Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Irena Rachůnková; Svatoslav Staněk

Zeros of derivatives of solutions to singular $(p, n-p)$ conjugate BVPs

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 43 (2004), No. 1, 137--141

Persistent URL: `http://dml.cz/dmlcz/132938`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **43** (2004) 137–141

Zeros of Derivatives of Solutions to Singular $(p, n-p)$ Conjugate BVPs *

Irena RACHŮNKOVÁ [1], Svatoslav STANĚK [2]

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: [1] *rachunko@inf.upol.cz*
[2] *stanek@inf.upol.cz*

(Received January 10, 2004)

Abstract

Positive solutions of the singular $(p, n-p)$ conjugate BVP are studied. The set of all zeros of their derivatives up to order $n-1$ is described. By means of this, estimates from below of the solutions and the absolute values of their derivatives up to order $n-1$ on the considered interval are reached. Such estimates are necessary for the application of the general existence principle to the BVP under consideration.

Key words: Singular conjugate BVP, positive solutions, zeros of derivatives, estimates from below.

2000 Mathematics Subject Classification: 34B15, 34B16, 34B18

1 Introduction

Let $n, p \in \mathbb{N}$, $n > 2$, $p \le n-1$, and T be a positive number. In [3] (for $p = 1$) and [6], the authors have considered the singular $(p, n-p)$ conjugate boundary value problem (BVP)

$$(-1)^p x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t)), \tag{1.1}$$

*Supported by Grant No. 201/04/1077 of the Grant Agency of the Czech Republic and by the Council of Czech Government J14/98 153100011

$$x^{(i)}(0)=0,\quad x^{(j)}(T)=0\quad 0\le i\le n-p-1,\ 0\le j\le p-1, \tag{1.2}$$

where f satisfies the local Carathéodory conditions on the set $\mathcal{D}=[0,T]\times((0,\infty)\times\mathbb{R}_0^{n-1})$ with $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$ and f is singular at the value 0 of each its phase variable. They have given conditions on f guaranteeing the existence of a positive (on $(0,T)$) solution to BVP (1.1), (1.2). The singularities of the function f in (1.1) 'appear' in any positive solution of BVP (1.1), (1.2) and some its derivatives at the fixed points $t=0$, $t=T$, and all its derivatives up to order $n-1$ 'pass through' singularities of f also at inner points of the interval $(0,T)$ which are not fixed. Therefore for proving the solvability of BVP (1.1), (1.2) in the class of positive functions on $(0,T)$ it is very important to give a localization analysis of zeros of derivatives up to order $n-1$ of positive solutions to BVP (1.1), (1.2). This analysis have been presented for $p=1$ in [3] and for $p=2$ in [6] under the assumption that $f\ge c$ on $\mathcal{D}$ with a positive constant c. The aim of this paper is to complete this analysis for all values of p. We note that the singular differential equation

$$(-1)^p x^{(n)}(t)=\phi(t)g(t,x(t)) \tag{1.3}$$

together with the boundary conditions (1.2) have been discussed for $\phi(t)g(t,x):(0,1)\times(0,\infty)\to(0,\infty)$ continuous in [1], [2], [4] and [5] (in [4] and [5] with $\phi=1$). But for BVP (1.3), (1.2) singularities of g 'appear' in its positive solutions only at the fixed points $t=0$ and $t=1$.

2 Localization analysis of zeros to solutions of BVP (1.1), (1.2)

Let c be a positive constant and let f in (1.1) satisfy $f\ge c$ on $\mathcal{D}$. Then the localization analysis of zeros to solutions of BVP (1.1), (1.2) and their derivatives up to order $n-1$ can be studied by the localization analysis of zeros to solutions of the differential inequality

$$(-1)^p x^{(n)}(t)\ge c \tag{2.1}$$

satisfying the boundary conditions (1.2). By a *solution of problem* (2.1), (1.2) we understand a function $x\in AC^{n-1}([0,T])$ (functions having absolutely continuous $(n-1)^{\text{st}}$ derivative on $[0,T]$) satisfying (2.1) for a.e. $t\in[0,T]$ and fulfilling (1.2).

Having a solution x of problem (2.1), (1.2) we are interested in zeros of $x^{(k)}$, $0\le k\le n-1$, belonging to $(0,T)$. Without loss of generality we can suppose

$$p-1\le n-p-1 \tag{2.2}$$

that is $p\le n/2$, because by replacing t by $T-t$ we can transform the case $n/2<p$ to (2.2).

For $p=1,2$ we have already studied zeros of $x^{(k)}$ and we have proved the following results:

Lemma 2.1 *Let x be a solution of problem (2.1), (1.2) for $p = 1$. Then $x > 0$ on $(0,T)$ and $x^{(k)}$ has just one zero in $(0,T)$, $1 \le k \le n-1$.*

Proof Lemma follows from [3], Lemmas 2.7 and 2.9. □

Lemma 2.2 *Let x be a solution of problem (2.1), (1.2) for $p = 2$. Then*

(i) $x > 0$ *on* $(0,T)$,

(ii) $x^{(k)}$ *has just one zero in* $(0,T)$ *for* $k = 1$ *and* $k = n-1$,

(iii) $x^{(k)}$ *has just two zeros in* $(0,T)$ *for* $2 \le k \le n-2$.

Proof See [6], Lemmas 2.2. □

Decomposition analysis of zeros to solutions of BVP (2.1), (1.2) with $p \ge 3$ is described in the next theorem.

Theorem 2.3 *Let x be a solution of problem (2.1), (1.2) for $p \ge 3$ and let (2.2) hold. Then*

(i) $x > 0$ *on* $(0,T)$,

(ii) $x^{(k)}$ *has just j zeros in* $(0,T)$ *for* $k = j$ *and* $k = n-j$ *where* $j = 1, 2, \dots, p-1$,

(iii) $x^{(k)}$ *has just p zeros in* $(0,T)$ *for* $p \le k \le n-p$.

Proof The proof is divided into three parts.

I. *Lower bounds for zeros.* By (1.2) we see that x' has at least one zero $t_1^1 \in (0,T)$. Hence $x'(0) = x'(t_1^1) = x'(T) = 0$, which implies that x'' has at least two zeros $t_1^2, t_2^2 \in (0,T)$. So, we have $x''(0) = x''(t_1^2) = x''(t_2^2) = x''(T) = 0$. By induction we conclude that $x^{(j)}$, $j = 3, \dots, p-1$, has at least j zeros $t_1^j, \dots, t_j^j \in (0,T)$ and, due to (1.2) and (2.2) $x^{(j)}(0) = x^{(j)}(t_1^j) = \dots = x^{(j)}(t_j^j) = x^{(j)}(T) = 0$, $j = 3, \dots, p-1$. Therefore $x^{(p)}$ hat at least p zeros in $(0,T)$. Now we will distinguish two cases: $p < n/2$ and $p = n/2$.

1. Let $p < n/2$. Then $p \le n-p-1$ and, by (1.2),

$$x^{(j)}(0) = 0, \quad j = p, \dots, n-p-1.$$

Thus $x^{(k)}$ has at least p zeros in $(0,T)$ for $k = p+1, \dots, n-p$.

2. Let $p = n/2$ (clearly n is even in this case). Then $p = n-p$ and $x^{(n-p)}$ has at least p zeros in $(0,T)$.

Hence we have shown that $x^{(n-p)}$ has at least p zeros in $(0,T)$ in the both cases. Since for $x^{(n-j)}$, $1 \le j \le p-1$, we cannot already use (1.2), we deduce that $x^{(n-j)}$ has at least j zeros in $(0,T)$ for $j = 1, \dots, p-1$. Particularly $x^{(n-1)}$ has at least one zero in $(0,T)$.

II. *Exact number of zeros.* By (2.1), $x^{(n-1)}$ is strictly monotonous and hence it has just one zero in $(0,T)$. Therefore, by I, we deduce that $x^{(n-k)}$ has just k zeros in $(0,T)$ for $2 \le k \le p-1$ and $x^{(k)}$ has just p zeros in $(0,T)$ for

$p \le k \le n-p$. Similarly, $x^{(k)}$ has just k zeros in $(0,T)$ for $1 \le k \le p-1$ and x has no zero in $(0,T)$.

III. *Positivity of* x. Denote by t_1^k the first zero of $x^{(k)}$ in $(0,T)$, $1 \le k \le n-1$. Inequality (2.1) implies that $(-1)^p x^{(n-1)} < 0$ on $[0, t_1^{n-1})$ and $(-1)^p x^{(n-2)} > 0$ on $[0, t_1^{n-2})$. Therefore $(-1)^{p+j} x^{(n-j)} > 0$ on $(0, t_1^{n-j})$ for $j = 3, \dots, p$. Particularly we have $x^{(n-p)} > 0$ on $(0, t_1^p)$, wherefrom, by virtue of (1.2), we obtain $x^{(k)} > 0$ on $(0, t_1^k)$, $1 \le k \le n-p-1$, and consequently $x > 0$ on $(0,T)$. □

Our next theorem provides estimates from below of solutions to problem (2.1), (1.2) and of the absolute value of their derivatives up to order $n-1$ on the interval $[0,T]$. These estimations are necessary to apply the general existence principle of [6] to problem (1.1), (1.2) with f in (1.1) satisfying the inequality $f \ge c$ on $\mathcal{D}$.

Theorem 2.4 *Let x be a solution of problem (2.1), (1.2). Then for any $i \in \{1, \dots, n-1\}$ there are $p_i + 1$ disjoint intervals (a_k, a_{k+1}), $0 \le k \le p_i$, $p_i < (n-1)p$, such that*

$$\bigcup_{k=0}^{p_i} [a_k, a_{k+1}] = [0,T] \tag{2.3}$$

and for each $k \in \{0, \dots, p_i\}$ one of the inequalities

$$|x^{(n-i)}(t)| \ge \frac{c}{i!}(t-a_k)^i \quad \textit{for } t \in [a_k, a_{k+1}] \tag{2.4}$$

or

$$|x^{(n-i)}(t)| \ge \frac{c}{i!}(a_{k+1}-t)^i \quad \textit{for } t \in [a_k, a_{k+1}] \tag{2.5}$$

is satisfied.

Proof Let x be a solution of problem (2.1), (1.2) and let $t_i^j \in (0,T)$ be zeros of $x^{(j)}$ described in Lemmas 2.1, 2.2 and Theorem 2.3. Integrating (2.1) we get

$$\begin{aligned} &(-1)^{p+1} x^{(n-1)}(t) \ge c(t_1^{n-1} - t) && \text{for } t \in [0, t_1^{n-1}] \\ &(-1)^{p} x^{(n-1)}(t) \ge c(t - t_1^{n-1}) && \text{for } t \in [t_1^{n-1}, T]. \end{aligned} \tag{2.6}$$

Now, integrate the first inequality in (2.6) from $t \in [0, t_1^{n-2})$ to t_1^{n-2}, we have

$$(-1)^p x^{n-2}(t) \ge \frac{c}{2}\Big(-(t_1^{n-1} - t_1^{n-2})^2 + (t_1^{n-1} - t)^2 \Big) \ge \frac{c}{2!}(t_1^{n-2} - t)^2.$$

Hence, we get in such a way

$$\begin{aligned} &(-1)^p x^{(n-2)}(t) \ge \tfrac{c}{2!}(t_1^{n-2} - t)^2 && \text{for } t \in [0, t_1^{n-2}] \\ &(-1)^{p+1} x^{(n-2)}(t) \ge \tfrac{c}{2!}(t - t_1^{n-1})^2 && \text{for } t \in [t_1^{n-2}, t_1^{n-1}] \\ &(-1)^{p+1} x^{(n-2)}(t) \ge \tfrac{c}{2!}(t_2^{n-2} - t)^2 && \text{for } t \in [t_1^{n-1}, t_2^{n-2}] \\ &(-1)^{p} x^{(n-2)}(t) \ge \tfrac{c}{2!}(t - t_2^{n-2})^2 && \text{for } t \in [t_2^{n-2}, T]. \end{aligned} \tag{2.7}$$

Choose $i \in \{1, \dots, n-1\}$ and take all different zeros of functions $x^{(n-1)}, \dots, x^{(n-i)}$, which are in $(0, T)$. By Lemmas 2.1, 2.2 and Theorem 2.3, there is a finite number $p_i < (n-1)p$ of these zeros. Let us put them in order and denote by $a_1, \dots, a_{p_i}$. Set $a_0 = 0$, $a_{p_i+1} = T$. In this way we get $p_i + 1$ disjoint intervals (a_k, a_{k+1}), $0 \le k \le p_i$, satisfying (2.3).

If $i = 1$, then for $a_1 = t_1^{n-1}$, $a_2 = T$, we get by (2.6) that $|x^{(n-1)}(t)| \ge c(a_1 - t)$ for $t \in [a_0, a_1]$ and $|x^{(n-1)}(t)| \ge c(t - a_1)$ for $t \in [a_1, a_2]$.

If $i = 2$, we put $t_1^{n-1} = a_1$, $t_1^{n-2} = a_2$, $t_2^{n-2} = a_3$, $T = a_4$, and then (2.7) gives (2.4) or (2.5).

If $i > 2$ and we integrate the inequalities in (2.7) $(i-2)$-times, we get that on each $[a_k, a_{k+1}]$, $k \in \{0, \dots, p_i\}$ either (2.4) or (2.5) has to be fulfilled. □

References

[1] Agarwal, R. P., O'Regan, D.: *Positive solutions for (p,n-p) conjugate boundary value problems.* J. Differential Equations **150** (1998), 462–473.

[2] Agarwal, R. P., O'Regan, D.: *Multiplicity results for singular conjugate, focal, and* (n, p) *problems.* J. Differential Equations **170** (2001), 142–156.

[3] Agarwal, R. P., O'Regan, D., Rachůnková, I., Staněk, S.: *Two-point higher order BVPs with singularities in phase variables.* Computers & Mathematics with Applications **46** (2003), 1799–1826.

[4] Eloe, P. W., Henderson, J.: *Singular nonlinear* $(n-1, 1)$ *conjugate boundary value problems.* Georgian Math. J. **4** (1997), 401–412.

[5] Eloe, P. W., Henderson, J.: *Singular nonlinear* $(k, n-k)$ *conjugate boundary value problems.* J. Differential Equations **133** (1997), 136–151.

[6] Rachůnková, I., Staněk, S.: *General existence principle for singular BVPs and its applications.* Georgian Math. J. **11** (2004), 549–565.