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ON ISOPART PARAMETERS OF COMPLETE BIPARTITE GRAPHS AND n-CUBES

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Introduction

If H is a graph and $H_1, H_2, \dots, H_n$ $(n>2)$ are non-empty, pairwise edge-disjoint subgraphs of H having the property that

$$E(H)=\bigcup_{i=1}^{n} E(H_i),$$

then we say that H is the *edge sum* of $H_1, H_2, \dots, H_n$ and write

$$H=H_1 \oplus H_2 \oplus \dots \oplus H_n.$$

In such a case we also say that H can be *decomposed* into the subgraphs $H_1, H_2, \dots, H_n$. If there is a graph G that is isomorphic to each of the subgraphs $H_1, H_2, \dots, H_n$, then we have a *G-decomposition* of H. The graph H is said to be *G-decomposable,* and G is an *isopart* of H. See [1] for undefined terms.

Most investigators (e.g. Kőnig [5], Petersen [6], Reiss [7]) of decompositions of a given graph H require that H be a regular graph. For example, Reiss [7] showed that, when p is even, the complete graph K_p can be decomposed into spanning 1-regular subgraphs called 1-*factors.* Wilson [8], Fink [2], and Fink and Ruiz [4] have shown, in different ways, that every nonempty graph G is an isopart of infinitely many connected regular graphs H.

With the results of [8], [2], and [4] in mind, Fink [3] introduced and investigated three "isopart parameters", $p_0(G)$, $r_0(G)$, and $f_0(G)$. The numbers $p_0(G)$ and $r_0(G)$ are respectively the minimum order and minimum degree of regularity among all connected, regular, G-decomposable graphs. The parameter $f_0(G)$ is the smallest number t (≥ 2) for which there exists a connected regular graph H decomposable into t copies of G. We write r_0, p_0 and f_0 rather than $r_0(G)$, $p_0(G)$ and $f_0(G)$ when the graph G is clear. In [3], Fink determines r_0, p_0 and f_0 for K_n, C_n and $K_{1,n}$. In this paper we find $r_0(K_{m,n})$, $p_0(K_{m,n})$ and $f_0(K_{m,n})$ for all positive integers m and n. Also, we determine the isopart parameters for $K_{m[n]}$, the complete m-partite graph with n vertices in each partite set. We end the paper by finding $p_0(Q_n)$, $r_0(Q_n)$ and $f_0(Q_n)$ where Q_n is the n-dimensional cube.

Main results. We begin by stating Fink's results for stars.

Theorem A ([3]) *If $n \geqslant 2$, then*

$$\begin{aligned} r_0(K_{1,n}) &= n, \\ p_0(K_{1,n}) &= 2n \quad \textit{and} \\ f_0(K_{1,n}) &= n. \end{aligned}$$

We now give the upper bounds for the isofactor parameters for complete bipartite graphs $K_{m,n}$ when $m \neq n$.

Theorem 1. *If $m < n$ and $l = \operatorname{lcm}(m, n)$, then*

$$\begin{aligned} r_0(K_{m,n}) &\leqslant l, \\ p_0(K_{m,n}) &\leqslant 2l \quad \textit{and} \\ f_0(K_{m,n}) &\leqslant \frac{l^2}{mn}. \end{aligned}$$

Proof. Let m and n be distinct positive integers and $l = \operatorname{lcm}(m, n)$. Let $G = K_{l,l}$ and denote by $A = \{u_1, u_2, \ldots, u_l\}$ and $B = \{v_1, v_2, \ldots, v_l\}$ the partite sets of G. Let $A_i = \{u_{(i-1)n+k} \mid k = 1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, \frac{l}{n}$ and $B_i = \{v_{(i-1)m+j} \mid j = 1, 2, \ldots, m\}$ for $i = 1, 2, \ldots, \frac{l}{m}$. Clearly, $A = \bigcup_{i=1}^{l/n} A_i$ and $B = \bigcup_{i=1}^{l/m} B_i$. For $1 \leqslant i \leqslant \frac{l}{n}$ and $1 \leqslant j \leqslant \frac{l}{m}$ define $H_{ij} = \langle a_i \cup B_j \rangle$. Clearly, $H_{ij} \cong K_{m,n}$ and thus this defines a $K_{m,n}$-decomposition of G. Hence it follows that $r_0(K_{m,n}) \leqslant l$, $p_0(K_{m,n}) \leqslant 2l$ and $f_0(K_{m,n}) \leqslant \frac{l^2}{mn}$.

We proceed by showing $r_0(K_{m,n}) = \operatorname{lcm}(m, n)$.

Theorem 2. *If $m < n$ and $l = \operatorname{lcm}(m, n)$, then $r_0(K_{m,n}) = l$.*

Proof. Let m and n be distinct positive integers and $l = \operatorname{lcm}(m, n)$. To show $r_0 = l$, we need only to show that $r_0 \geqslant l$ since we showed $r_0 \leqslant l$ in Theorem 1. Assume, to the contrary, that for some $r < l$, there exists an r-regular connected graph H which is $K_{m,n}$-decomposable. Clearly, since vertices of $K_{m,n}$ are of degree m or n, $r = am + bn$ for some non-negative integers a and b. This is a unique representation, since if there exist non-negative integers c and d with $am + bn = cm + dn = r < l$, then it follows that $a - c = d - b = 0$. Hence it follows that in any $K_{m,n}$-decomposition of H every vertex must be a vertex of degree m in a copies of $K_{m,n}$ and a vertex of degree n in b copies of $K_{m,n}$. If p is of the order of H, and there are t copies of $K_{m,n}$ in the decomposition, then $\frac{pa}{n}=t=\frac{pb}{m}$. Hence it follows that $am=bn$. But this implies that l divides r and $r \geq l$, which is a contradiction. Hence it follows that $r_0=l$.

Lemma 3. *Let H be a connected r-regular $K_{m,n}$-decomposable graph of order p. If $r>r_0(K_{m,n})=l$, then $p>2l$.*

Proof. Let H be a connected r-regular, $K_{m,n}$-decomposable graph of order p. Clearly, if $r \geq 2l$, then $p>2l$. Thus assume that $l<r<2l$. As we saw in the previous proof, if r can be written uniquely as $r=am+bn$ for some non-negative integers a and b, then l must divide r. But since $l<r<2l$, it follows that r can be written in at least two distinct ways as a linear combination of m and n over the positive integers. A straightforward arithmetic argument reveals that r can be written in precisely two ways. Namely, $r=am+\left(\frac{r-am}{n}\right)n$ and $r=\left(a+\frac{l}{m}\right)m+\left(\frac{r-am-l}{n}\right)n$ or alternatively, $r=\left(\frac{r-bn}{n}\right)m+bn$ and $r=\left(\frac{r-bn-l}{n}\right)m+\left(b+\frac{l}{n}\right)n$ where $a=\frac{r-bn-l}{n}$, $b=\frac{r-am-l}{n}$ and $r=am+bn+l$. Note that since $r>l$, at least one of a or b is non-zero. For convenience, we will refer to a vertex of degree d in an isopart as a d-vertex. Let $A=\{u \in V(H) | u$ occurs as an m-vertex in a different isoparts, $K_{m,n}\}$ and $B=\{v \in V(H) | v$ occurs as an m-vertex in $a+\frac{l}{m}$ different isoparts, $K_{m,n}\}$. If $|A|=p_1$ and $|B|=p_2$, then clearly

$$p_1+p_2=p. \tag{1}$$

If we consider the m-vertices in the $\frac{rp}{2mn}$ copies of $K_{m,n}$, it follows that

$$ap_1+\left(a+\frac{l}{m}\right)p_2=\frac{rp}{2m} \tag{2}$$

since each isopart contains n m-vertices. Equations (1) and (2) give $p_2=\left(\frac{r-2am}{l}\right)p$ and $p_1=\left(\frac{2l+2am-r}{2l}\right)p$. Consequently, $p_1=\left(\frac{2l+2am-r}{r-2am}\right)p_2$.

Consider now the maximum number of edges in $\langle A \rangle$. Since each edge in $\langle A \rangle$ is an edge of some $K_{m,n}$, it joins an m-vertex to an n-vertex. Since each vertex in $\langle A \rangle$ is an m-vertex of am edges, it follows that there are at most $p_1 am$ edges in $\langle A \rangle$. Hence it follows that there must be a vertex u in A adjacent to at most am vertices in A and thus adjacent to at least $r-am$ vertices in B.

Therefore $p_2 \geq r-am$ and $p_1 \geq \left(\frac{2l+2am-r}{r-2am}\right)(r-am)$, which implies that

$p_1 \geqslant 2l + 2am - r$ and consequently $p = p_1 + p_2 \geqslant 2l + am$. A symmetrical argument gives $p_1 \geqslant r - bn$, $p_2 \geqslant 2l + 2b - r$ and $p \geqslant 2l + bn$. It now follows that $p > 2l$.

Theorem 4. *If* $m < n$ and $l = \operatorname{lcm}(m, n)$, then $p_0(K_{m,n}) = 2l$.

Furthermore, the only connected regular $K_{m,n}$-decomposable graph of order $2l$ is $K_{l,l}$.

Proof. Let m and n be distinct positive integers and $l = \operatorname{lcm}(m, n)$. Let H be a connected r-regular $K_{m,n}$-decomposable graph of order p_0. Clearly, from Lemma 3 it follows that H is an l-regular graph and from Theorems 1 and 2 that $p_0 \leqslant 2l$.

Since $m \neq n$, there are precisely two distinct ways that l can be written as a linear combination of m and n, namely $l = \left(\frac{l}{n}\right) n = \left(\frac{l}{m}\right) m$. Let A be the subset of $V(H)$ whose vertices are m-vertices in some isopart and B be the set of vertices that are n-vertices in some $K_{m,n}$. Clearly, $V(H) = A \cup B$ and it follows that both sets, A and B, are independent. Since each vertex of A and B has degree l it must be the case that $|A| \geqslant l$ and $|B| \geqslant l$.

Consequently, it follows that $p_0 \geqslant 2l$ and thus $p_0 = 2l$. Therefore $|A| = l$, $|B| = l$, each vertex of A is adjacent to every vertex of B, and $H \cong K_{l,l}$.

Theorem 5. *If* $m < n$ *and* $l = \operatorname{lcm}(m, n)$, *then* $f_0(K_{m,n}) = \dfrac{l^2}{mn}$,

Proof. Let m and n be distinct positive integers and $l = \operatorname{lcm}(m, n)$. Let H be a connected r-regular $K_{m,n}$-decomposable graph of order p containing precisely f_0 copies of $K_{m,n}$. It follows that

$$f_0 = \frac{rp}{2mn} \geqslant \frac{r_0 p_0}{2mn} = \frac{l^2}{mn},$$

and the proof is complete.

We get as a corollary of these results, Theorem A.

Corollary 6. *If* $n \geqslant 2$, *then*

$$r_0(K_{1,n}) = n,$$
$$p_0(K_{1,n}) = 2n \quad \text{and}$$
$$f_0(K_{1,n}) = n.$$

The only case that remains for complete bipartite graphs is when the partite sets are of the same order. For convenience we denote by $K_{n[m]}$ the complete n-partite graph, with partite sets each of order m. We prove the following:

Theorem 7. *If* m, n *are positive integers* $(n \geqslant 2)$, *then*

$$r_0(K_{n[m]} = 2m(n-1),$$
$$p_0(K_{n[m]}) = m\binom{n+1}{2} \quad \text{and}$$

$$f_0(K_{n[m]}) = n + 1.$$

Proof. First, we show that there exists a regular graph G which can be decomposed into $K_{n[m]}$'s, which gives the desired upper bounds. Let F_i be a family of m distinct vertices for $i = 1, 2, \ldots, \binom{n+1}{2}$. Let

$$V(G) = \bigcup_{i=1}^{\binom{n+1}{2}} F_i. \quad \text{Clearly,} \quad |V(G)| = m\binom{n+1}{2}.$$

Consider the following ordered classes of families; $C_1 = \{F_1, F_2, \ldots, F_n\}$, $C_{i+1} = \{$the i^{th} family of C_m; $m = 1, 2, \ldots, i\} \cup \{$the next $n - i$ families$\}$, (Ex., $C_2 = \{F_1, F_{n+1}, F_{n+2}, \ldots, F_{2n-1}\}$ and $C_3 = \{F_2, F_{n+1}, F_{2n}, F_{2n+1}, \ldots, F_{3n-3}\}$), $i = 1, 2, \ldots, n$. It follows that there are $n + 1$ classes of families with the properties that every family is in exactly two classes and any pair of families is in at most one common class. Let $xy \in E(G)$ iff $x \in F_i$ and $y \in F_j$ with $i \neq j$ and both F_i and $F_j \in C_k$ for some k. Hence $\langle C_j \rangle \cong K_{n[m]}$ for $j = 1, 2, \ldots, n + 1$, with $E(\langle C_i \rangle) \cap E(\langle C_j \rangle) = \emptyset$ for $i \neq j$. Also by the above observation, each family, and thus each vertex, is in 2 classes. So each vertex is in two $K_{n[m]}$'s and has degree $2m(n-1)$. Therefore, $r_0(K_{n[m]}) \leqslant 2m(n-1)$, $p_0(K_{n[m]}) \leqslant m\binom{n+1}{2}$, and $f_0(K_{n[m]}) \leqslant n + 1$.

Since any $K_{n[m]}$-decomposable connected graph G must contain at least $2K_{n[m]}$'s, it follows that $r_0(K_{n[m]}) \geqslant 2m(n-1)$ and consequently $r_0(K_{n[m]}) = 2m(n-1)$.

Finally, suppose G is an r-regular $K_{n[m]}$-decomposable graph. Let C_1 be one copy of $K_{n[m]}$ with partite sets $A_1, A_2, \ldots, A_n$. It follows that x_1 in A_1 must be contained in a second copy of $K_{n[m]}$, C_2. Also, C_2 can contain at most m vertices of C_1, which implies that G contains at least $(n-1)m$ additional vertices. For x_2 in $C_1 - C_2$, x_2 must be in a third copy of $K_{n[m]}$, C_3. Again C_3 can contain at most m vertices of C_1 and m vertices of C_2, which implies G contains at least $(n-2)m$ additional vertices. By continuing this argument, we conclude that

$$|V(G)| \geqslant nm + (n-1)m + (n-2)m + \ldots + 1m = \binom{n+1}{2} m$$

and that G contains at least $n + 1$ copies of $K_{n[m]}$. Therefore the result follows. For completeness we state the following:

Theorem 8. *Let m, n be positive integer with $l = \operatorname{lcm}(m, n)$. If $n < n$, then $r_0(K_{m,n}) = l$, $p_0(K_{m,n}) = 2l$ and $f_0(K_{m,n}) = \dfrac{l^2}{mn}$. If $m = n$, then $r_0(K_{m,m}) = 2m$, $p_0(K_{m,m}) = 3m$ and $f_0(K_{m,m}) = 3$.*

Finally, we determine the isofactor parameters for another class of bipartite graphs, the n-dimensional cubes, Q_n. Since Q_1 and Q_2 are isomorphic to $K_{1,1}$ and $K_{2,2}$, respectively, the isofactor parameters for these graphs are easily calculated using Theorem 8. Thus we only need to find $r_0(Q_n)$, $p_0(Q_n)$ and $f_0(Q_n)$ when $n \geqslant 3$.

Theorem 9. *If $n \geqslant 3$, then $r_0(Q_n) = 2n$, $p_0(Q_n) = 2^n$ and $f_0(Q_n) = 2$.*

Proof. The n-cube, Q_n is an n-regular bipartite graph of order 2^n with 2^{n-1} vertices in each of its partite sets. Let U and V denote these partite sets. Since Q_n is a proper subgraph of $K_{2^{n-1}, 2^{n-1}}$ and $n \geqslant 3$, there is a perfect matching from U to V in the complement $\bar{Q}_n$. Thus, since U and V induce complete graphs in $\bar{Q}_n$, it follows that $K_{2^{n-1}} \times Q_1$ is a subgraph of $\bar{Q}_n$. Hence $Q_n \cong Q_{n-1} \times Q_1 \subseteq \bar{Q}_n$. Consequently, by identifying the two edge disjoint copies of Q_n, it follows that $r_0(Q_n) \leqslant 2n$, $p_0(Q_n) \leqslant 2^n$ and $f_0(Q_n) \leqslant 2$. However, equality follows since there are at least 2 copies of Q_n, at least 2^n vertices, and, clearly, regularity at least $2n$ in any connected regular Q_n-decomposable graph.

Conclusion. There appear to be a number of feasible questions that these parameters present. Two key problems would be to determine r_0, p_0 and f_0 for any bipartite graph and subsequently for any n-partite graph.

The authors are (currently) preparing an article on the dependence and independence of the parameters on one another.

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О ПАРАМЕТРАХ ИЗОЧАСТЕЙ ПОЛНЫХ ДВУДОЛЬНЫХ ГРАФОВ И n-КУБОВ

Richard M. Davitt—John Frederick Fink—Michael S. Jacobson

Резюме

Пусть H-граф с $H=H_1, H_2, \ldots, H_n$. Если $G=H_i$ для всякого i, $i=1,2,\ldots,n$, тогда H разлагается на G, а G является изочастью H.

Доказано, что каждый не пустой граф G является изочастью бесконечного множества связных обычных графов H.

Числа $p_0(G)$ и $r_0(G)$ — это минимальные порядок и степень регулярности (соответственно) среди всех связных обычных разлагаемых на G графов.

Параметр $f_0(G)$ — наименьшее число t $(t=2)$, для которого существует связный обычный граф H, разлагаемый на t изоморфных копий G.

Цель этой работы определить параметры этих изочастей для всех польных двудольных графов и n-кубов.