is convenient for invariant estimating the parameters $\vartheta_1, \dots, \vartheta_p$.

Since only the quadratic estimators are taken into account, the second tensor power of the maximum invariant, i.e., $(\mathbf{M_X}\boldsymbol{Y})^{2\otimes} = (\mathbf{M_X} \otimes \mathbf{M_X})\boldsymbol{Y}^{2\otimes}$, in the model from Definition 1.7, can be considered as the observation vector. In this way, a model "linear" with respect to its structure

$$\left[(\mathbf{M_X} \otimes \mathbf{M_X})\boldsymbol{Y}^{2\otimes}, (\mathbf{M_X} \otimes \mathbf{M_X})\tilde{\mathbf{V}}\boldsymbol{\vartheta}, (\mathbf{M_X} \otimes \mathbf{M_X})(\mathbf{I} + \mathbf{C})(\Sigma_0 \otimes \Sigma_0)(\mathbf{M_X} \otimes \mathbf{M_X})\right]$$

is obtained; the covariance matrix is implied by the assumption on normality of the vector $\boldsymbol{Y}$ and Lemma 1.8.

LEMMA 1.10. *For the model from Definition 1.7 the vector $\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\boldsymbol{Y}$ is also the maximum invariant.*

Proof. The matrix $\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}$ transforms the maximum invariant into $\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\boldsymbol{Y}$, and the matrix $\mathbf{M_X}\Sigma_0^{1/2}$ transforms the invariant $\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\boldsymbol{Y}$ into $\mathbf{M_X}\boldsymbol{Y}$. □

Thus, we can start from another "linear" model

$$\left\{\boldsymbol{Z}, \left[\left(\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right) \otimes \left(\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right)\right]\tilde{\mathbf{V}}\boldsymbol{\vartheta}, \operatorname{Var}(\boldsymbol{Z} \mid \boldsymbol{\vartheta}_0)\right\}$$

created by the second tensor power

$$\boldsymbol{Z} = \left(\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\boldsymbol{Y}\right)^{2\otimes} = \left[\left(\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right) \otimes \left(\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right)\right]\boldsymbol{Y}^{2\otimes}.$$

Lemma 1.11.

(i) *The OLS-estimator (cf. Definition* 1.4) *of* $\boldsymbol{\vartheta}$ *(from Definition* 1.7) *based on the second tensor power of the maximum invariant* $(\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\boldsymbol{Y}^{2\otimes}$ *from the model*

$$\left[(\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\boldsymbol{Y}^{2\otimes}, (\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\tilde{\mathbf{V}}\boldsymbol{\vartheta}, (\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})(\mathbf{I}+\mathbf{C})(\Sigma_0 \otimes \Sigma_0)(\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\right]$$

(see Remark 1.9) *is*

$$\tilde{\vartheta} = \left[\tilde{\mathbf{V}}'(\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\tilde{\mathbf{V}}\right]^{-1}\tilde{\mathbf{V}}'(\mathbf{M}_\mathbf{X} \otimes \mathbf{M}_\mathbf{X})\boldsymbol{Y}^{2\otimes} = \mathbf{S}_{\mathbf{M}_\mathbf{X}}^{-1}\begin{pmatrix} \boldsymbol{Y}'\mathbf{M}_\mathbf{X}\mathbf{V}_1\mathbf{M}_\mathbf{X}\boldsymbol{Y} \\ \vdots \\ \boldsymbol{Y}'\mathbf{M}_\mathbf{X}\mathbf{V}_p\mathbf{M}_\mathbf{X}\boldsymbol{Y} \end{pmatrix}.$$

(ii) *The OLS-estimator of* $\boldsymbol{\vartheta}$ *(from Definition* 1.7) *based on the second tensor power of the maximum invariant* $\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\boldsymbol{Y}$ *is*

$$\begin{aligned}
\tilde{\vartheta} &= \left\{\tilde{\mathbf{V}}'\left[\left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right) \otimes \left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\Sigma_0^{-1/2}\right)\right]\tilde{\mathbf{V}}\right\}^{-1} \\
&\quad \cdot \tilde{\mathbf{V}}'\left[\left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\right) \otimes \left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\right)\right] \\
&\quad \cdot \left[\left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\right) \otimes \left(\Sigma_0^{-1/2}\mathbf{M}_{\Sigma_0^{-1/2}\mathbf{X}}\right)\right]\boldsymbol{Y}^{2\otimes} \\
&= \left\{\tilde{\mathbf{V}}'\left[(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+ \otimes (\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\right]\tilde{\mathbf{V}}\right\}^{-1}\tilde{\mathbf{V}}' \\
&\quad \cdot \left[(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+ \otimes (\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\right]\boldsymbol{Y}^{2\otimes} \\
&= \mathbf{S}_{(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+}^{-1}\begin{pmatrix} \boldsymbol{Y}'(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\mathbf{V}_1(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\boldsymbol{Y} \\ \vdots \\ \boldsymbol{Y}'(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\mathbf{V}_p(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\boldsymbol{Y} \end{pmatrix}.
\end{aligned}$$

Here

$$\begin{aligned}
\{\mathbf{S}_{\mathbf{M}_\mathbf{X}}\}_{i,j} &= \operatorname{Tr}(\mathbf{M}_\mathbf{X}\mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_j), & i,j &= 1,\dots,p, \\
\left\{\mathbf{S}_{(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+}\right\}_{i,j} &= \operatorname{Tr}\left[(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\mathbf{V}_i(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\mathbf{V}_j\right], & i,j &= 1,\dots,p.
\end{aligned}$$

P r o o f . If the relations

$$\mathbf{M}_{\mathbf{X}}\mathbf{M}_{\mathbf{X}} = \mathbf{M}_{\mathbf{X}}\,, \qquad \mathbf{M}_{\mathbf{\Sigma}_0^{-1/2}\mathbf{X}}\mathbf{M}_{\mathbf{\Sigma}_0^{-1/2}\mathbf{X}} = \mathbf{M}_{\mathbf{\Sigma}_0^{-1/2}\mathbf{X}}\,,$$

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD})\,,$$

$$\big[\mathrm{vec}(\mathbf{V}_i)\big]'(\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})\,\mathrm{vec}(\mathbf{V}_j) = \big[\mathrm{vec}(\mathbf{V}_i)\big]'\,\mathrm{vec}(\mathbf{M}_{\mathbf{X}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}) = \mathrm{Tr}(\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}})\,,$$

$$(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+ = \mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}_0^{-1} = \mathbf{\Sigma}_0^{-1/2}\mathbf{M}_{\mathbf{\Sigma}_0^{-1}\mathbf{X}}\mathbf{\Sigma}_0^{-1/2}\,,$$

$$\begin{aligned}\big[\mathrm{vec}(\mathbf{V}_i)\big]'&\big[(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\otimes(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\big]\,\mathrm{vec}(\mathbf{V}_j)\\ &= \mathrm{Tr}\big[\mathbf{V}_i(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\mathbf{V}_j(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\,\big]\,,\end{aligned}$$

$$\begin{aligned}\big[\mathrm{vec}(\mathbf{V}_i)\big]'(\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})\boldsymbol{Y}^{2\otimes} &= \big[\mathrm{vec}(\mathbf{V}_i)\big]'\,\mathrm{vec}(\mathbf{M}_{\mathbf{X}}\boldsymbol{YY}'\mathbf{M}_{\mathbf{X}})\\ &= \mathrm{Tr}(\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\boldsymbol{YY}'\mathbf{M}_{\mathbf{X}}) = \boldsymbol{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\boldsymbol{Y}\,,\end{aligned}$$

and

$$\begin{aligned}\big[\mathrm{vec}(\mathbf{V}_i)\big]'&\big[(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\otimes(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\big]\,\mathrm{vec}(\mathbf{V}_j)\\ &= \mathrm{Tr}\big[\mathbf{V}_i(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\mathbf{V}_j(\mathbf{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathbf{M}_{\mathbf{X}})^+\big]\end{aligned}$$

are taken into account, then the proof can be finished straightforwardly. □

R e m a r k 1.12. Lemma 1.11 demonstrates the fact that all invariant estimators of $\boldsymbol{\vartheta}$ in the model from Definition 1.7 can be derived from the linear theory (least squares procedures) applied to models

$$\big[(\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})\boldsymbol{Y}^{2\otimes},\ (\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})\tilde{\mathbf{V}}\boldsymbol{\vartheta},\ (\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})(\mathbf{I}+\mathbf{C})(\Sigma_0\otimes\Sigma_0)(\mathbf{M}_{\mathbf{X}}\otimes\mathbf{M}_{\mathbf{X}})\big]$$

and

$$\Big[\boldsymbol{Z},\ \big[\big(\mathbf{M}_{\mathbf{\Sigma}_0^{-1/2}\mathbf{X}}\mathbf{\Sigma}_0^{-1/2}\big)\otimes\big(\mathbf{M}_{\mathbf{\Sigma}_0^{-1/2}\mathbf{X}}\mathbf{\Sigma}_0^{-1/2}\big)\big]\tilde{\mathbf{V}}\boldsymbol{\vartheta},\ \mathrm{Var}(\boldsymbol{Z})\Big]\,,$$

respectively. It is to be said that, in the case $\boldsymbol{Y}$ is not normally distributed, another covariance matrix of the second tensor power of the observation vector occurs in the "linear" models considered, however, this is of no importance for the further consideration since the OLS-estimator only will be demonstrated (in more detail, cf. [7] and [8]).

LEMMA 1.13. *In the model from Lemma* 1.5*,*

$$\mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big] >_L \mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\big]\,.$$

P r o o f . It follows from the equivalence

$$\begin{aligned}&\big[\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1 + (\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)'\mathbf{\Sigma}_{22.1}^{-1}(\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)\big]^{-1}\\ &= (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1} - (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)'\big[\mathbf{\Sigma}_{22.1} + (\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)\\ &\qquad\cdot(\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}(\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)'\big]^{-1}(\mathbf{X}_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)(\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\end{aligned}$$

and the obvious fact that the second term on the right hand side of the equality is a p.s.d. matrix. □

Remark 1.14. If $\tilde{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ and $\tilde{\tilde{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ are any unbiased estimators which differ from $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ and $\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$, respectively, then the inequality from Lemma 1.13 need not be valid; cf. the following example:

Example 1.15. Let us consider the model

$$\left[\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \boldsymbol{Y}_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \sigma_{11}, & \sigma_{12}, & \sigma_{13} \\ \sigma_{21}, & \sigma_{22}, & \sigma_{23} \\ \sigma_{31}, & \sigma_{32}, & \sigma_{33} \end{pmatrix}\right].$$

Let $\tilde{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \frac{1}{2}(\boldsymbol{Y}_1 + \boldsymbol{Y}_2)$ (an OLS-estimator), then $Var[\tilde{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)] = \frac{1}{4}(\boldsymbol{\sigma}_{11} + 2\boldsymbol{\sigma}_{12} + \boldsymbol{\sigma}_{22})$. If $\tilde{\tilde{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3) = \frac{1}{3}(\boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Y}_3)$ (an OLS-estimator), then $\mathrm{Var}[\tilde{\tilde{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3)] = \frac{1}{9}(\boldsymbol{\sigma}_{11} + 2\boldsymbol{\sigma}_{12} + 2\boldsymbol{\sigma}_{13} + \boldsymbol{\sigma}_{22} + 2\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{33})$. Thus

$$\begin{aligned} &\mathrm{Var}[\tilde{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)] > \mathrm{Var}[\tilde{\tilde{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3)] \\ &\qquad\qquad \iff 5(\boldsymbol{\sigma}_{11} + \boldsymbol{\sigma}_{22} + 2\boldsymbol{\sigma}_{12}) > 4(2\boldsymbol{\sigma}_{13} + 2\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{33}). \end{aligned}$$

The last consideration leads to the conclusion: Under some circumstances, a sequence of non-efficient estimators may have decreasing variances. If the estimators considered are from the numerical viewpoint simple, then it seems to be reasonable to use this sequence instead of the sequence of the efficient estimators which are, from the numerical viewpoint, complicated.

Therefore, in the following, the $\mathbf{I}$-MINQUE procedure for the estimation of variance components in KF is used as these estimators seem to be the most simple in the class of quadratic estimators.

The notation $\mathbf{I}$-MINQUE means that either $\boldsymbol{\vartheta}_0$ in Definition 1.7 fulfils the equality $\mathbf{I} = \sum_{i=1}^{p} \vartheta_{i0}\mathbf{V}_i$ (if such $\boldsymbol{\vartheta}_0$ exists), or the matrix $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_{i0}\mathbf{V}_i$ is substituted by $\mathbf{I}$ (if such $\boldsymbol{\vartheta}_0$ does not exist).

2. Estimators of variance components in KF

The KF is considered in the form

$$\begin{pmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_j \\ \cdots \\ \boldsymbol{v}_{j+1} \end{pmatrix} = \begin{pmatrix} \mathbf{H}_{0,j} \\ \cdots \\ \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0} \end{pmatrix} \boldsymbol{x}_0 + \begin{pmatrix} \boldsymbol{\varepsilon}_{0,j} \\ \cdots \\ \boldsymbol{\varepsilon}_j \end{pmatrix},$$

where

$$\operatorname{Var}\begin{pmatrix}\varepsilon_{0,j}\\ \dots\\ \varepsilon_j\end{pmatrix} = \begin{pmatrix}\sigma^2_{\mathbf{Q}}\mathbf{V}_j, & \mathbf{B}_j\\ \mathbf{B}'_j, & \mathbf{C}_{j+1}\mathbf{K}_{j+1,j+1}\mathbf{C}'_{j+1}\end{pmatrix} + \sigma^2_{\mathbf{R}}\begin{pmatrix}\overline{\overline{\mathbf{R}}}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{R}_{j+1}\end{pmatrix},$$

$$\overline{\overline{\mathbf{R}}} = \begin{pmatrix}\mathbf{R}_0, & \mathbf{O}, & \dots, & \mathbf{O}\\ \mathbf{O}, & \mathbf{R}_1, & \dots, & \mathbf{O}\\ \dots & \dots & \dots & \dots\\ \mathbf{O}, & \mathbf{O}, & \dots, & \mathbf{R}_j\end{pmatrix},$$

$$\mathbf{H}_{0,j} = \begin{pmatrix}\mathbf{C}_0\\ \mathbf{C}_1\mathbf{\Phi}_{1,0}\\ \vdots\\ \mathbf{C}_j\mathbf{\Phi}_{j,0}\end{pmatrix}, \quad \mathbf{\Phi}_{j,0} = \mathbf{A}_{j-1}\mathbf{A}_{j-2}\dots\mathbf{A}_0, \qquad j = 1,2,\dots,$$

$$\mathbf{V}_j = \begin{pmatrix}\mathbf{O}, & \mathbf{O}, & \mathbf{O}, & \dots, & \mathbf{O}\\ \mathbf{O}, & \mathbf{C}_1\mathbf{K}_{11}\mathbf{C}'_1, & \mathbf{C}_1\mathbf{K}_{12}\mathbf{C}'_2, & \dots, & \mathbf{C}_1\mathbf{K}_{1j}\mathbf{C}'_j\\ \mathbf{O}, & \mathbf{C}_2\mathbf{K}_{21}\mathbf{C}'_1, & \mathbf{C}_2\mathbf{K}_{22}\mathbf{C}'_2, & \dots, & \mathbf{C}_2\mathbf{K}_{2j}\mathbf{C}'_j\\ \dots & \dots & \dots & \dots & \dots\\ \mathbf{O}, & \mathbf{C}_j\mathbf{K}_{j1}\mathbf{C}'_1, & \mathbf{C}_j\mathbf{K}_{j2}\mathbf{C}'_2, & \dots, & \mathbf{C}_j\mathbf{K}_{jj}\mathbf{C}'_j\end{pmatrix},$$

$$\mathbf{K}_{i+r,i} = \mathbf{\Phi}_{i+r,1}\mathbf{\Gamma}_0\mathbf{Q}_0\mathbf{\Gamma}'_0\mathbf{\Phi}'_{i,1} + \mathbf{\Phi}_{i+r,2}\mathbf{\Gamma}_1\mathbf{Q}_1\mathbf{\Gamma}'_1\mathbf{\Phi}'_{i,2} + \dots + \mathbf{\Phi}_{i+r,i}\mathbf{\Gamma}_{i-1}\mathbf{Q}_{i-1}\mathbf{\Gamma}'_{i-1},$$
$$\mathbf{K}_{i,i+r} = \mathbf{\Phi}_{i,1}\mathbf{\Gamma}_0\mathbf{Q}_0\mathbf{\Gamma}'_0\mathbf{\Phi}'_{i+r,1} + \mathbf{\Phi}_{i,2}\mathbf{\Gamma}_1\mathbf{Q}_1\mathbf{\Gamma}'_1\mathbf{\Phi}'_{i+r,2} + \dots + \mathbf{\Gamma}_{i-1}\mathbf{Q}_{i-1}\mathbf{\Gamma}'_{i-1}\mathbf{\Phi}'_{i+r,i},$$
$$i = 1,2,\dots,$$

$$\overline{\mathbf{Q}}_j = \sigma^2_{\mathbf{Q}}\mathbf{Q}_j, \quad \overline{\mathbf{R}}_j = \sigma^2_{\mathbf{R}}\mathbf{R}_j, \qquad j = 0,1,2,\dots,$$

$$\mathbf{B}_j = \begin{pmatrix}\mathbf{O}\\ \mathbf{C}_1\mathbf{K}_{1,j+1}\mathbf{C}'_{j+1}\\ \vdots\\ \mathbf{C}_j\mathbf{K}_{j,j+1}\mathbf{C}'_{j+1}\end{pmatrix}.$$

THEOREM 2.1. *In the given KF, the iterative procedure for* $\mathbf{I}$*-MINQUE of the parameters* $\sigma^2_{\mathbf{Q}}$, $\sigma^2_{\mathbf{R}}$ *is given by the sequence of relations:*

(1)
$$\mathbf{P}_j = (\mathbf{H}'_{0,j}\mathbf{H}_{0,j})^{-1},$$

(2)
$$\tilde{\mathbf{x}}_{0|j} = \mathbf{P}_j\mathbf{H}'_{0,j}\begin{pmatrix}\mathbf{v}_0\\ \vdots\\ \mathbf{v}_j\end{pmatrix},$$

(3)

$$\tilde{\gamma}_{1,j} = \left[\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j}\right]' \mathbf{V}_j \left[\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j}\right],$$

(4)

$$\tilde{\gamma}_{2,j} = \left[\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j}\right]' \mathbf{R}_{jj} \left[\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j}\right],$$

where

$$\mathbf{R}_{jj} = \begin{pmatrix} \mathbf{R}_0, & \mathbf{O}, & \ldots, & \mathbf{O} \\ \mathbf{O}, & \mathbf{R}_1, & \ldots, & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{O}, & \boldsymbol{O}, & \ldots, & \mathbf{R}_j \end{pmatrix},$$

(5)

$$\mathbf{M}_{\mathbf{H}_{0,j}} = \mathbf{I} - \mathbf{H}_{0,j}\mathbf{P}_j\mathbf{H}'_{0,j},$$

(6)

$$\mathbf{C}_j^{(\mathbf{I})} = \begin{pmatrix} \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}) \\ \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}) \end{pmatrix},$$

(7)

$$\begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j} \\ \tilde{\sigma}^2_{\mathbf{R},j} \end{pmatrix} = \left(\mathbf{C}_j^{(\mathbf{I})}\right)^{-1} \begin{pmatrix} \tilde{\gamma}_{1,j} \\ \tilde{\gamma}_{2,j} \end{pmatrix},$$

(8)

$$\mathbf{G}_{j+1} = \mathbf{P}_j\mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\left(\mathbf{I} + \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\mathbf{P}_j\mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\right)^{-1},$$

(9)

$$\tilde{\boldsymbol{x}}_{0|j+1} = \tilde{\boldsymbol{x}}_{0|j} + \mathbf{G}_{j+1}\left(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\boldsymbol{x}}_{0|j}\right),$$

(10)

$$\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\boldsymbol{x}}_{0|j+1} = \left(\mathbf{I} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\mathbf{G}_{j+1}\right)\left(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\boldsymbol{x}}_{0|j}\right),$$

(11)

$$\begin{aligned} &\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j+1} \\ &\qquad = \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j} - \mathbf{H}_{0,j}\mathbf{G}_{j+1}\left(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\boldsymbol{x}}_{0|j}\right), \end{aligned}$$

(12)

$$\tilde{\gamma}_{1,j+1} = \tilde{\gamma}_{1,j} - 2\big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)'\mathbf{G}'_{j+1}\mathbf{H}_{0,j}\mathbf{G}_{j+1}\left[\begin{pmatrix}\boldsymbol{v}_0\\ \vdots\\ \boldsymbol{v}_j\end{pmatrix} - \mathbf{H}_{0,j}\tilde{\mathbf{x}}_{0|j}\right]$$
$$+ \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)'\mathbf{G}'_{j+1}\mathbf{H}_{0,j}\mathbf{V}_j\mathbf{H}_{0,j}\mathbf{G}_{j+1}$$
$$\cdot \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)$$
$$+ 2\left[\begin{pmatrix}\boldsymbol{v}_0\\ \vdots\\ \boldsymbol{v}_j\end{pmatrix}\mathbf{H}_{0,j}\tilde{\mathbf{x}}_{0|j+1}\right]'\mathbf{B}_j\big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j+1}\big)$$
$$+ \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j+1}\big)'\mathbf{C}_{j+1}\mathbf{K}_{j+1,j+1}\mathbf{C}'_{j+1}$$
$$\cdot \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j+1}\big),$$

(13)

$$\tilde{\gamma}_{2,j+1} = \tilde{\gamma}_{2,j} - 2\big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)'\mathbf{G}'_{j+1}\mathbf{H}_{0,j}\mathbf{R}_{jj}\left[\begin{pmatrix}\boldsymbol{v}_0\\ \vdots\\ \boldsymbol{v}_j\end{pmatrix} - \mathbf{H}_{0,j}\tilde{\mathbf{x}}_{0|j}\right]$$
$$+ \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)'\mathbf{G}'_{j+1}\mathbf{H}_{0,j}\mathbf{R}_{jj}\mathbf{H}_{0,j}\mathbf{G}_{j+1}$$
$$\cdot \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j}\big)$$
$$+ \big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j+1}\big)'\mathbf{R}_{j+1}\big(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\tilde{\mathbf{x}}_{0|j+1}\big),$$

(14)

$$\mathbf{P}_{j+1} = \big(\mathbf{I} - \mathbf{G}_{j+1}\mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\big)\mathbf{P}_j,$$

(15)

$$\mathbf{M}_{\mathbf{H}_{0,j+1}} = \mathbf{M}_{\begin{pmatrix}\mathbf{H}_{0,j}\\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\end{pmatrix}}$$
$$= \begin{pmatrix}\mathbf{M}_{\mathbf{H}_{0,j}}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix} + \begin{pmatrix}\mathbf{H}_{0,j}(\mathbf{P}_j - \mathbf{P}_{j+1})\mathbf{H}'_{0,j}, & -\mathbf{H}_{0,j}\mathbf{G}_{j+1}\\ -\mathbf{G}'_{j+1}\mathbf{H}'_{0,j}, & (\mathbf{I} + \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\mathbf{P}_j\mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1})^{-1}\end{pmatrix},$$

(16)

$$\mathbf{V}_{j+1} = \begin{pmatrix}\mathbf{V}_j, & \mathbf{B}_j\\ \mathbf{B}'_j, & \mathbf{C}_{j+1}\mathbf{K}_{j+1,j+1}\mathbf{C}'_{j+1}\end{pmatrix}, \qquad \mathbf{R}_{j+1,j+1} = \begin{pmatrix}\mathbf{R}_{jj}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{R}_{j+1}\end{pmatrix},$$

(17)

$$\mathbf{C}^{(\mathbf{I})}_{j+1} =$$
$$\begin{pmatrix}\mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{V}_{j+1}\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{V}_{j+1}), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{V}_{j+1}\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{R}_{j+1,j+1})\\ \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{R}_{j+1,j+1}\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{V}_{j+1}), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{R}_{j+1,j+1}\mathbf{M}_{\mathbf{H}_{0,j+1}}\mathbf{R}_{j+1,j+1})\end{pmatrix},$$

(18)
$$\begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j+1} \\ \tilde{\sigma}^2_{\mathbf{R},j+1} \end{pmatrix} = \left(\mathbf{C}^{(\mathbf{I})}_{j+1}\right)^{-1} \begin{pmatrix} \tilde{\gamma}_{1,j+1} \\ \tilde{\gamma}_{2,j+1} \end{pmatrix}, \dots \quad etc.$$

P r o o f. With respect to Lemma 1.11, the $\mathbf{I}$-MINQUE of $\begin{pmatrix} \sigma^2_{\mathbf{Q}} \\ \sigma^2_{\mathbf{R}} \end{pmatrix}$ in the model

$$\left[\begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix}, \mathbf{H}_{0,j}\boldsymbol{x}_0, \sigma^2_{\mathbf{Q}}\mathbf{V}_j + \sigma^2_{\mathbf{R}}\mathbf{R}_{jj}\right]$$

is

$$\begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j} \\ \tilde{\sigma}^2_{\mathbf{R},j} \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}) \\ \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j), & \mathrm{Tr}(\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\gamma}_{1,j} \\ \tilde{\gamma}_{2,j} \end{pmatrix},$$

where

$$\tilde{\gamma}_{1,j} = (\boldsymbol{v}'_0, \dots, \boldsymbol{v}'_j)\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{V}_j\mathbf{M}_{\mathbf{H}_{0,j}} \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix},$$

$$\tilde{\gamma}_{2,j} = (\boldsymbol{v}'_0, \dots, \boldsymbol{v}'_j)\mathbf{M}_{\mathbf{H}_{0,j}}\mathbf{R}_{jj}\mathbf{M}_{\mathbf{H}_{0,j}} \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix}.$$

Obviously,

$$\mathbf{M}_{\mathbf{H}_{0,j}} \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\mathbf{P}_j\mathbf{H}'_{0,j} \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix} - \mathbf{H}_{0,j}\tilde{\boldsymbol{x}}_{0|j}.$$

Now, from Lemma 1.5, for

$$\boldsymbol{Y}_1 = \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_j \end{pmatrix}, \quad \mathbf{X}_1 = \mathbf{H}_{0,j}, \qquad \boldsymbol{\beta} = \boldsymbol{x}_0, \quad \mathbf{X}_2 = \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0},$$

$$\boldsymbol{\Sigma}_{11} = \mathbf{I}, \quad \boldsymbol{\Sigma}_{12} = \mathbf{O}, \quad \boldsymbol{\Sigma}_{22} = \mathbf{I}$$

we obtain

$$\tilde{\boldsymbol{x}}_{0|j+1} = \tilde{\boldsymbol{x}}_{0|j} + \mathbf{G}_{j+1}\left(\boldsymbol{v}_{j+1} - \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\tilde{\boldsymbol{x}}_{0|j}\right).$$

(10) and (11) are consequences of this relation.

As far as (12) and (13) are concerned, they arise from the formula

$$
\begin{aligned}
\tilde{\gamma}_{1,j+1} &= \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix}' \mathbf{M}_{\mathbf{H}_{0,j+1}} \mathbf{V}_{j+1} \mathbf{M}_{\mathbf{H}_{0,j+1}} \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix} \\
&= \left[\begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix} - \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \end{pmatrix} \tilde{\mathbf{x}}_{0|j+1} \right]' \\
&\qquad \cdot \mathbf{V}_{j+1} \left[\begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix} - \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \end{pmatrix} \tilde{\mathbf{x}}_{0|j+1} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\gamma}_{2,j+1} &= \left[\begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix} - \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \end{pmatrix} \tilde{\mathbf{x}}_{0|j+1} \right]' \\
&\qquad \cdot \mathbf{R}_{j+1,j+1} \left[\begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j \\ \mathbf{v}_{j+1} \end{pmatrix} - \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \end{pmatrix} \tilde{\mathbf{x}}_{0|j+1} \right],
\end{aligned}
$$

respectively. Here (9) and (10) must be taken into account.

(14) follows from the relations

$$
\begin{aligned}
\mathbf{P}_{j+1} &= \left[(\mathbf{H}'_{0,j}, \mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}) \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \end{pmatrix} \right]^{-1} \\
&= \left(\mathbf{P}_{j+1}^{-1} + \mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \right)^{-1} \\
&= \mathbf{P}_j - \mathbf{P}_j \mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1} \left[\mathbf{I} + \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\mathbf{P}_j\mathbf{\Phi}'_{j+1,0}\mathbf{C}'_{j+1} \right]^{-1} \mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0}\mathbf{P}_j \\
&= \left(\mathbf{I} - \mathbf{G}_{j+1}\mathbf{C}_{j+1}\mathbf{\Phi}_{j+1,0} \right) \mathbf{P}_j \,.
\end{aligned}
$$

As

$$\begin{aligned}
\mathbf{M}_{\mathbf{H}_{0,j+1}} &= \mathbf{M}_{\begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0} \end{pmatrix}} \\
&= \begin{pmatrix} \mathbf{I}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} \end{pmatrix} - \begin{pmatrix} \mathbf{H}_{0,j} \\ \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0} \end{pmatrix} \\
&\qquad \cdot \left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\left(\mathbf{H}'_{0,j},\ \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\right) \\
&= \begin{pmatrix} \mathbf{U}, & \mathbf{V} \\ \mathbf{V}', & \mathbf{Z} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{U} &= \mathbf{I} - \mathbf{H}_{0,j}\left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\mathbf{H}'_{0,j}, \\
\mathbf{V} &= -\mathbf{H}_{0,j}\left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}, \\
\mathbf{Z} &= \mathbf{I} - \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{I} - \mathbf{H}_{0,j}\left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\mathbf{H}'_{0,j} \\
&\quad = \mathbf{I} - \mathbf{H}_{0,j}\mathbf{P}_j\mathbf{H}'_{0,j} + \mathbf{H}_{0,j}\mathbf{P}_j\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1} \\
&\qquad \cdot \left(\mathbf{I} + \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\right)^{-1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j\mathbf{H}'_{0,j} \\
&\quad = \mathbf{M}_{\mathbf{H}_{0,j}} + \mathbf{H}_{0,j}\mathbf{G}_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j\mathbf{H}'_{0,j},
\end{aligned}$$

and, from (14),

$$\mathbf{G}_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j = \mathbf{P}_j - \mathbf{P}_{j+1},$$

we obtain the $(1,1)$st block of the matrix $\mathbf{M}_{\mathbf{H}_{0,j}}$ in (15).

The $(1,2)$nd block can be expressed as

$$-\mathbf{H}_{0,j}\mathbf{P}_j\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\left(\mathbf{I} + \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\right)^{-1},$$

what, with respect to (8), equals $-\mathbf{H}_{0,j}\mathbf{G}_{j+1}$. Analogously, we obtain the $(2,1)$st block. The $(2,2)$nd block is implied by the relationship

$$\begin{aligned}
\mathbf{I} - \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}&\left(\mathbf{P}_j^{-1} + \boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\right)^{-1}\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1} \\
&= \left(\mathbf{I} + \mathbf{C}_{j+1}\boldsymbol{\Phi}_{j+1,0}\mathbf{P}_j\boldsymbol{\Phi}'_{j+1,0}\mathbf{C}'_{j+1}\right)^{-1}.
\end{aligned}$$

Now, the proof can easily be finished. □

Remark 2.2. With respect to Remark 1.14 and Example 1.15 on the one side and with respect to Lemma 1.11 on the other, it is not clear if the sequence of estimators

$$\dots, \begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j} \\ \tilde{\sigma}^2_{\mathbf{R},j} \end{pmatrix}, \begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j+1} \\ \tilde{\sigma}^2_{\mathbf{R},j+1} \end{pmatrix}, \dots$$

has decreasing variances. The following lemma is useful for recognizing this fact.

LEMMA 2.3. *If the random vectors $\xi_0, \xi_1, \dots$ and $\eta_0, \eta_1, \dots$ in the KF are normally distributed, then*

$$\operatorname{Var}\left[\begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j} \\ \tilde{\sigma}^2_{\mathbf{R},j} \end{pmatrix} \,\Big|\, \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}\right] = (\mathbf{C}_j^{(\mathbf{I})})^{-1} \operatorname{Var}\left[\begin{pmatrix} \tilde{\gamma}_{1,j} \\ \tilde{\gamma}_{2,j} \end{pmatrix} \,\Big|\, \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}\right] (\mathbf{C}_j^{(\mathbf{I})})^{-1},$$

where

$$\begin{aligned} &\operatorname{Var}(\tilde{\gamma}_{1,j} \mid \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}) \\ &= \sigma^4_{\mathbf{Q}} 2 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j)^4\right] + \sigma^2_{\mathbf{Q}} \sigma^2_{\mathbf{R}} \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j)^3 \mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj}\right] \\ &\qquad + \sigma^4_{\mathbf{R}} 2 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j \mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^2\right], \end{aligned}$$

$$\begin{aligned} &\operatorname{cov}(\tilde{\gamma}_{1,j}, \tilde{\gamma}_{2,j} \mid \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}) \\ &= \sigma^4_{\mathbf{Q}} 2 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j)^3 \mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj}\right] + \sigma^2_{\mathbf{Q}} \sigma^2_{\mathbf{R}} 4 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j \mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^2\right] \\ &\qquad + \sigma^4_{\mathbf{R}} 2 \operatorname{Tr}\left[\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j (\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^3\right], \end{aligned}$$

$$\begin{aligned} &\operatorname{Var}(\tilde{\gamma}_{2,j} \mid \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}) \\ &= \sigma^4_{\mathbf{Q}} 2 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j \mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^2\right] + \sigma^2_{\mathbf{Q}} \sigma^2_{\mathbf{R}} 4 \operatorname{Tr}\left[\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{V}_j (\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^3\right] \\ &\qquad + \sigma^4_{\mathbf{R}} 2 \operatorname{Tr}\left[(\mathbf{M}_{\mathbf{H}_{0,j}} \mathbf{R}_{jj})^4\right]. \end{aligned}$$

Proof. It is implied by the relationship

$$\operatorname{cov}(\mathbf{M_X T}_1 \mathbf{M_X} \boldsymbol{Y}, \mathbf{M_X T}_2 \mathbf{M_X} \boldsymbol{Y} \mid \vartheta) = 2 \operatorname{Tr}\left[\mathbf{M_X T}_1 \mathbf{M_X} \Sigma(\vartheta) \mathbf{M_X T}_2 \mathbf{M_X} \Sigma(\vartheta)\right],$$

which is valid under the conditions

$$\boldsymbol{Y} \sim N(\mathbf{X}\beta, \Sigma(\vartheta)),$$

and $\mathbf{T}_1 = \mathbf{T}_1'$, $\mathbf{T}_2 = \mathbf{T}_2'$, which can easily be proved (see, e.g., [5]). □

Remark 2.4. If the estimates $\tilde{\sigma}^2_{\mathbf{Q},j}$, $\tilde{\sigma}^2_{\mathbf{R},j}$ are used instead of the unknown values $\sigma^2_{\mathbf{Q}}$, $\sigma^2_{\mathbf{R}}$, then we obtain an estimate of the covariance matrix

$$\operatorname{Var}\left[\begin{pmatrix} \tilde{\sigma}^2_{\mathbf{Q},j} \\ \tilde{\sigma}^2_{\mathbf{R},j} \end{pmatrix} \,\Big|\, \sigma^2_{\mathbf{Q}}, \sigma^2_{\mathbf{R}}\right].$$

If the sequence of such estimates demonstrates a tendency to decrease (e.g., in the Loewner sense), then it is reasonable to use the procedure given by Theorem 2.1. A desirable situation may occur when this sequence is oscillating around the null matrix after some step j. In this case, the estimates $\tilde{\sigma}^2_{\mathbf{Q},j}$ and $\tilde{\sigma}^2_{\mathbf{R},j}$ are, from the practical point of view, equal to the actual values of the parameters $\sigma^2_{\mathbf{Q}}$

and $\sigma^2_{\mathbf{R}}$, and there is no necessity to continue in the iteration procedure. The filtering and predicting, respectively, of the state vectors $\boldsymbol{x}_k$ can be continued with values $\tilde{\sigma}^2_{\mathbf{Q},j}$ and $\tilde{\sigma}^2_{\mathbf{R},j}$.

Remark 2.5. Till now, the problem whether the quantities $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ are unbiasedly estimable was not investigated. This depends on the regularity of the matrix $\mathbf{C}_j^{(\mathbf{I})}$ (cf. [5]). For this reason, it is sufficient to investigate the case $j = 1$, i.e., to solve the problem within the model

$$\left[\begin{pmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{C}_0 \\ \mathbf{C}_1\mathbf{A}_0 \end{pmatrix} \boldsymbol{x}_0, \sigma^2_{\mathbf{Q}} \begin{pmatrix} \mathbf{O}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1' \end{pmatrix} + \sigma^2_{\mathbf{R}} \begin{pmatrix} \mathbf{R}_0, & \mathbf{O} \\ \mathbf{O}, & \mathbf{R}_1 \end{pmatrix}\right].$$

The matrix $\mathbf{C}_1^{(\mathbf{I})}$ for this model is

$$\mathbf{C}_1^{(\mathbf{I})} = \begin{pmatrix} \mathbf{C}_{11}, & \mathbf{C}_{12} \\ \mathbf{C}_{21}, & \mathbf{C}_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{C}_{11} &= \mathrm{Tr}\big[(\mathbf{I}-\mathbf{U})\mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1'(\mathbf{I}-\mathbf{U})\mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1'\big], \\
\mathbf{C}_{12} &= \mathrm{Tr}(\mathbf{C}_0\mathbf{S}^{-1}\mathbf{A}_0'\mathbf{C}_1'\mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1'\mathbf{C}_1\mathbf{A}_0\mathbf{S}^{-1}\mathbf{C}_0'\mathbf{R}_0) \\
&\qquad + \mathrm{Tr}\big[(\mathbf{I}-\mathbf{U})\mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1'(\mathbf{I}-\mathbf{U})\mathbf{R}_1\big], \\
\mathbf{C}_{22} &= \mathrm{Tr}\big[(\mathbf{I}-\mathbf{C}_0\mathbf{S}^{-1}\mathbf{C}_0')\mathbf{R}_0(\mathbf{I}-\mathbf{C}_0\mathbf{S}^{-1}\mathbf{C}_0')\mathbf{R}_0 + \mathbf{C}_0\mathbf{S}^{-1}\mathbf{A}_0'\mathbf{C}_1'\mathbf{R}_1\mathbf{C}_1\mathbf{A}_0\mathbf{S}^{-1}\mathbf{C}_0'\mathbf{R}_0\big] \\
&\qquad + \mathrm{Tr}\big[\mathbf{C}_1\mathbf{A}_0\mathbf{S}^{-1}\mathbf{C}_0'\mathbf{R}_0\mathbf{C}_0\mathbf{S}^{-1}\mathbf{A}_0'\mathbf{C}_1'\mathbf{R}_1 + (\mathbf{I}-\mathbf{U})\mathbf{R}_1(\mathbf{I}-\mathbf{U})\mathbf{R}_1\big], \\
\mathbf{S}^{-1} &= (\mathbf{C}_0'\mathbf{C}_0 + \mathbf{A}_0'\mathbf{C}_1'\mathbf{C}_1\mathbf{A}_0)^{-1}, \\
\mathbf{U} &= \mathbf{C}_1\mathbf{A}_0\mathbf{S}^{-1}\mathbf{A}_0'\mathbf{C}_1'.
\end{aligned}$$

As

$$\mathbf{I}-\mathbf{U} = \mathbf{I} - \mathbf{C}_1\mathbf{A}_0(\mathbf{C}_0'\mathbf{C}_0 + \mathbf{A}_0'\mathbf{C}_1'\mathbf{C}_1\mathbf{A}_0)^{-1}\mathbf{A}_0'\mathbf{C}_1' = \big[\mathbf{I} + \mathbf{C}_1\mathbf{A}_0(\mathbf{C}_0'\mathbf{C}_0)^{-1}\mathbf{A}_0'\mathbf{C}_1'\big]^{-1}$$

is obviously a p.d. matrix and $\mathbf{C}_1\boldsymbol{\Gamma}_0\mathbf{Q}_0\boldsymbol{\Gamma}_0'\mathbf{C}_1'$ is a p.s.d. matrix, $\mathbf{C}_{11} \neq \mathbf{O}$. As $\sigma^2_{\mathbf{R}}$ is unbiasedly estimable from the vector $\boldsymbol{v}_0 = \mathbf{C}_0\boldsymbol{x}_0 + \boldsymbol{\eta}_0$, and at least one linear combination $g_1\sigma^2_{\mathbf{Q}} + g_2\sigma^2_{\mathbf{R}}$ has the property

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{M}(\mathbf{C}_1^{(\mathbf{I})}) \ \ \& \ \ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \ \& \ \ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (\Longleftarrow \ \mathbf{C}_{11} \neq \mathbf{O}),$$

what implies that $g_1\sigma^2_{\mathbf{Q}} + g_2\sigma^2_{\mathbf{R}}$ and, simultaneously, $\sigma^2_{\mathbf{R}}$ are unbiasedly estimable, both of $\sigma^2_{\mathbf{Q}}$ and $\sigma^2_{\mathbf{R}}$ are unbiasedly estimable.

If the sequence considered in Remark 2.4 has no desirable properties, then it is necessary to develop the iterative procedure for determining estimates $\hat{\sigma}^2_{\mathbf{Q},j}$, $\hat{\sigma}^2_{\mathbf{R},j}$ which are at least $(\sigma^2_{\mathbf{Q},0}, \sigma^2_{\mathbf{R},0})$-locally efficient. In this case, the sequence

$$\mathrm{Var}\left[\begin{pmatrix} \hat{\sigma}^2_{\mathbf{Q},j} \\ \hat{\sigma}^2_{\mathbf{R},j} \end{pmatrix} \,\middle|\, \sigma^2_{\mathbf{Q},0}, \sigma^2_{\mathbf{R},0}\right], \qquad j = 0, 1, \ldots$$

is decreasing in the Loewner sense.

Such an iterative procedure is much more complicated than that given by Theorem 2.1. Nevertheless, the idea used in Theorem 2.1 can be used as well.

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