Mathematica Slovaca

Stanley P. Gudder

Tensor products of sequential effect algebras

Mathematica Slovaca, Vol. 54 (2004), No. 1, 1--11

Persistent URL: http://dml.cz/dmlcz/132911

Math. Slovaca, **54** (2004), No. 1, 1–11

Mathematica Slovaca

Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

TENSOR PRODUCTS OF SEQUENTIAL EFFECT ALGEBRAS

Stan Gudder

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A sequential effect algebra (SEA) is an effect algebra on which a sequential product with natural properties is defined. It is first shown that the tensor product of a Boolean algebra with an arbitrary SEA exists. We then characterize pairs of SEA's that admit a tensor product. As a corollary we show that a pair of commutative SEA's admit a tensor product if they admit a bimorphism.

1. Introduction

Sequential effect algebras (SEA's) were recently introduced to study general properties of sequential measurements ([9], [11], [12]). Important physical models for SEA's can be constructed from fuzzy set systems and Hilbert space operators ([2], [3], [12], [13], [14]). It is relevant to study tensor products of SEA's because they describe combined physical systems. For example, the tensor product of a Boolean algebra and a SEA describes the interaction of a measuring apparatus with a quantum mechanical system. Our work parallels the pioneering results in [4] and some of our methods are similar. The basis for all of this research goes back to the original work in [6].

This paper begins with the basic definitions of effect algebras and SEA's. Our first result shows that the tensor product of a Boolean algebra with an arbitrary SEA exists. We then characterize pairs of SEA's that admit a tensor product. As a corollary we show that a pair of commutative SEA's admit a tensor product if they admit a single bimorphism. We note that this result is significant because

2000 Mathematics Subject Classification: effect algebra, sequential product, tensor product, quantum effect.
Keywords: Primary 03G12, 81P10.

there are important examples of commutative nonboolean SEA's. The paper closes with some unsolved problems.

2. Basic definitions

This section summarizes the basic definitions concerning effect algebras ([1], [5], [7], [8], [15]) and sequential effect algebras ([9], [11], [12]). For motivation and further details the reader is referred to the cited literature. If $\oplus$ is a partial binary operation, we write $a \perp b$ if $a \oplus b$ is defined.

An *effect algebra* is a system $(E, 0, 1, \oplus)$ where 0, 1 are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

(E1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.

(E2) If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$ and $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \perp a'$ and $a \oplus a' = 1$.

(E4) If $a \perp 1$, then $a = 0$.

In the sequel, whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$. We define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. It can be shown that $(E, \leq, ')$ is a partially ordered set with $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$, and $a \leq b$ implies $b' \leq a'$. Moreover, we have $a \perp b$ if and only if $a \leq b'$.

If E and F are effect algebras, we say that $\phi\colon E \to F$ is *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

If $\phi\colon E \to F$ is additive and $\phi(1) = 1$, then ϕ is a *morphism.* If $\phi\colon E \to F$ is a morphism and $\phi(a) \perp \phi(b)$ implies that $a \perp b$, then ϕ is a *monomorphism.* A surjective monomorphism is an *isomorphism.*

It is easy to check that a morphism ϕ is an isomorphism if and only if ϕ is bijective and ϕ^{-1} is a morphism.

If G is also an effect algebra, a map $\beta\colon E \times F \to G$ is a *bimorphism* if

(B1) $\beta(1, 1) = 1$.

(B2) $\beta(a \oplus b, c) = \beta(a, c) \oplus \beta(b, c)$ whenever $a, b \in E$ with $a \perp b$.

(B3) $\beta(a, b \oplus c) = \beta(a, b) \oplus \beta(a, c)$ whenever $b, c \in F$ with $b \perp c$.

We thus see that $\beta(\cdot, c)$ and $\beta(a, \cdot)$ are additive for all $a \in E$, $c \in F$. Moreover, $\beta(\cdot, 1)$ and $\beta(1, \cdot)$ are morphisms.

Roughly speaking, an effect algebra is a structure that is designed to study parallel combinations of simple yes-no measurements called *effects.* We now consider a richer structure that also enables us to study series combinations of effects. For a binary operation $\circ$, if $a \circ b = b \circ a$, we write $a \mid b$.

A *sequential effect algebra* (SEA) is a system $(E,0,1,\oplus,\circ)$ where $(E,0,1,\oplus)$ is an effect algebra and $\circ\colon E\times E\to E$ is a binary operation that satisfies the following conditions.

(S1) $b\mapsto a\circ b$ is additive for every $a\in E$.
(S2) $1\circ a=a$ for all $a\in E$.
(S3) If $a\circ b=0$, then $a\mid b$.
(S4) If $a\mid b$, then $a\mid b'$ and $a\circ(b\circ c)=(a\circ b)\circ c$ for all $c\in E$.
(S5) If $c\mid a$ and $c\mid b$, then $c\mid a\circ b$ and $c\mid(a\oplus b)$.

We call an operation that satisfies (S1)–(S5) a *sequential product* on E. If $a\mid b$ for all $a,b\in E$, we call E a *commutative* SEA.

There are many examples of SEA's ([11], [12]), but we shall only consider the four most important ones here. For a Boolean algebra $\mathcal{B}$, define $a\perp b$ if $a\wedge b=0$ and in this case $a\oplus b=a\vee b$. Defining $a\circ b=a\wedge b$, we have that $(\mathcal{B},0,1,\oplus,\circ)$ is a commutative SEA. A particularly simple commutative SEA is the unit interval $[0,1]\subseteq\mathbb{R}$. For $a,b\in[0,1]$, define $a\circ b=ab$ and define $a\perp b$ if $a+b\le 1$, in which case $a\oplus b=a+b$. For the function space $[0,1]^X$ define the functions f_0, f_1 by $f_0(x)=0$, $f_1(x)=1$ for all $x\in X$.

We call $\mathcal{F}\subseteq[0,1]^X$ a *fuzzy set system* on X if

$f_0,f_1\in\mathcal{F}$,
if $f\in\mathcal{F}$, then $f_1-f\in\mathcal{F}$,
if $f,g\in\mathcal{F}$ with $f+g\le 1$, then $f+g\in\mathcal{F}$,
if $f,g\in\mathcal{F}$, then $fg\in\mathcal{F}$.

Then $\mathcal{F}$ becomes a commutative SEA when $f\oplus g=f+g$ for $f+g\le 1$ and $f\circ g=fg$. The most important noncommutative SEA is obtained from the set $\mathcal{E}(H)$ of self-adjoint operators on a Hilbert space H satisfying $0\le A\le I$.

For $A,B\in\mathcal{E}(H)$ we define $A\perp B$ if $A+B\in\mathcal{E}(H)$ and in this case $A\oplus B=A+B$. The sequential product on $\mathcal{E}(H)$ is defined by $A\circ B=A^{1/2}BA^{1/2}$ where $A^{1/2}$ is the unique positive square root of A.

It is shown in [11] that $\big(\mathcal{E}(H),O,I,\oplus,\circ\big)$ is a SEA. This Hilbert space SEA is useful for studying the foundations of quantum mechanics ([2], [3], [13], [14]).

Let E and F be SEA's. A *SEA-morphism* $\phi\colon E\to F$ is an effect algebra morphism that satisfies $\phi(a\circ b)=\phi(a)\circ\phi(b)$ for every $a,b\in E$. A SEA-morphism that is an effect algebra isomorphism is a *SEA-isomorphism*.

If G is also a SEA, a *SEA-bimorphism* is a map $\beta\colon E\times F\to G$ that is an effect algebra bimorphism satisfying

$$\beta(a,b)\circ\beta(c,d)=\beta(a\circ c,b\circ d)$$

for all $a,c\in E$ and $b,d\in F$.

The *SEA tensor product* of E and F is a SEA T and a SEA-bimorphism $\tau\colon E \times F \to T$ such that

(T1) Every $a \in T$ has the form $a = \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$.
(T2) If $\beta\colon E \times F \to G$ is a SEA-bimorphism, then there exists a SEA-morphism $\phi\colon T \to G$ such that $\beta = \phi \circ \tau$.

It can be shown that the SEA tensor product is unique up to a SEA-isomorphism if it exists. Examples of SEA-bimorphisms and SEA tensor products will be given in the next section. The definition of an effect algebra tensor product is obtained from our previous definition by replacing SEA with effect algebra whenever it appears.

3. Tensor products

It is shown in [4] that the effect algebra tensor product of a Boolean algebra with an arbitrary effect algebra always exists. We now show that this result holds for SEA's. Our proof is similar to the special case proved in [11].

THEOREM 3.1. *If $\mathcal{B}$ is a Boolean algebra and E is a SEA, then the SEA tensor product of $\mathcal{B}$ and E exists.*

Proof. By the Stone representation theorem, we can (and will) assume that $\mathcal{B}$ is the set of clopen subsets of a totally disconnected topological space X.

We call a function $f\colon X \to E$ *simple* if f is continuous for the discrete topology on E and f has a finite number of values.

Define

$$T = \{f \in E^X : \ f \text{ is simple}\}.$$

On T define $f \perp g$ if $f(x) \perp g(x)$ for all $x \in X$ and if $f \perp g$ define $(f \oplus g)(x) = f(x) \oplus g(x)$. Defining $0(x) = 0$ and $1(x) = 1$ for every $x \in X$, it is easy to check that $(T, 0, 1, \oplus)$ is an effect algebra. For $f, g \in T$ define $(f \circ g)(x) = f(x) \circ g(x)$. Again, it is easy to check that $(T, 0, 1, \oplus, \circ)$ is a SEA. Define $\tau\colon \mathcal{B} \times E \to T$ by

$$\tau(A, a)(x) = \begin{cases} a & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

It is clear that τ is an effect algebra bimorphism. Since

$$\begin{aligned}\big(\tau(A, a) \circ \tau(B, b)\big)(x) &= \tau(A, a)(x) \circ \tau(B, b)(x) = \tau(A \cap B, a \circ b)(x) \\ &= \tau(A \circ B, a \circ b)(x),\end{aligned}$$

we see that τ is a SEA-bimorphism.

Suppose $f \in T$ and f has the distinct values $a_1, \ldots, a_n$. Then $f^{-1}(a_i) = A_i \in \mathcal{B}$ and we have $A_i \cap A_j = \emptyset$, $i \neq j$, and $\bigcup A_i = X$. Moreover, we have the representation

$$f = \bigoplus_{i=1}^{n} \tau(A_i, a_i)\,.$$

This representation is unique because suppose

$$f = \bigoplus_{j=1}^{m} \tau(B_j, b_j)$$

where $b_i \neq b_j$, $B_i \cap B_j = \emptyset$, $i \neq j$, and $\bigcup B_j = X$. Then $x \in B_j$ if and only if $f(x) = b_j$. Hence, $b_j = a_i$ for some i and then $B_j = A_i$. The uniqueness now follows. Let $\beta\colon \mathcal{B} \times E \to F$ be a SEA-bimorphism. Define $\phi\colon T \to F$ as follows. If $f = \bigoplus \tau(A_i, a_i)$ is the unique representation of f, then $\phi(f) = \bigoplus \beta(A_i, a_i)$. Notice that $\bigoplus \beta(A_i, a_i)$ is defined because $\bigoplus \beta(A_i, 1) = 1$ and $\beta(A_i, a_i) \leq \beta(A_i, 1)$. To show that ϕ is a SEA-morphism, we have

$$\phi(1) = \beta(X, 1) = 1\,.$$

If $f \perp g$ and $g = \bigoplus \tau(B_j, b_j)$ is the unique representation of g, then

$$f \oplus g = \bigoplus_{i,j} \tau(A_i \cap B_j, a_i \oplus b_j)\,.$$

Assuming for simplicity that $a_i \oplus b_j \neq a_r \oplus b_s$, $i \neq r$, $j \neq s$, we have

$$\begin{aligned}
\phi(f \oplus g) &= \bigoplus_{i,j} \beta(A_i \cap B_j, a_i \oplus b_j) \\
&= \bigoplus_{i,j} \beta(A_i \cap B_j, a_i) \oplus \bigoplus_{i,j} \beta(A_i \cap B_j, b_j) \\
&= \bigoplus \beta(A_i, a_i) \oplus \bigoplus \beta(B_j, b_j) = \phi(f) \oplus \phi(g)\,.
\end{aligned}$$

If $a_i \oplus b_j = a_r \oplus b_s$, we can group these terms together and obtain a similar result. In a similar way we obtain

$$\begin{aligned}
\phi(f \circ g) &= \bigoplus_{i,j} \beta(A_i \cap B_j, a_i \circ b_j) = \bigoplus_{i,j} \beta(A_i, a_i) \circ \beta(B_j, b_j) \\
&= \bigoplus \beta(A_i, a_i) \circ \bigoplus \beta(B_j, b_j) = \phi(f) \circ \phi(g)\,.
\end{aligned}$$

We conclude that ϕ is a SEA-morphism. Moreover,

$$\beta(A, a) = \phi\big(\tau(A, a)\big) = \phi \circ \tau(A, a)$$

for every $A \in \mathcal{B}$, $a \in E$ so that $\beta = \phi \circ \tau$. □

We now characterize pairs E and F of SEA's that admit a tensor product.

A finite sequence $A = \{(a_i, b_i)\}$ in $E \times F$ is *orthosummable* if $\bigoplus \beta(A) := \bigoplus \beta(a_i, b_i)$ is defined for every SEA-bimorphism β.

We say that E and F are *tensoral* if

(1) there exists a SEA-bimorphism $\beta \colon E \times F \to G$ for some SEA G,

(2) for any orthosummable sequence A in $E \times F$ and $(c, d) \in E \times F$ there exists an orthosummable sequence C in $E \times F$ such that

$$\bigoplus \beta(C) = \left[\bigoplus \beta(A)\right] \circ \beta(c, d)$$

for every SEA-bimorphism β.

LEMMA 3.2. *If E and F are commutative and satisfy* (1), *then E and F are tensoral.*

Proof. To show that (2) holds, suppose that $A = \{(a_i, b_i)\}$ is orthosummable in $E \times F$ and $(c, d) \in E \times F$. Let $C = \{(a_i \circ c, b_i \circ d)\}$. Then for any SEA-bimorphism β on $E \times F$ we have

$$\begin{aligned} \bigoplus \beta(C) &= \bigoplus \beta(a_i \circ c, b_i \circ d) = \bigoplus \beta(c \circ a_i, d \circ b_i) \\ &= \bigoplus [\beta(c, d) \circ \beta(a_i, b_i)] \\ &= \beta(c, d) \circ \bigoplus \beta(a_i, b_i). \end{aligned}$$

Since

$$\beta(c, d) \circ \beta(a_i, b_i) = \beta(c \circ a_i, d \circ b_i) = \beta(a_i \circ c, b_i \circ d) = \beta(a_i, b_i) \circ \beta(c, d),$$

we have

$$\bigoplus \beta(C) = \left[\bigoplus \beta(a_i, b_i)\right] \circ \beta(c, d) = \left[\bigoplus \beta(A)\right] \circ \beta(c, d).$$

□

THEOREM 3.3. *The SEA tensor product of E and F exists if and only if E and F are tensoral.*

Proof. Suppose the SEA tensor product of E and F exists. Then clearly (1) holds. To show that (2) holds, suppose $A = \{(a_i, b_i)\}$ is orthosummable in $E \times F$ and $(c, d) \in E \times F$. By (T1) there exists a finite sequence $C = \{(c_j, d_j)\}$ in $E \times F$ such that

$$\bigoplus \tau(c_j, d_j) = \left[\bigoplus \tau(a_i, b_i)\right] \circ \tau(c, d).$$

If $\beta\colon E\times F\to G$ is a SEA-bimorphism, by (T2) there exists a SEA-morphism $\phi\colon T\to G$ such that $\beta=\phi\circ\tau$. Hence,

$$\begin{aligned}\bigoplus\beta(C)&=\bigoplus\beta(c_j,d_j)=\bigoplus\phi\circ\tau(c_j,d_j)=\phi\left[\bigoplus\tau(c_j,d_j)\right]\\&=\phi\left\{\left[\bigoplus\tau(a_i,b_i)\right]\circ\tau(c,d)\right\}=\left[\bigoplus\phi\circ\tau(a_i,b_i)\right]\circ\phi\circ\tau(c,d)\\&=\left[\bigoplus\beta(A)\right]\circ\beta(c,d)\,.\end{aligned}$$

We conclude that E and F are tensoral.

Conversely, suppose E and F are tensoral. Let $\mathcal{K}$ be the set of all finite sequences K in $E\times F$ such that $\bigoplus\beta(K)=1$ for every SEA-bimorphism β on $E\times F$. Now $\mathcal{K}\neq\emptyset$ because $\{(1,1)\}\in\mathcal{K}$. Let $\mathcal{E}(\mathcal{K})$ be the set of all finite sequences $\{(a_i,b_i)\}_{i=1}^n$ in $E\times F$ for which there exists a finite sequence $\{(c_j,d_j)\}_{j=1}^m$ in $E\times F$ such that

$$\{(a_1,b_1),\dots,(a_n,b_n),(c_1,d_1),\dots(c_m,d_m)\}\in\mathcal{K}\,.$$

On $\mathcal{E}(\mathcal{K})$ we define a relation $\sim$ by $A\sim B$ if $\bigoplus\beta(A)=\bigoplus\beta(B)$ for every SEA-bimorphism β on $E\times F$. Then $\sim$ is an equivalence relation and for $A\in\mathcal{E}(\mathcal{K})$ we define

$$\pi(A)=\{B\in\mathcal{E}(\mathcal{K}):\ B\sim A\}\,.$$

Let $\pi(\mathcal{K})=\{\pi(A):\ A\in\mathcal{E}(\mathcal{K})\}$ and define $0\in\pi(\mathcal{K})$ by $0=\pi[\{(0,0)\}]$ and $1\in\pi(\mathcal{K})$ by $1=\pi[\{(1,1)\}]$.

We now organize $\pi(\mathcal{K})$ into a SEA as follows. For $A=\{(a_i,b_i)\}_{i=1}^n\in\mathcal{E}(\mathcal{K})$ and $B=\{(c_j,d_j)\}_{j=1}^m\in\mathcal{E}(\mathcal{K})$ define $\pi(A)\perp\pi(B)$ if

$$C=\{(a_1,b_1),\dots,(a_n,b_n),(c_1,d_1),\dots,(c_m,d_m)\}\in\mathcal{E}(\mathcal{K})\,.$$

Then there exists a $D\in\mathcal{E}(\mathcal{K})$ such that

$$\bigoplus\beta(A)\oplus\bigoplus\beta(B)\oplus\bigoplus\beta(D)=1$$

for every SEA-bimorphism β on $E\times F$. The relation $\perp$ is well defined because if $A'\sim A$ and $B'\sim B$, then for any SEA-bimorphism β on $E\times F$ we have

$$\bigoplus\beta(A')\oplus\bigoplus\beta(B')\oplus\bigoplus\beta(D)=1\,.$$

Hence, for $A'=\{(a_i',b_i')\}_{i=1}^{n'}$ and $B'=\{(c_j',d_j')\}_{j=1}^{m'}$ we have

$$C'=\{(a_1',b_1'),\dots,(a_{n'}',b_{n'}'),(c_1',d_1'),\dots,(c_{m'}',d_{m'}')\}\in\mathcal{E}(\mathcal{K})\,.$$

If $\pi(A)\perp\pi(B)$, we define $\pi(A)\oplus\pi(B)=\pi(C)$ and this is well defined because $\bigoplus\beta(C)=\bigoplus\beta(C')$ for every SEA-bimorphism β on $E\times F$. It is easy to check that $(\pi(\mathcal{K}),0,1,\oplus)$ is an effect algebra. For $A,B\in\mathcal{E}(\mathcal{K})$ given previously there exists a $C\in\mathcal{E}(\mathcal{K})$ such that

$$\bigoplus\beta(C)=\bigoplus\beta(A)\circ\bigoplus\beta(B)$$

for every SEA-bimorphism on $E\times F$. Indeed, by (2) there exists a finite sequence C_j in $E\times F$ that is orthosummable and satisfies

$$\bigoplus\beta(C_j)=\left[\bigoplus\beta(A)\right]\circ\beta(c_j,d_j)\,,\qquad j=1,\ldots,m\,,$$

for every β. Let C be the finite sequence in $E\times F$ formed by concatenating $C_1,\ldots,C_m$. Then for every β we have

$$\bigoplus\beta(C)=\bigoplus_{j=1}^{m}\left[\bigoplus\beta(A)\right]\circ\beta(c_j,d_j)=\bigoplus\beta(A)\circ\bigoplus\beta(B)\,.$$

Letting $B'\in\mathcal{E}(\mathcal{K})$ such that $\bigoplus\beta(B)\oplus\bigoplus\beta(B')=1$, there exists a finite sequence C' in $E\times F$ satisfying

$$\bigoplus\beta(C')=\bigoplus\beta(A)\circ\bigoplus\beta(B')$$

for every β. Hence, for every SEA-bimorphism β on $E\times F$ we have

$$\begin{aligned}\bigoplus\beta(C)\oplus\bigoplus\beta(C')&=\bigoplus\beta(A)\circ\bigoplus\beta(B)\oplus\bigoplus\beta(A)\circ\bigoplus\beta(B')\\&=\bigoplus\beta(A)\,.\end{aligned}$$

It follows that $C\in\mathcal{E}(\mathcal{K})$. Define $\pi(A)\circ\pi(B)=\pi(C)$ and it is clear that this is well defined.

We now show that $\circ$ is a sequential product on $\pi(\mathcal{K})$. Suppose that $\pi(B)\perp\pi(C)$. Now there exists $D,E\in\mathcal{E}(\mathcal{K})$ such that for every β we have

$$\begin{aligned}\bigoplus\beta(D)&=\bigoplus\beta(A)\circ\bigoplus\beta(B)\,,\\\bigoplus\beta(E)&=\bigoplus\beta(A)\circ\bigoplus\beta(C)\,.\end{aligned}$$

Hence, for every β we have

$$\bigoplus\beta(D)\oplus\bigoplus\beta(E)=\bigoplus\beta(A)\circ\left[\bigoplus\beta(B)\oplus\bigoplus\beta(C)\right]\,.$$

It follows that $\pi(A)\circ\pi(B)\perp\pi(A)\circ\pi(C)$ and

$$\pi(A)\circ\left[\pi(B)\oplus\pi(C)\right]=\pi(D)\oplus\pi(E)=\pi(A)\circ\pi(B)\oplus\pi(A)\circ\pi(C)$$

so that (S1) holds. For $K\in\mathcal{K}$ and $A\in\mathcal{E}(\mathcal{K})$ we have

$$\bigoplus\beta(A)=\bigoplus\beta(K)\circ\bigoplus\beta(A)$$

for every β. Hence,

$$1 \circ \pi(A) = \pi(K) \circ \pi(A) = \pi(A)$$

so that (S2) holds. Now it is easy to show that $\pi(A) \perp \pi(B)$ if and only if $\bigoplus \beta(A) \mid \bigoplus \beta(B)$ for every β. Suppose that $\pi(A) \circ \pi(B) = 0$. Then $\bigoplus \beta(A) \circ \bigoplus \beta(B) = 0$ so that $\bigoplus \beta(A) \mid \bigoplus \beta(B)$ for every β and hence $\pi(A) \mid \pi(B)$. Thus, (S3) holds. If $\pi(A) \perp \pi(B)$, then $\bigoplus \beta(A) \mid \bigoplus \beta(B)$ and it follows that $\pi(A) \mid \pi(B)'$. Moreover, for every $C \in \mathcal{E}(\mathcal{K})$ and every β we have that

$$\bigoplus \beta(A) \circ \left[\bigoplus \beta(B) \circ \bigoplus \beta(C)\right] = \left[\bigoplus \beta(A) \circ \bigoplus \beta(B)\right] \circ \bigoplus \beta(C).$$

Hence,

$$\pi(A) \circ \left[\pi(B) \circ \pi(C)\right] = \left[\pi(A) \circ \pi(B)\right] \circ \pi(C)$$

and (S4) holds. The last property (S5) is similar. We conclude that $\left(\pi(\mathcal{K}), 0, 1, \oplus, \circ\right)$ is a SEA.

Notice that for every $(a,b) \in E \times F$ we have that $\{(a,b)\} \in \mathcal{E}(\mathcal{K})$. This is because

$$\left\{(a,b),(a,b'),(a',1)\right\} \in \mathcal{K}.$$

Define $\tau\colon E \times F \to \pi(\mathcal{K})$ by $\tau(a,b) = \pi\left[\{(a,b)\}\right]$. Clearly, $\tau(1,1) = 1$. For $b \perp c$, we have $\left\{(a, b \oplus c)\right\} \sim \left\{(a,b),(a,c)\right\}$. Hence,

$$\tau(a, b \oplus c) = \tau(a,b) \oplus \tau(a,c)$$

and similarly

$$\tau(a \oplus d, b) = \tau(a,b) \oplus \tau(d,b).$$

For $(a,b),(c,d) \in E \times F$ we have

$$\beta(a,b) \circ \beta(c,d) = \beta(a \circ c, b \circ d)$$

for every β. Hence,

$$\pi\left[\{(a,b)\}\right] \circ \pi\left[\{c,d)\}\right] = \pi\left[\{a \circ c, b \circ d)\}\right]$$

and we have

$$\tau(a,b) \circ \tau(c,d) = \tau(a \circ c, b \circ d).$$

Thus, τ is a SEA-bimorphism. Any element in $\pi(\mathcal{K})$ has the form

$$\begin{aligned}\pi\left[\{(a_1,b_1),\dots,(a_n,b_n)\}\right] &= \pi\left[\{(a_1,b_1)\}\right] \oplus \cdots \oplus \pi\left[\{(a_n,b_n)\}\right] \\ &= \tau(a_1,b_1) \oplus \cdots \oplus \tau(a_n,b_n)\end{aligned}$$

for $\left\{(a_1,b_1),\dots,(a_n,b_n)\right\} \in \mathcal{E}(\mathcal{K})$. Finally, let $\beta\colon E \times F \to G$ be a SEA-bimorphism. Define $\phi\colon \pi(\mathcal{K}) \to G$ by $\phi\left[\pi(A)\right] = \bigoplus \beta(A)$. Then ϕ is well defined and it is easy to check that ϕ is a SEA-morphism. Moreover,

$$\beta(a,b) = \phi\left[\pi\left(\{(a,b)\}\right)\right] = \phi \circ \tau(a,b)$$

for every $(a,b) \in E \times F$ so that $\beta = \phi \circ \tau$. □

It is shown in [4] that the effect algebra tensor product of two effect algebras exists if they admit an effect algebra bimorphism. The following corollary is the analogous result for commutative SEA's.

COROLLARY 3.4. *If E and F are commutative SEA's satisfying Condition* (1)*, then the SEA tensor product of E and F exists.*

EXAMPLE 1. Define $\beta\colon [0,1]\times[0,1]\to[0,1]$ by $\beta(a,b)=ab$. It is easy to check that β is a SEA-bimorphism so by Corollary 3.4, the SEA tensor product of $[0,1]$ with itself exists.

EXAMPLE 2. For fuzzy set systems $\mathcal{F}\subseteq[0,1]^X$, $\mathcal{G}\subseteq[0,1]^Y$ define $\beta\colon \mathcal{F}\times\mathcal{G}\to[0,1]^{X\times Y}$ by $\beta(f,g)(x,y)=f(x)g(y)$. It is easy to check that β is a SEA-bimorphism so by Corollary 3.4, the SEA tensor product of $\mathcal{F}$ and $\mathcal{G}$ exists.

EXAMPLE 3. Let $\mathcal{E}(H_1)\otimes\mathcal{E}(H_2)$ be the standard Hilbert space tensor product and define $A\circ B=A^{1/2}BA^{1/2}$ as usual in both $\mathcal{E}(H_1)$ and $\mathcal{E}(H_2)$. Define

$$\beta\colon \mathcal{E}(H_1)\times\mathcal{E}(H_2)\to\mathcal{E}(H_1)\otimes\mathcal{E}(H_2)$$

by $\beta(A_1,A_2)=A_1\otimes A_2$. Then β is an effect algebra bimorphism and

$$\begin{aligned}\beta(A_1,A_2)\circ\beta(B_1,B_2)&=A_1\otimes A_2\circ B_1\otimes B_2\\&=(A_1\otimes A_2)^{1/2}B_1\otimes B_2(A_1\otimes A_2)^{1/2}\\&=A_1^{1/2}\otimes A_2^{1/2}B_1\otimes B_2A_1^{1/2}\otimes A_2^{1/2}\\&=A_1^{1/2}B_1A_1^{1/2}\otimes A_2^{1/2}B_2A_2^{1/2}\\&=\beta(A_1\circ B_1,A_2\circ B_2)\,.\end{aligned}$$

Hence, β is a SEA-bimorphism so Condition (1) holds. We do not know whether Condition (2) holds; i.e., whether the SEA-tensor product of $\mathcal{E}(H_1)$ with $\mathcal{E}(H_2)$ exists.

It is shown in [10] that the effect algebra tensor product of two effect algebras need not exist. We do not know the corresponding state of affairs for the SEA tensor product of two arbitrary SEA's.

REFERENCES

[1] BENNETT, M. K.—FOULIS, D. J.: *Interval and scale effect algebras*, Adv. Appl. Math. **91** (1997), 200–215.

[2] BUSCH, P.—LAHTI, P. J.—MIDDLESTAEDT, P.: *The Quantum Theory of Measurements*, Springer-Verlag, Berlin, 1991.

[3] BUSCH, P.—GRABOWSKI, M.—LAHTI, P. J.: *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[4] DVUREČENSKIJ, A.: *Tensor product of difference posets*, Trans. Amer. Math. Soc. **347** (1995), 1043–1057.

[5] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.

[6] FOULIS, D. J.—BENNETT, M. K.: *Tensor products of orthoalgebras*, Order **10** (1993), 271–282.

[7] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[8] GIUNTINI, R.—GREULING, H.: *Toward a formal language for unsharp properties*, Found. Phys. **19** (1989), 931–945.

[9] GUDDER, S.: *A histories approach to quantum mechanics*, J. Math. Phys. **39** (1998), 5772–5788.

[10] GUDDER, S.—GREECHIE, R.: *Effect algebra counterexamples*, Math. Slovaca **46** (1996), 317–325.

[11] GUDDER, S.—GREECHIE, R.: *Sequential products on effect algebras*, Rep. Math. Phys. **49** (2002), 87–111.

[12] GUDDER, S.—GREECHIE, R.: *Uniqueness and order in sequential effect algebras*, Internat. J. Theoret. Phys. (To appear).

[13] GUDDER, S.—NAGY, G.: *Sequentially independent effects*, Proc. Amer. Math. Soc. **130** (2001), 1125–1130.

[14] GUDDER, S.—NAGY, G.: *Sequential quantum measurements*, J. Math. Phys. **42** (2001), 5212–5222.

[15] KÔPKA, F.—CHOVANEC, F.: *D-Posets*, Math. Slovaca **44** (1994), 21–34.

Received January 28, 2003

Department of Mathematics
University of Denver
Denver, Colorado 80208
U. S. A.
E-mail: sgudder@math.du.edu