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Dually Residuated ℓ-monoids Having No Non-trivial Convex Subalgebras*

Jan KÜHR

*Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: kuhr@inf.upol.cz*

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Abstract

In this note we describe the structure of dually residuated ℓ-monoids ($DR\ell$-monoids) that have no non-trivial convex subalgebras.

Key words: $DR\ell$-monoid; $GPMV$-algebra; Archimedean property.

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A *dually residuated ℓ-monoid*, a *$DR\ell$-monoid* for short, is an algebra

$$(A, \oplus, 0, \vee, \wedge, \oslash, \otimes)$$

of type $\langle 2, 0, 2, 2, 2, 2\rangle$ such that

(a) $(A, \oplus, 0, \vee, \wedge)$ is a lattice-ordered monoid, i.e., $(A, \oplus, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and $\oplus$ distributes over both $\vee$ and $\wedge$,

(b) for any $a, b \in A$, $a \oslash b$ is the least element $x \in A$ with $x \oplus b \geq a$, and $a \otimes b$ is the least element $y \in A$ with $b \oplus y \geq a$, and

(c) A satisfies the identities

$$((x \oslash y) \vee 0) \oplus y \leq x \vee y, \quad y \oplus ((x \otimes y) \vee 0) \leq x \vee y,$$
$$x \oslash x \geq 0, \quad x \otimes x \geq 0.$$

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If the operation $\oplus$ is commutative then A is called a *commutative DRℓ-monoid*. In such a case, the operations $\oslash$ and $\otimes$ coincide, and also conversely, A is commutative whenever $\oslash = \otimes$.

Commutative $DR\ell$-monoids were originally introduced by K. L. N. Swamy [10] in order to capture the common features of Abelian ℓ-groups and Boolean algebras. The above definition, omitting the commutativity of $\oplus$, is due to T. Kovář [6] and allows us to consider all ℓ-groups in the setting of $DR\ell$-monoids. Indeed, given an arbitrary ℓ-group $(G, +, -, 0, \vee, \wedge)$, then $(G, +, 0, \vee, \wedge, \oslash, \otimes)$ is a $DR\ell$-monoid in which $x \oslash y := x - y$ and $x \otimes y := -y + x$.

The reader familiar with residuated lattices easily recognizes that the name "dually residuated ℓ-monoid" says less than the definition since $DR\ell$-monoids are equivalent to a certain proper subclass of residuated lattices. To be more precise, by a *residuated lattice* we mean an algebra $(L, \cdot, e, \vee, \wedge, \rightarrow, \rightsquigarrow)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$, where $(L, \cdot, e)$ is a monoid, $(L, \vee, \wedge)$ is a lattice and the equivalences

$$a \cdot b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \quad \text{iff} \quad b \leq a \rightsquigarrow c \tag{1}$$

hold for all $a, b, c \in L$. Though it need not be evident at once, it not hard to show that our $DR\ell$-monoids are termwise equivalent to those residuated lattices satisfying the identities

$$x \wedge y = ((x \rightarrow y) \wedge e) \cdot x = x \cdot ((x \rightsquigarrow y) \wedge e). \tag{2}$$

Residuated lattices that fulfil (2) were considered e.g. in [2], [5] under the name *GBL-algebras*.

Now, we shortly review some relevant concepts from [7]. Given a $DR\ell$-monoid A, we define the *absolute value* of $x \in A$ by

$$|x| := x \vee (0 \oslash x) = x \vee (0 \otimes x).$$

A non-empty subset I of A is called an *ideal* if

(I1) $a \oplus b \in I$ for all $a, b \in I$,

(I2) $a \in I$ and $|b| \leq |a|$ imply $b \in I$.

By a *non-trivial* ideal of we mean an ideal I with $\{0\} \subset I \subset A$.

The set $\mathscr{I}(A)$ of all ideals of A partially ordered by set-inclusion forms an algebraic distributive lattice in which infima agree with set-theoretical intersections. Hence for every $X \subseteq A$ there exists the smallest ideal $I(X)$ containing X; for $\emptyset \neq X$ we have

$$I(X) = \{a \in A : |a| \leq |x_1| \oplus \cdots \oplus |x_n| \text{ for some } x_1, \ldots, x_n \in X, n \in \mathbb{N}\}.$$

It can be easily proved that $I \subseteq A$ is an ideal if and only if I is a convex subalgebra of A.

The congruence kernels are characterized as the so-called *normal ideals*: An ideal I of A is said to be *normal* if

$$a \oslash b \in I \quad \text{iff} \quad a \otimes b \in I$$

for every $a, b \in A$. If I is a normal ideal then the relation Θ_I defined via

$$(a, b) \in \Theta_I \quad \text{iff} \quad (a \oslash b) \vee (b \oslash a) \in I$$

is a congruence with $[0]_{\Theta_I} = I$, and conversely, for any congruence Θ on A, $I = [0]_\Theta$ is a normal ideal such that $\Theta_I = \Theta$. Therefore, the congruence lattice of A is isomorphic to the lattice of all normal ideals of A. For the sake of brevity, we write A/I for the quotient algebra A/Θ, where $I = [0]_\Theta$, and the elements of A/I are denoted by a/I rather than $[a]_\Theta$.

There are two basic kinds of $DR\ell$-monoids from which every $DR\ell$-monoid can be built using direct products: ℓ-groups and lower bounded $DR\ell$-monoids, i.e., $DR\ell$-monoids having 0 as a least element.

Let A be an arbitrary $DR\ell$-monoid. Put

$$G_A := \{a \in A : a \oplus (0 \oslash a) = 0 = (0 \oslash a) \oplus a\}$$

and

$$S_A := \{a \in A : 0 \oslash a = 0\} = \{a \in A : 0 \otimes a = 0\}.$$

Both G_A and S_A are ideals of A; obviously, the first one is an ℓ-group and the second one is a lower bounded $DR\ell$-monoid. T. Kovář proved in [6] that A is the direct sum of G_A and S_A. The same result for *GBL*-algebras was independently obtained by N. Galatos and C. Tsinakis (see [2]).

Assume that a $DR\ell$-monoid A has no non-trivial ideals. Since both G_A and S_A are (normal) ideals of A, it is clear that either $A = G_A$ or $A = S_A$. In the former case, A is an ℓ-group having no non-trivial convex ℓ-subgroups, and hence it is an Archimedean totally ordered group which is isomorphic to a subgroup of the additive group of reals equipped with the usual order. Therefore, in the sequel we concentrate on lower bounded $DR\ell$-monoids which have no non-trivial ideals.

For every $x, y \in A$ and $n \in \mathbb{N}_0$, we inductively define

$$0 \odot x := 0, \qquad (n+1) \odot x := n \odot x \oplus x,$$

and

$$x \oslash^0 y := x, \qquad x \oslash^{n+1} y := (x \oslash^n y) \oslash y;$$

$x \otimes^n y$ is defined analogously.

Lemma 1 *Let A be a lower bounded $DR\ell$-monoid. The following are equivalent:*

(a) *A has no non-trivial ideals;*

(b) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \leq n \odot a$;*

(c) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \oslash^n a = 0$;*

(d) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \otimes^n a = 0$.*

Proof Obviously, (b)–(d) are equivalent. Moreover, since

$$I(a) := I(\{a\}) = \{b \in A : b \leq n \odot a \text{ for some } n \in \mathbb{N}\},$$

it follows that each of these conditons is equivalent to (a). □

Lemma 2 *Let A be a lower bounded DRℓ-monoid and H be its normal ideal. Then the ideal lattice $\mathscr{I}(A/H)$ of the quotient DRℓ-monoid A/H is isomorphic to the interval $[H, A]$ of the lattice $\mathscr{I}(A)$.*

Proof If $I \in \mathscr{I}(A)$ and $H \subseteq I$ then

$$\phi(I) := \{x/H : x \in I\}$$

is an ideal of A/H. Conversely, if $J \in \mathscr{I}(A/H)$ then

$$\psi(J) := \{x \in A : x/H \in J\}$$

is an ideal of A such that $H \subseteq \psi(J)$. It is easily seen that the mappings ϕ and ψ are mutually inverse order-preserving bijections between $\mathscr{I}(A/H)$ and $[H, A]$ ordered by set-theoretical inclusion. □

An ideal $I \in \mathscr{I}(A)$ is called *maximal* if $I \subset A$ and there is no ideal $J \in \mathscr{I}(A)$ such that $I \subset J \subset A$. In view of Lemma 2 we have:

Proposition 3 *Let A be a lower bounded DRℓ-monoid and H be a normal ideal with $H \subset A$. Then H is maximal if and only if the quotient DRℓ-monoid A/H has no non-trivial ideals.*

Lemma 4 *Let A be a lower bounded DRℓ-monoid that has no non-trivial ideals. Then for every $a, b \in A$, $a \neq 0$,*

$$a \oslash b = a \Longrightarrow b = 0, \qquad a \otimes b = a \Longrightarrow b = 0.$$

Proof We show that the set

$$J_a := \{x \in A : a \oslash x = a\}$$

is an ideal of A. Clearly, $0 \in J_a$. If $x, y \in J_a$ then $a \oslash (x \oplus y) = (a \oslash y) \oslash x = a \oslash x = a$, so that $x \oplus y \in J_a$. Finally, if $x \in J_a$ and $y \leq x$ then $a = a \oslash x \leq a \oslash y \leq a$, and hence $a = a \oslash y$.

However, since $a \notin J_a$ and A has no non-trivial ideals, it follows that $J_a = \{0\}$, and consequently, $a \oslash b = a$ entails $b = 0$ as claimed. □

Lemma 5 *Let A be a lower bounded DRℓ-monoid having no non-trivial ideals. If $0 < x \leq y < a$ and $a \oslash x = a \oslash y$ or $a \otimes x = a \otimes y$, then $x = y$.*

Proof We have $y = x \vee y = (y \oslash x) \oplus x$, so that $a \oslash x = a \oslash y = a \oslash ((y \oslash x) \oplus x) = (a \oslash x) \oslash (y \oslash x)$. Since $a \oslash x \neq 0$, we obtain $y \oslash x = 0$ by Lemma 4, yielding $y \leq x$, so $x = y$. □

Theorem 6 *Let A be a $DR\ell$-monoid that has no non-trivial ideals. Then A satisfies the identities*

$$x \wedge y = x \oslash ((x \otimes y) \vee 0) = x \otimes ((x \oslash y) \vee 0). \tag{3}$$

Proof In the case when A is an ℓ-group the identities (3) evidently hold. Hence assume that A is a lower bounded $DR\ell$-monoid. Note that $x \oslash ((x \otimes y) \vee 0) = x \oslash (x \otimes y)$ and $x \otimes ((x \oslash y) \vee 0) = x \otimes (x \oslash y)$. If $x \leq y$ then $x \oslash (x \otimes y) = x \oslash 0 = x = x \wedge y$ and also $x \otimes (x \oslash y) = x = x \wedge y$. Further, let $x \not\leq y$, i.e., $x \wedge y < x$. Since both $x \oslash (x \otimes y)$ and $x \otimes (x \oslash y)$ are common lower bounds of $\{x, y\}$, we may suppose that $0 < x \wedge y < x$. In this case we have $0 < x \oslash (x \otimes y) \leq x \wedge y < x$ because $x \oslash (x \otimes y) = 0$ would mean $x = x \oslash y$ yielding $y = 0$ which is impossible due to $0 < x \wedge y$. Finally, we have $x \otimes (x \oslash (x \otimes y)) = x \otimes y = x \otimes (x \wedge y)$ which entails $x \oslash (x \otimes y) = x \wedge y$ by Lemma 5. By replacing $\oslash$ and $\otimes$ we get $x \otimes (x \oslash y) = x \wedge y$. □

Therefore, a $DR\ell$-monoid without non-trivial ideals is either an ℓ-group or is lower bounded and verifies the identities

$$x \wedge y = x \oslash (x \otimes y) = x \otimes (x \oslash y). \tag{4}$$

Such $DR\ell$-monoids were investigated in [8], [9] and called here *generalized pseudo MV-algebras* (*GPMV-algebras* for short). The name is motived by the fact that bounded $GPMV$-algebras are termwise equivalent to pseudo MV-algebras. In the literature, there exist two classes of algebras that are equivalent to $GPMV$-algebras, namely, *integral GMV-algebras* and *Wajsberg pseudo-hoops* (see [2] and [3], respectively).

By [9], every $GPMV$-algebra A can be embedded into the positive cone $G(A)^+$ of an ℓ-group $G(A)$ such that, assuming $A \subseteq G(A)$, A is a lattice ideal of $G(A)^+$ which generates $G(A)^+$ as a semigroup, and the operations $\oslash$, $\otimes$ on A are given as follows:

$$a \oslash b := (a - b) \vee 0, \qquad a \otimes b := (-b + a) \vee 0.$$

Moreover, the ideal lattice $\mathscr{I}(A)$ of A and the lattice $\mathscr{C}(G(A))$ of all convex ℓ-subgroups of $G(A)$ are isomorphic under the mapping assigning to each $I \in \mathscr{I}(A)$ the convex ℓ-subgroup of $G(A)$ generated by I. In view of the well-known fact that an ℓ-group is totally ordered exactly if its lattice of all convex ℓ-subgroups is a chain, this means that A is totally ordered if and only if so is $G(A)$, and hence we gain:

Corollary 7 *Every $DR\ell$-monoid which has no non-trivial ideals is totally ordered.*

In [9], the Archimedean property for $GPMV$-algebra is defined in the following way. Given a $GPMV$-algebra A, we introduce a partial addition $+$ by setting $a + b := a \oplus b$ iff $(a \oplus b) \oslash b = a$, or equivalently, $(a \oplus b) \otimes a = b$. Observe that if $A \subseteq G(A)$, then $+$ is the restriction of the group addition to those pairs of elements of A whose sum belongs to A.

This partial operation is associative in the sense that $a + b$ and $(a + b) + c$ exist iff $b + c$ and $a + (b + c)$ exist and $(a + b) + c = a + (b + c)$, and therefore, for any $a \in A$, $n \in \mathbb{N}_0$, we may define

$$0 \cdot a := 0, \qquad (n + 1) \cdot a := n \cdot a + a.$$

Accordingly, we write $a \ll b$ whenever $n \cdot a$ exists and $n \cdot a \leq b$ for all $n \in \mathbb{N}$. Now, we say that a *GPMV*-algebra A is *Archimedean* if $a \not\ll b$ for all $a, b \in A \setminus \{0\}$.

As proved in [9], a *GPMV*-algebra A is Archimedean if and only if $G(A)$ is an Archimedean ℓ-group, hence all Archimedean *GPMV*-algebras are commutative. Therefore we conclude:

Theorem 8 *Let A be a DRℓ-monoid having no non-trivial ideals. Then A is either an Archimedean totally ordered group or A is Archimedean totally ordered GPMV-algebra.*

In fact, if A is a totally ordered Archimedean *GPMV*-algebra then the ℓ-group $G(A)$ is isomorphic to a subgroup of the additive group $\mathbb{R}$ of real numbers with the usual order, and consequently, we may always assume that A is a subset of $\mathbb{R}^+$; the operations $\oslash$ and $\varobslash$ agree and we have $a \oslash b = a \varobslash b = \max\{a - b, 0\}$.

Corollary 9 *Let A be a lower bounded DRℓ-monoid. If H is a normal ideal of A which is simultaneously a maximal ideal, then A/H is a totally ordered Archimedean GPMV-algebra.*

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