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# MATHEMATICAL MODELLING OF ROCK BOLT SYSTEMS I

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*Abstract.* The main goal of the paper is to give a variational formulation of the behaviour of bolt systems in rock mass. The problem arises in geomechanics where bolt systems are applied to reinforce underground openings by inserting steel bars or cables. After giving a variational formulation, we prove the existence and uniqueness and some other properties.

*Keywords*: Boundary value problems for differential equations, variational methods, existence and uniqueness theorems, dependence of solutions of PDE on parameters, rock bolt systems

*MSC 2000*: 35A05, 35A15, 35B30

## 1. Introduction

The term rock bolting is defined in geomechanics as any form of mechanical support that is inserted into the rock mass with the primary objective of increasing its stiffness and/or strength with respect to tensile shear loads. We refer the reader interested in the technical aspect of the procedure to [1], [2]. After describing the variational formulation of the behaviour of isolated bolts, we will deal with the variational formulation of bolt systems. The existence and uniqueness of those two problems as well as the relations between them will be given. But first of all let us describe rock bolts and the way they are applied. We will deal with the following type of rock bolts. The bolts are steel bars or cables which are inserted in rock holes and fixed to the rock at both end points of the bolt with a special cement (Fig. 1).

There is a special technology how to do it and we refer the reader to [1], [2] for the details of that technique.

Bolts are applied to stabilize tunnels and underground openings which can be schematically described in the following three steps:

1. A chamber is made in the rock mass (Fig. 2a).

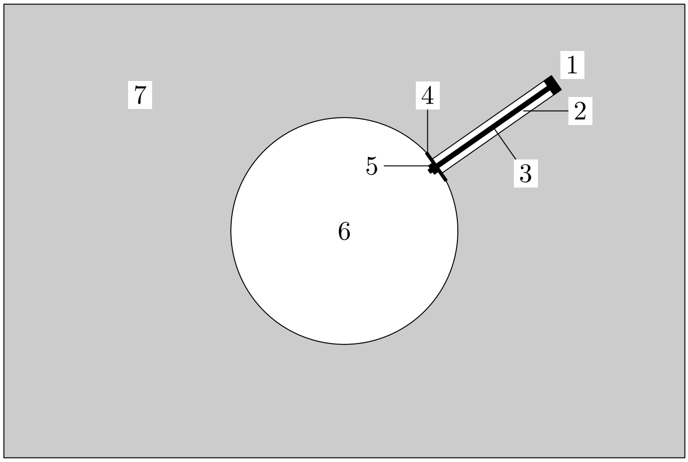


Fig. 1. 1—special cement, 2—bore hole, 3—bolt, 4—bearing plate, 5—nut, 6—tunnel or underground opening, 7—rock mass

2. After the bore holes are made, the bolts are inserted into them and fixed at the end points (Fig. 2b).
3. In the third step the chamber is enlarged (Fig. 2c).

Because of the initial stress state in the rock mass, the bolts come into contact with the rock surrounding which results in the stability of the underground opening. It is very well known in practice that the stress-strain field in the area occupied by bolts considerably differs from the stress-strain field in the same area which is not occupied by any bolts. It is also impossible to achieve a proper result by applying isolated bolts but bolts have to be inserted in sufficient numbers to be able to act as a system. It is evident that it is possible to apply a model with isolated bolts to calculate the stress-strain field. But the number of bolts, applied to support an underground opening, obviously reaches a few hundreds. If we consider this number and the fact that bolts are from two to three metres long and only from two to three centimetres thick so it is almost impossible to solve such problems with finite elements because of the difficulties arising from the construction and regularity of such a finite element mesh as well as because of numerical difficulties. The solution of the problem will be found in the three steps which correspond to those mentioned above. The details of the problem will be specified in the following sections.

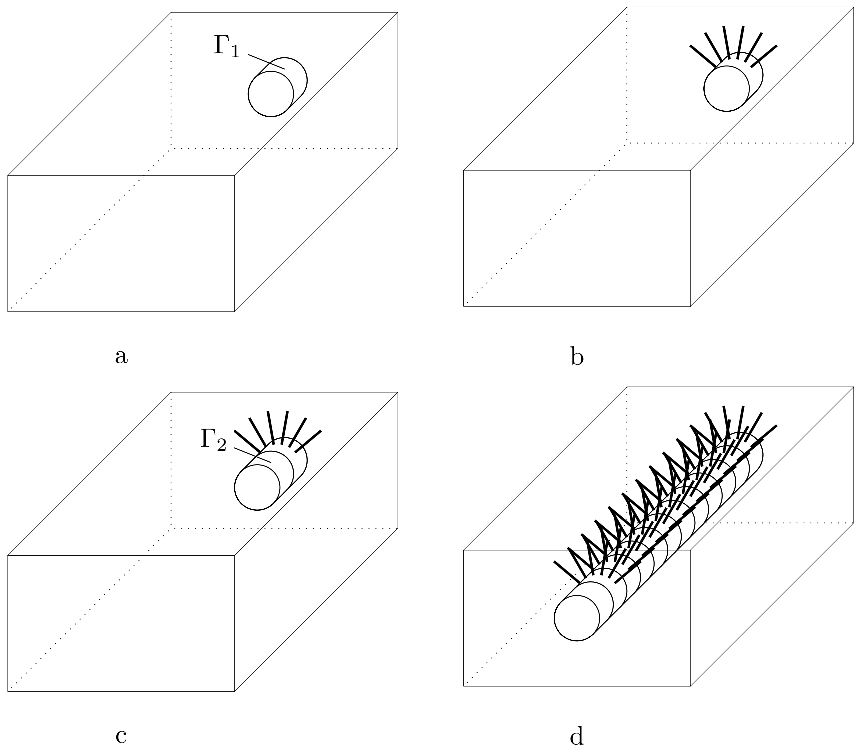


Fig. 2.

## 2. Variational formulation of the problem with individual bolts

Let us start with the variational formulation of the problem which makes up the essential part of the solution to the whole problem. But first of all we will set the conditions we will deal with through the rest of this paper.

1. Linear elastic behaviour of the rock mas.
2. Linear elastic behaviour of the bolts.
3. No contacts between the bolts and the rock mass except at the end points of each bolt.
4. The volume of the bore holes is small in comparison with the underground opening dimensions so that it can be neglected.

To make our explanations clearer the subsequent figures will be two-dimensional and will represent the cross sections of the bodies in Figures 2a, 2b, 2c.

Let an elastic body occupy a bounded region $\Omega$ with a Lipschitz boundary and let $x = (x_1, x_2, x_3)$ be Cartesian coordinates of the point $x$. Let us denote by

$u = (u_1, u_2, u_3)$ the displacement vector field and the associated strain tensor field

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3. \tag{2.1}$$

The stress tensor is related to the strain tensor by means of the following generalized Hook's law

$$\tau_{ij} = c_{ijkl}\, e_{kl}, \qquad i, j = 1, 2, 3. \tag{2.2}$$

We use the following summation convention: whereas a subscript is repeated in a term, summation is required to be taken over that subscript from 1 to 3.

Assume that $c_{ijkl}$ are bounded and measurable functions in $\Omega$ satisfying the conditions

$$c_{ijkl} = c_{jikl} = c_{klij}. \tag{2.3}$$

Moreover, there exists a positive constant $K_0$ such that the inequality

$$c_{ijkl}(x)\, e_{ij}\, e_{kl} \geqslant K_0\, e_{ij}\, e_{ij} \tag{2.4}$$

holds almost everywhere in $\Omega$ for any symmetric $e_{ij}$. Let us have the following decomposition of the boundary $\partial\Omega$:

$$\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_0 \cup \mathscr{R}, \tag{2.5}$$

where $\Gamma_u, \Gamma_\tau, \Gamma_0$ are mutually disjoint open parts and the surface measure of $\mathscr{R}$ is zero. Let the body $\Omega$ be fixed on $\Gamma_u$:

$$u(x) = 0, \quad x \in \Gamma_u \tag{2.6}$$

and let the tractions be prescribed on $\Gamma_\tau$:

$$T_i(u)(x) = \tau_{ij}(x)\, \nu_j(x) = P_i(x), \qquad i = 1, 2, 3, \quad x \in \Gamma_\tau, \tag{2.7}$$

where $\nu(x) = \big(\nu_1(x), \nu_2(x), \nu_3(x)\big)$ is the unit outward normal to $\partial\Omega$ at $x$. Define the normal and tangential components of the displacement and stress vectors by

$$\begin{aligned} u_\nu &= u_j\, \nu_j, & (u_t)_i &= u_i - u_\nu\, \nu_i, \\ T_\nu &= \tau_{jk}\, \nu_j\, \nu_k, & (T_t)_i &= \tau_{ij}\, \nu_j - T_\nu\, \nu_i, \qquad i = 1, 2, 3 \end{aligned} \tag{2.8}$$

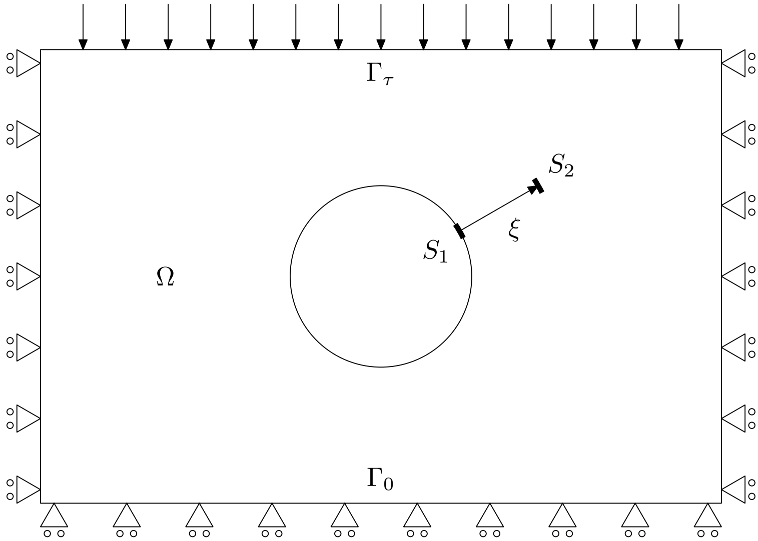


Fig. 3.

and on $\Gamma_0$ put

$$u_\nu = 0, \qquad (T_t)_i = 0, \quad i = 1, 2, 3. \tag{2.9}$$

Let us consider the situation in Fig. 3 which corresponds to the cross section of the body in Fig. 2b. In Fig. 3 the bolt is described by a one-to-one transformation $\xi\colon\ S_1 \mapsto S_2$, where $S_1$, $S_2$ are Lipschitz 2D surfaces which correspond to the two end "points" where the bolt is fixed. The bore hole is neglected due to the condition 4 at the beginning of this chapter. Assume

$$K_1\,|x_1 - x_2| \leqslant |\xi(x_1) - \xi(x_2)| \leqslant K_2\,|x_1 - x_2|,$$
$$x_1, x_2 \in S_1,$$

where $K_1$, $K_2$ are positive constants and $|.\,|$ is the Euclidean norm in $\mathbb{R}^3$. Let us define a function $\gamma\colon\ S_1 \mapsto \mathbb{R}_3$,

$$\gamma(x) = \frac{\xi(x) - x}{|\xi(x) - x|}.$$

Because we deal with small deformations the deformation of the whole bolt length can be approximated by the term $\big\langle u\big(\xi(x)\big) - u(x), \gamma(x)\big\rangle$, where $\langle .,.\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$, $u(x)$ is the displacement in $x$. In our model transversal deformations of the bolt will be neglected.

Denote by

$$V = \Big\{ u \in [H^1(\Omega)]^3 \mid u = 0 \text{ on } \Gamma_u,\ u_n = 0 \text{ on } \Gamma_0 \Big\}$$

the space of virtual displacements and assume that $F \in [L_2(\Omega)]^3$ and $P \in [L_2(\Gamma_\tau)]^3$ are prescribed body forces and surface loads. Let us introduce the forms

$$\begin{aligned} A(u,v) &= \int\limits_\Omega c_{ijkl}\, e_{ij}(u)\, e_{kl}(v)\,\mathrm{d}x, \\ a(u,v) &= \int\limits_{S_1} c(x) \big\langle u\big(\xi(x)\big) - u(x), \gamma(x)\big\rangle \big\langle v\big(\xi(x)\big) - v(x), \gamma(x)\big\rangle\,\mathrm{d}\Gamma, \\ L(v) &= \int\limits_\Omega F_i\, v_i\,\mathrm{d}x + \int\limits_{\Gamma_\tau} P_i\, v_i\,\mathrm{d}\Gamma, \end{aligned} \tag{2.10}$$

where $c = E/d$, $E$ is Young's modulus of the bolt material and $d(x) = |\xi(x) - x|$ is the bolt length. Under the conditions considered above the second form $a(u,v)$ corresponds to the bilinear form of elastic deformation energy of the bolt. Let us define the functional of total potential energy

$$\mathscr{L}(u) = \frac{1}{2}\, A(u,u) + \frac{1}{2}\, a(u,u) - L(u). \tag{2.11}$$

Now we turn to the particular task of finding the weak solutions of some boundary value problems with bolts.

**Definition 2.1.** An element $u \in V$ will be called the solution to the bolt problem if $\mathscr{L}(u) \leqslant \mathscr{L}(v)$ for all $v \in V$.

Let us consider the subspace $R \subset [H^1(\Omega)]^3$ defined by

$$R = \Big\{ v \in [H^1(\Omega)]^3 \mid v(x) = a + b \times x \Big\},$$

where $a, b$ are vectors from $\mathbb{R}^3$ and $\times$ is the vector product. This subspace corresponds to the rigid-body translations and the rigid-body rotations.

**Theorem 2.1.** *Let $F \in [L_2(\Omega)]^3$ and $P \in [L_2(\Gamma_\tau)]^3$ and $\Gamma_u \neq \emptyset$ or $\Gamma_u = \emptyset$ and $R \cap V = \{0\}$. Further let the conditions* (2.3) *be fulfilled and let the function $c(x)$ in* (2.10) *be non-negative. Then there exists one and only one weak solution $u$ of the bolt problem and*

$$\|u\|_{[H^1(\Omega)]^3} \leqslant K \left( \|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3} \right), \tag{2.12}$$

*where $K$ is a positive constant.*

P r o o f. Considering that $a(u,u) \geqslant 0$ for every $u \in V$ the proof is a consequence of Korn's inequality [4] and can be given in the same way as the second basic problem in the theory of elasticity [3]. $\square$

**Theorem 2.2.** *Let $F \in [L_2(\Omega)]^3$, $\Gamma_\tau = \partial\Omega$, $P \in [L_2(\Gamma_\tau)]^3$ and let the conditions of total equilibrium*

$$\int_\Omega F_i \,\mathrm{d}x + \int_{\Gamma_\tau} P_i \,\mathrm{d}\Gamma = 0, \quad i = 1,2,3, \tag{2.13}$$

$$\int_\Omega (x \times F)_i \,\mathrm{d}x + \int_{\Gamma_\tau} (x \times P)_i \,\mathrm{d}\Gamma = 0, \quad i = 1,2,3 \tag{2.14}$$

*be fulfilled. Then there exists a weak solution $u$ of the bolt problem ($V = [H^1(\Omega)]^3$) and $u'$ is another solution to that problem if and only if $u - u' \in R$.*

P r o o f. Let $Q$ be the orthogonal complement of $R$ with respect to the scalar product in $[H^1(\Omega)]^3$. Then from Korn's inequality [3], [4] we can obtain in the same way as in the proof of Theorem 2.1 that there exists a weak solution $u \in Q$. To prove that the weak solution $u$ is a solution on the whole space $[H^1(\Omega)]^3$, it is sufficient to check $A(u,v) = a(u,v) = 0$ for $\forall u \in Q$ and $\forall v \in R$. These two equalities together with the conditions (2.13), (2.14) guarantee that $u$ is a minimum of $\mathscr{L}$. The equality $A(u,v) = 0$ holds because of the very well known fact that $e_{ij}(v) = 0$ if and only if $v \in R$ [3]. For the validity of the equation $a(u,v) = 0$ it is sufficient to prove $\langle v(\xi(x)) - v(x), \gamma(x)\rangle = 0$ for $v \in R$. Let us consider $\gamma(x) = (\xi(x) - x)/|\xi(x) - x|$ and $v \in R$ if and only if $v = Dx + c$ where $D$ is a $3 \times 3$ skew-symmetric matrix and $c$ is a vector in $\mathbb{R}^3$. Then

$$\begin{aligned}
\langle v(\xi(x)) - v(x), \gamma(x)\rangle &= \langle D\xi(x) - Dx, \xi(x) - x\rangle/|\xi(x) - x| \\
&= \langle \xi(x) - x, D^T\xi(x) - D^T x\rangle/|\xi(x) - x| \\
&= -\langle \xi(x) - x, D\xi(x) - Dx\rangle/|\xi(x) - x| \\
&= -\langle v(\xi(x)) - v(x), \gamma(x)\rangle,
\end{aligned}$$

which results in the equality

$$\langle v(\xi(x)) - v(x), \gamma(x)\rangle = 0.$$

If $u'$ is another weak solution of the bolt problem then $u, u'$ satisfy the equations

$$\begin{aligned}
A(u, u - u') + a(u, u - u') &= L(u - u'), \\
A(u', u - u') + a(u', u - u') &= L(u - u').
\end{aligned}$$

Subtracting them we have

$$A(u-u',u-u')+a(u-u',u-u')=0,$$

which yields $A(u-u',u-u')=0$ and consequently $e_{ij}(u-u')=0$. Because of the fact mentioned above $u-u'\in R$. $\square$

## 3. Modelling of bolts as continuous systems

So far we have studied the model of a single bolt which can be easily generalized to several bolts. In this chapter we shall put forward a new model which describes the behaviour of bolts as the behaviour of a "continuous" system. Then we shall compare the new model with the one dealing with distinct bolts.

Let us have a look at Figures 4a, 4b. There is a subarea $\Omega'\subset\Omega$ which is occupied by two different sets of bolts and the surfaces $S_1$, $S_2$ form a part of the boundary $\partial\Omega'$. Unlike in Chapter 2 the surfaces $S_1$, $S_2$ do not correspond to the areas where bolts are fixed but they are wider. The two transformations $\xi_1,\xi_2\colon S_1\mapsto S_2$ in Figures 4a-b geometrically describe the two sets of bolts. On the parts of $S_1$ where the bolts are fixed these transformations are defined in the usual way and on the rest of $S_1$ they are defined so as to be continuous on the whole surface $S_1$. Let us define other two functions $c_1,c_2\colon S_1\mapsto R$ in the following way: $c_1(x)=E/d$ if there is a bolt which is fixed at the point $x$ and $c_1(x)=0$ in the rest of the surface $S_1$. We have considered the first set of bolts and the function $c_2$ is defined in the same way with respect to the second set of bolts. Then we can introduce for $n=1,2$ the forms

$$a_n(u,v)=\int_{S_1} c_n(x)\big\langle u(\xi_n(x))-u(x),\gamma_n(x)\big\rangle\big\langle v(\xi_n(x))-v(x),\gamma_n(x)\big\rangle\,\mathrm{d}\Gamma,\tag{3.1}$$

which correspond to the two bilinear forms of elastic deformation energy of the two sets of bolts.

We can consider $\xi_1$, $c_1$ and $\xi_2$, $c_2$ as the first two steps of a process describing a "spreading" of bolts over the subdomain $\Omega'$. Let us start with the exact definition of this process.

**Definition 3.1.** We say that $\xi_n\colon S_1\mapsto S_2$, $c_n\colon S_1\mapsto R$ *b*-converge to $\xi\colon S_1\mapsto S_2$, $c\colon S_1\mapsto R$ if the following conditions are fulfilled:

1. $\exists K_1,K_2>0$, $\forall n,\forall x,y\in S_1$:

$$K_1|x-y|\leqslant|\xi_n(x)-\xi_n(y)|\leqslant K_2|x-y|,$$
$$K_1|x-y|\leqslant|\,\xi(x)-\xi(y)|\leqslant K_2|x-y|;$$

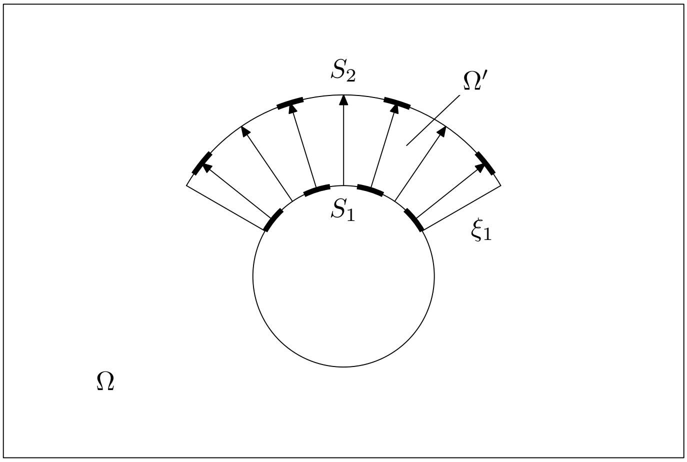


Fig. 4a

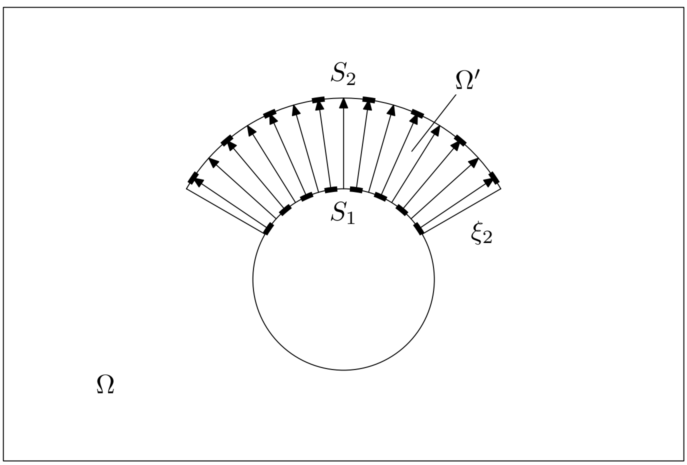


Fig. 4b

2. $\xi_n$ uniformly converges to $\xi$ on $S_1$;
3. $\exists K_3 > 0$, $\forall n$: $\|c_n\|_{L_\infty(S_1)} < K_3$, $\|c\|_{L_\infty(S_1)} < K_3$. $L_\infty(S_1)$ is the space of bounded measurable functions on $S_1$ with the essential norm;
4. $\forall f \in C(S_1)$

$$\int_{S_1} c_n f \,\mathrm{d}\Gamma \to \int_{S_1} c f \,\mathrm{d}\Gamma;$$

   $C(S_1)$ is the space of continuous functions on $S_1$.

Let us present a simple example to demonstrate this type of convergence.

E x a m p l e 3.1. First let us define $S_1, S_2$ and $\xi_n, \xi$: $S_1 = \langle 0,1\rangle \times \langle 0,1\rangle \times \{0\}$, $S_2 = \langle 0,1\rangle \times \langle 0,1\rangle \times \{1\}$, $\xi_n = \xi$ and $\xi(x,y,0) = (x,y,1)$. Now let us define functions $c_n$:

$$c_n(x,y,0) = \begin{cases} 1 \quad \text{on } \left\langle \dfrac{j}{n}, \dfrac{j+\frac{1}{2}}{n} \right\rangle \times \left\langle \dfrac{k}{n}, \dfrac{k+\frac{1}{2}}{n} \right\rangle \times \{0\}\,, \\ \quad j = 0,\ldots,n-1,\ k = 0,\ldots,n-1, \\ \\ 0 \quad \text{on the rest of the surface } S_1. \end{cases}$$

If we consider Definition 3.1, it is evident that $\xi_n, c_n$ $b$-converge to $\xi$, $c$, where $c = \frac{1}{4}$ on the whole $S_1$.

R e m a r k 3.1. Let us notice that in spite of the fact that the functions $c_n$ are discontinuous the limit function $c$ is continuous. Due to the definition we can consider the function $c$ to be equal to $Es\varrho/d$, where $E$ is Young's modulus, $d$ is the length of the bolt, $s$ is the area of the bolt cross section and $\varrho$ is the "density" of bolts on $S_1$.

Let us have $\xi_n, c_n$ which $b$-converge to $\xi, c$; then we have weak solutions $u_n, u$ of the bolt problems corresponding to $\xi_n, c_n$ and $\xi, c$. Then a natural question arises what we can say about the convergence of $u_n$ to $u$. This question will be answered in this chapter but first we define a convergence of bilinear forms and prove an auxiliary lemma.

**Definition 3.2.** Let $a_n(u,v)$, $a(u,v)$ be continuous bilinear forms on the Hilbert space $V$. Then we say that $a_n(u,v)$ converges to $a(u,v)$ if

$$\forall \varepsilon > 0\ \exists n_0,\ \forall n > n_0,\ \forall u,v \in V\colon\ \bigl|a_n(u,v) - a(u,v)\bigr| \leqslant \varepsilon\,\|u\|\,\|v\|,$$

where $\|.\|$ is the norm in the space $V$.

**Lemma 3.1.** *Let $\xi_n, c_n$ $b$-converge to $\xi, c$ and let $a_n(u,v)$, $a(u,v)$ be bilinear forms defined by the relation* (3.1). *Then $a_n(u,v)$ converge to $a(u,v)$.*

P r o o f. It is sufficient to prove that

$$\begin{aligned} &\forall \varepsilon > 0\ \exists\, n_0,\ \forall n > n_0,\ \forall u,v \in [H^1(\Omega)]^3,\ \|u\| \leqslant 1 \wedge \|v\| \leqslant 1 \Rightarrow \\ &\qquad \bigl|a_n(u,v) - a(u,v)\bigr| \leqslant \varepsilon. \end{aligned} \tag{3.2}$$

According to the Kondrachov theorem [5] the trace operator

$$T\colon\ [H^1(\Omega)]^3 \mapsto [L_2(S_1 \cup S_2)]^3$$

transforms the unit ball $B \subset [H^1(\Omega)]^3$ into a precompact set in $[L_2(S_1 \cup S_2)]^3$. So there is a finite set $\{u_1, \dots, u_l\} \subset B$ satisfying

$$\forall u \in B\ \exists\, i \in \{1, \dots, l\}\colon\ \|u - u_i\|_{[L_2(S_1 \cup S_2)]^3} < \varepsilon. \tag{3.3}$$

For the sake of simplicity we will omit the sign of the trace operator $T$. The functions $u_1, \dots, u_l$ can be chosen to be continuous because the set of continuous functions is dense in the space $[H^1(\Omega)]^3$ (see [6]). Due to this fact and the condition 2 from Definition 3.1 the sequences $\bigl\{u_i(\xi_n(x))\bigr\}_{n=1}^{\infty}$, $i = 1, \dots, l$, $\bigl\{\gamma_n(x)\bigr\}_{n=1}^{\infty}$ uniformly converge to $u_i\bigl(\xi(x)\bigr)$, $i = 1, \dots, l$, $\gamma(x)$, from which together with the condition 4 of Definition 3.1 we can derive

$$\exists\, n_0,\ \forall\, i, j \in \{1, \dots, l\}\ \forall n > n_0 : \quad \bigl|a_n(u_i, u_j) - a(u_i, u_j)\bigr| < \varepsilon. \tag{3.4}$$

Now let us estimate $\bigl|a_n(v, w) - a(v, w)\bigr|$ for every $v, w \in B$ and $n \in \mathbb{N}$:

$$\begin{aligned} \bigl|a_n(v, w) - a(v, w)\bigr| \leqslant{} & \bigl|a_n(u_i, u_j) - a(u_i, u_j)\bigr| + \bigl|a_n(v - u_i, u_j)\bigr| \\ & + \bigl|a(v - u_i, u_j)\bigr| + \bigl|a_n(u_i, w - u_j)\bigr| \\ & + \bigl|a(u_i, w - u_j)\bigr| + \bigl|a_n(v - u_i, w - u_j)\bigr| \\ & + \bigl|a(v - u_i, w - u_j)\bigr|. \end{aligned} \tag{3.5}$$

If $n > n_0$ the first term on the right hand side can be estimated by (3.4). Because of (3.1) we can estimate the second term in the following way:

$$\begin{aligned} & \bigl|a_n(v - u_i, u_j)\bigr| \\ & \leqslant K \Bigl(\bigl\|v(\xi_n(x)) - u_i(\xi_n(x))\bigr\|_{[L_2(S_1)]^3} + \bigl\|v(x) - u_i(x)\bigr\|_{[L_2(S_1)]^3}\Bigr) \\ & \qquad \times \Bigl(\bigl\|u_j(\xi_n(x))\bigr\|_{[L_2(S_1)]^3} + \bigl\|u_j(x)\bigr\|_{[L_2(S_1)]^3}\Bigr), \end{aligned} \tag{3.6}$$

where $K$ is independent of $v, u_i, u_j$ and $n$. Considering $u_j \in B$ and the condition 1 of Definition 3.1 we can reformulate the inequality (3.6) into

$$\bigl|a_n(v - u_i, u_j)\bigr| \leqslant K' \bigl\|v(x) - u_i(x)\bigr\|_{[L_2(S_1 \cup S_2)]^3}, \tag{3.7}$$

where $K'$ is independent of $v, u_i, u_j$ and $n$. Similar estimates can be derived for the other terms on the right hand side of (3.5) and consequently that estimate can be rewritten in the following form:

$$\begin{aligned} \bigl|a_n(v, w) - a(v, w)\bigr| & \leqslant \bigl|a_n(u_i, u_j) - a(u_i, u_j)\bigr| \\ & + K''\Bigl(\bigl\|v(x) - u_i(x)\bigr\|_{[L_2(S_1 \cup S_2)]^3} + \bigl\|w(x) - u_j(x)\bigr\|_{[L_2(S_1 \cup S_2)]^3}\Bigr), \end{aligned} \tag{3.8}$$

where $K''$ is independent of $u_i$, $u_j$, $v$, $w$ and $n$. This estimate together with (3.3) and (3.4) give the desired relation (3.2). □

**Theorem 3.1.** *Let the assumptions of Theorem* 2.1 *be fulfilled and let* $\xi_n, c_n$ *b-converge to* $\xi, c$. *Then the sequence* $u_n$ *of the bolt problem solutions, corresponding to* $\xi_n, c_n$*, converges to the bolt problem solution* $u$*, corresponding to* $\xi, c$*, in the space* $[H^1(\Omega)]^3$.

P r o o f. First let us prove that the sequence $u_n$ is bounded in the norm of the space $V \subset [H^1(\Omega)]^3$, which is the same space as the one in Theorem 2.1. Consider the variational equality which reflects the fact that $u_n$ is the bolt problem solution with $\xi_n, c_n$:

$$A(u_n, v) + a_n(u_n, v) = L(v) \qquad \forall v \in V. \tag{3.9}$$

Applying Korn's inequality and replacing $v$ by $u_n$ we have

$$K\,\|u_n\|_V^2 \leqslant A(u_n, u_n) + a_n(u_n, u_n) = L(u_n) \leqslant \|L\|_{V^*}\,\|u_n\|_V;$$

then

$$\|u_n\|_V \leqslant \frac{\|L\|_{V^*}}{K} \tag{3.10}$$

and there exists a subsequence $u_{n_k}$ which converges weakly to $u^*$. Denoting this subsequence by $u_n$ we can rewrite (3.9) into

$$A(u_n, v) + a_n(u_n, v) - a(u_n, v) + a(u_n, v) = L(v). \tag{3.11}$$

If we consider the weak convergence of $u_n$ to $u^*$ and Lemma 3.1 for the term $a_n(u_n, v) - a(u_n, v)$, then according to (3.11) $u^*$ is a solution of the bolt problem with $\xi, c$ and therefore $u^* = u$.

Let us consider other equalities corresponding to the solutions $u_n$, $u$:

$$\begin{aligned} A(u_n, u_n - u) + a_n(u_n, u_n - u) &= L(u_n - u),\\ A(u, u_n - u) + a(u, u_n - u) &= L(u_n - u). \end{aligned} \tag{3.12}$$

Subtracting them we obtain

$$\begin{aligned} &A(u_n - u, u_n - u) + a_n(u_n, u_n - u)\\ &\qquad - a(u_n, u_n - u) + a(u_n - u, u_n - u) = 0. \end{aligned} \tag{3.13}$$

According to Lemma 3.1 and the inequality (3.10) the term $a_n(u_n, u_n - u) - a(u_n, u_n - u)$ converges to 0. Because of the compactness of the trace operator $T$ (see (3.3)) and the weak convergence of $u_n$ to $u$, $a(u_n - u, u_n - u)$ converges to 0 and therefore $A(u_n - u, u_n - u)$ converges to 0, too. Then the desired result is a consequence of Korn's inequality. □

R e m a r k 3.2. In a similar way we can prove the same result for the boundary condition considered in Theorem 2.2. But we have restricted ourselves to a proper subspace of $[H^1(\Omega)]^3$ which guarantees the uniqueness of the problem. For instance the subspace shown in the proof of Theorem 2.2.

R e m a r k 3.3. Theorem 3.1 provides an asymptotic result which encourages us to replace a real bolt system by a "continuous" one which is easier for us to approximate numerically. Moreover, there is a very well known fact from practice that the efficiency of bolting increases if the bolts are inserted regularly with sufficient density.

## 4. Solution of the initial problem

Let us return to the initial problem formulated in Chapter 1. The solution of this problem corresponds to Figures 2a–c. Let us introduce the following domains and surfaces. The domain $\Omega_1$ corresponds to the domain in Fig. 2a. It is the whole parallelepiped without the cylindrical domain corresponding to the original chamber. The domain $\Omega_2$ corresponds to the whole parallelepiped without the enlarged chamber in Fig. 2c. The surface $\Gamma_\tau$ corresponds to the upper surface of the parallelepiped and the surface $\Gamma_0$ to the rest of the boundary of that parallelepiped. The surface $\Gamma_1$ (Fig. 2a) consists of the cylindrical surface and the two front surfaces and represents the boundary of the chamber. The surface $\Gamma_2$ (Fig. 2b) consists of the cylindrical surface and the front surface and corresponds to the part of the boundary of the enlarged chamber, which comes into existence due to the extension of the original chamber.

Let us introduce two subspaces of $[H^1(\Omega_1)]^3$ and $[H^1(\Omega_2)]^3$,

$$V_1 = \Big\{u \in [H^1(\Omega_1)]^3 \mid u_n = 0 \text{ on } \Gamma_0\Big\},$$
$$V_2 = \Big\{u \in [H^1(\Omega_2)]^3 \mid u_n = 0 \text{ on } \Gamma_0\Big\},$$

where $u_n$ is defined by (2.8). Let us consider the forms

$$A_1(u,v) = \int\limits_{\Omega_1} c_{ijkl}\, e_{ij}(u)\, e_{kl}(v)\,\mathrm{d}x, \quad u, v \in V_1,$$

$$A_2(u,v) = \int\limits_{\Omega_2} c_{ijkl}\, e_{ij}(u)\, e_{kl}(v)\,\mathrm{d}x, \quad u, v \in V_2,$$

$$L_1(u) = \int\limits_{\Omega_1} F_i\, u_i\,\mathrm{d}x + \int\limits_{\Gamma_\tau} P_i\, u_i\,\mathrm{d}\Gamma, \quad u, v \in V_1,$$

where $F_i \in [L_2(\Omega_1)]^3$ represents the body forces corresponding to the gravitational force and $P_i \in [L_2(\Gamma_\tau)]^3$ are the loads corresponding to the weight of the rock cover. The following form, like in Chapter 3, models the behaviour of the bolt system:

$$a(u,v) = \int\limits_{S_1} c(x)\big\langle u(\xi(x)) - u(x), \gamma(x)\big\rangle \big\langle v(\xi(x)) - v(x), \gamma(x)\big\rangle \,\mathrm{d}\Gamma.$$

Define the functional of total potential energy

$$\mathscr{L}_1(u) = \frac{1}{2}\, A_1(u,u) - L_1(u), \quad u \in V_1. \tag{4.1}$$

Now we shall describe the solution to our problem in three steps.

1. Let us find a minimum of $\mathscr{L}_1(u)$ on the space $V_1$. Due to the boundary conditions there exists a unique minimum of the functional. Let us denote this minimum by $u_1$, which is a solution of the boundary value problem of elasticity depicted in Fig. 2a (problem without bolts).
2. Let us introduce another form

$$\begin{aligned} L_2(v) = &-\int\limits_{\Omega_2} c_{ijkl}\, e_{ij}(u_1)\, e_{kl}(v)\,\mathrm{d}x + \int\limits_{\Omega_2} F_i\, v_i\,\mathrm{d}x \\ &+ \int\limits_{\Gamma_\tau} P_i v_i \,\mathrm{d}\Gamma, \quad v \in V_2. \end{aligned} \tag{4.2}$$

   Let $u_2$ be the minimum of the functional

$$\mathscr{L}_2(u) = \frac{1}{2}\, A_2(u,u) + \frac{1}{2}\, a(u,u) - L_2(u) + a(u_1\big|_{\Omega_2}, u), \quad u \in V_2. \tag{4.3}$$

3. Let us set $u = u_1\big|_{\Omega_2} + u_2$, where $u_1\big|_{\Omega_2}$ is the restriction of $u_1$ to the subdomain $\Omega_2 \subset \Omega_1$.

The strain-stress field induced by the displacement field $u$ corresponds to the strain-stress field in the rock mass after the whole process (Figures 2a–c) took place, which is the solution to our problem.

Remark 4.1. In this chapter we have described three steps typical for tunnelling. A tunnel is made by the gradual extraction of rock mass accompanied by the gradual installation of bolts for the stabilization of the whole tunnel. The three steps described above give an idea how to go on with the modelling in a more realistic situation, which is shown in Fig. 2d.

R e m a r k 4.2. Let us assume that all functions in (4.2) are sufficiently smooth. Then applying Green's formula, the boundary condition and the fact that $u_1$ is the minimum of $\mathscr{L}_1(u)$, we can write

$$L_2(v) = -\int_{\Gamma_2} T_i(u_1)\, v_i \,\mathrm{d}\Gamma, \quad v \in V_2, \tag{4.4}$$

where $T_i(u)$ is defined by (2.7). Then we can interpret $L_2(v)$ as the loads induced on $\Gamma_1$ by enlarging the chamber. Due to these loads, the bolts come into contact with the rock surrounding, as was described in Introduction. Then $u_2$ together with the corresponding stress field describe the part of displacement and stress fields in $\Omega_2$, which arise because of the enlarging of the chamber and the contact between rock and bolts.

## 5. Some other properties of the modelling of a single bolt

So far we have been interested in the behaviour of bolts like "continuous" systems. Now we pay our attention to some properties of a single bolt. Sometimes in geomechanical literature this type of bolts is modelled in the following way. After the body is approximated by a finite element grid the bolt is described by a relation between the two points of this grid corresponding to the end points of that bolt. The result of this chapter will show that such a modelling can bring about some difficulties. Let us start with the situation described in Fig. 1. Consider the sequences $\xi_n\colon\ S_1^n \mapsto S_2^n$, $c_n\colon\ S_1^n \mapsto R$ possessing the following properties:

$$S_1^n \supset S_1^{n+1}, \quad S_2^n \supset S_2^{n+1}, \tag{5.1}$$

$$\lim_{n\to\infty} \operatorname{diam}(S_1^n) = 0, \quad \lim_{n\to\infty} \operatorname{diam}(S_2^n) = 0. \tag{5.2}$$

The functions $c_n\colon\ S_1^n \mapsto R$ are constant and the following relation is fulfilled:

$$\operatorname{mes}(S_1^n)\, c_n = K_0, \tag{5.3}$$

where $K_0$ is a constant common for all $n$ and mes is the surface measure.

R e m a r k 5.1. The properties (5.1)–(5.3) demonstrate the fact that we gradually replace the bolt by a new one, which is thinner but made of harder material, in the way that the "whole" stiffness of the bolts remains identical.

**Theorem 5.1.** *Let the assumptions of Theorem* 2.1 *be fulfilled and let* $\xi_n, c_n$ *satisfy* (5.1)–(5.3). *Assume that* $c_{ijkl}(x)$ *are of class* $C^1(\overline{\Omega})$ *and the boundary of* $\Omega$ *is of class* $C^2$. *Let* $u_n$ *be the sequence of solutions to the bolt problems corresponding to* $\xi_n, c_n$. *Then* $u_n$ *converges to* $u$ *in the space* $[H^1(\Omega)]^3$, *which is a solution to the elasticity problem (without bolts) with the same boundary conditions.*

Proof. Let $x_1$, $x_2$ be two points which belong to $\Omega$ and satisfy the conditions

$$\forall n \in \mathbb{N}\colon\ x_1 \in S_1^n,\ x_2 \in S_2^n. \tag{5.4}$$

Such points are uniquely determined because of (5.2). Let us consider a function $g\colon \mathbb{R}^3 \mapsto R$ defined by

$$\begin{aligned} g &= 1 \quad \text{if} \quad |x| > 2,\\ g &= 0 \quad \text{if} \quad |x| < 1,\\ g &\in \langle 0, 1\rangle \quad \text{if} \quad 1 \leqslant |x| \leqslant 2. \end{aligned} \tag{5.5}$$

Moreover, this function is of class $C^\infty(\mathbb{R}^3)$. Now let us consider the sequence $g_n(x) = g(x/d_n)$, where $d_n$ is a sequence of positive real numbers chosen in the way that the following conditions are fulfilled: $\operatorname{diam}(S_1^n) < d_n$, $\operatorname{diam}(S_2^n) < d_n$ and $d_n$ converges to 0, which is possible because of (5.2). These conditions imply

$$\begin{aligned} \forall n \in \mathbb{N}\ \forall x \in S_1^n \quad g_n(x - x_1) = 0,\\ \forall n \in \mathbb{N}\ \forall x \in S_2^n \quad g_n(x - x_2) = 0. \end{aligned} \tag{5.6}$$

Let $u$ be the solution to the elasticity problem and consider the sequence

$$\widetilde{u}_n(x) = g_n(x - x_1)\, g_n(x - x_2)\, u(x). \tag{5.7}$$

Define sequences of subdomains $\Omega'_n, \Omega''_n \subset \Omega$

$$\begin{aligned} \Omega'_n &= \bigl\{x \in \Omega \mid |x - x_1| < 2d_n \text{ or } |x - x_2| < 2d_n\bigr\},\\ \Omega''_n &= \Omega \setminus \Omega'_n. \end{aligned}$$

Then the following equality holds:

$$\begin{aligned} \|u - \widetilde{u}_n\|^2_{[H^1(\Omega)]^3} &= \|u - \widetilde{u}_n\|^2_{[H^1(\Omega'_n)]^3} + \|u - \widetilde{u}_n\|^2_{[H^1(\Omega''_n)]^3}\\ &= \|u - \widetilde{u}_n\|^2_{[H^1(\Omega'_n)]^3}. \end{aligned} \tag{5.8}$$

The last equality is a consequence of the fact that $u(x) = \widetilde{u}_n(x)$ on $\Omega''_n$. For this reason, the following inequality holds:

$$\|u - \widetilde{u}_n\|_{[H^1(\Omega)]^3} \leqslant \|u\|_{[H^1(\Omega'_n)]^3} + \|\widetilde{u}_n\|_{[H^1(\Omega'_n)]^3}. \tag{5.9}$$

The first term on the right hand side converges to 0, which we can get from the absolute continuity of the integral [6]. Now we also note that the smoothness hypotheses put on $\Omega$ and the coefficients $c_{ijkl}(x)$ imply $u \in [H^2(\Omega)]^3$ (see [7]).

Moreover, $u \in \left[C\left(\overline{\Omega}\right)\right]^3$ which follows from the Sobolev imbedding theorem [6]. Considering some properties of distributional derivatives we can get the inequality

$$
\begin{aligned}
\left\|\frac{\partial}{\partial x_i}\widetilde{u}_n(x)\right\|_{[L_2(\Omega'_n)]^3} \leqslant{}& \left\|\frac{\partial}{\partial x_i} g_n(x-x_1)\, g_n(x-x_2)\, u(x)\right\|_{[L_2(\Omega'_n)]^3} \\
&+\left\|g_n(x-x_1)\,\frac{\partial}{\partial x_i}\, g_n(x-x_2)\, u(x)\right\|_{[L_2(\Omega'_n)]^3} \\
&+\left\|g_n(x-x_1)\, g_n(x-x_2)\,\frac{\partial}{\partial x_i}\, u(x)\right\|_{[L_2(\Omega'_n)]^3}.
\end{aligned} \tag{5.10}
$$

The third term on the right hand side of (5.10) converges to 0 because of the absolute continuity of the integral. The convergence of the first and the second term is a consequence of the following inequality and the continuity of $u$:

$$
\left\|\frac{\partial}{\partial x_i}\, g_n(x)\right\|_{L_2(\mathbb{R}^3)} < K\sqrt{d_n}, \tag{5.11}
$$

where $K$ is a constant independent of $u$. This result easily follows from the definition of $g_n(x)$. We have just proved that $\widetilde{u}_n$ converges to $u$ in $[H^1(\Omega)]^3$. We also note that the following relation holds for any $v \in V$:

$$
\begin{aligned}
a_n(\widetilde{u}_n, v) = \int_{S_1^n} & c_n(x)\big\langle \widetilde{u}_n(\xi_n(x)) - \widetilde{u}_n(x),\, \gamma_n(x)\big\rangle \\
&\times\big\langle v(\xi_n(x) - v(x),\, \gamma_n(x)\big\rangle\, \mathrm{d}\Gamma = 0.
\end{aligned} \tag{5.12}
$$

This result is a simple consequence of the definition of $g_n(x)$ and the relations (5.6). The fact that $u$ is a solution to the elasticity problem, results in

$$
A(u, v) = L(v) \qquad \forall v \in V. \tag{5.13}
$$

The continuity of $A(.,.)$ implies

$$
|A(u - \widetilde{u}_n, v)| < K\,\|u - \tilde{u}_n\|_{[H^1(\Omega)]^3}\,\|v\|_{[H^1(\Omega)]^3}, \tag{5.14}
$$

where $K$ is a constant independent of $u, \widetilde{u}_n, v$. Subtracting (5.13), (5.14) and applying (5.12) we get the inequality

$$
A(\widetilde{u}_n, v) + a_n(\widetilde{u}_n, v) - L(v) \leqslant K\|u - \widetilde{u}_n\|_{[H^1(\Omega)]^3}\,\|v\|_{[H^1(\Omega)]^3}. \tag{5.15}
$$

Let $u_n$ be the sequence of solutions to the bolt problems, then

$$A(u_n, v) + a_n(u_n, v) - L(v) = 0 \tag{5.16}$$

holds for any $v \in V$. Subtracting (5.15) and (5.16) we obtain

$$A(\widetilde{u}_n - u_n, v) + a_n(\widetilde{u}_n - u_n, v) \leqslant K\|u - \widetilde{u}_n\|_{[H^1(\Omega)]^3}\, \|v\|_{[H^1(\Omega)]^3}. \tag{5.17}$$

Replacing $v$ by $\widetilde{u}_n - u_n$ and applying Korn's inequality we get the relation

$$\begin{aligned} K_1\|\widetilde{u}_n - u_n\|^2_{[H^1(\Omega)]^3} &\leqslant A(\widetilde{u}_n - u_n, \widetilde{u}_n - u_n) + a_n(\widetilde{u}_n - u_n, \widetilde{u}_n - u_n) \\ &\leqslant K\|u - \widetilde{u}_n\|_{[H^1(\Omega)]^3}\, \|\widetilde{u}_n - u_n\|_{[H^1(\Omega)]^3}, \end{aligned} \tag{5.18}$$

where $K_1$, $K$ are constants independent of $u$, $\widetilde{u}_n$, $u_n$. This result together with the convergence of $\widetilde{u}_n$ to $u$ gives the convergence of $u_n$ to $u$. □

Remark 5.2. We also note that some of the smoothness hypotheses on $\Omega$ and the coefficients $c_{ijkl}(x)$ can be weakened. We can restrict the validity of these smoothness conditions to any neighbourhoods of the points $x_1, x_2$.

Remark 5.3. The condition (5.3) was not exploited in the proof of Theorem 5.1 so this convergence result remains valid without this condition. The essential condition is (5.2) and the stiffness of the bolt material can increase without any limit.

## 6. Some other properties of solutions to the bolt problem

So far we have been interested in the existence, uniqueness, and continuous dependence of the solution on the data $(F, P)$. Now we will prove a continuous dependence of the solution on the data characterizing the bolt system.

**Theorem 6.1.** *Let the assumptions of Theorem 2.1 be fulfilled. Moreover, let $\xi(x)\colon S_1 \mapsto S_2$, $c(x)\colon S_1 \mapsto R$, $c'(x)\colon S_1 \mapsto R$ be given, which characterize two different bolt systems and satisfy the usual conditions. Let $u, u'$ be two solutions to the bolt problems corresponding to the data $(F, P, \xi, c)$ and $(F, P, \xi, c')$. Then there exists a constant $K$ independent of $F, P, \xi, c, c'$ and such that the inequality*

$$\|u - u'\|_{[H^1(\Omega)]^3} \leqslant K\, \|c(x) - c'(x)\|_{L_\infty(S_1)} \left(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3}\right) \tag{6.1}$$

*holds.*

P r o o f. Considering that $u, u'$ are solutions to the bolt problems we have the following equations:

$$A(u, u-u') + a\,(u,\, u-u') - L(u-u') = 0, \tag{6.2}$$

$$A(u', u-u') + a'(u', u-u') - L(u-u') = 0, \tag{6.3}$$

where $a(.,.)$, $a'(.,.)$, $L(.)$ are the forms defined by (2.10) and the first two forms correspond to $\xi, c$ and $\xi, c'$. Subtracting the equalities (6.2), (6.3) and applying Korn's inequality we obtain

$$\begin{aligned} K_1 \|u-u'\|^2_{[H^1(\Omega)]^3} &\leqslant A(u-u', u-u') + a(u-u', u-u') \\ &= a'(u', u-u') - a(u', u-u'), \end{aligned} \tag{6.4}$$

where the constant $K_1$ is independent of the data mentioned in Theorem 6.1. Let us estimate the right hand side of (6.4):

$$\begin{aligned} &a'(u', u-u') - a(u', u-u') \\ &\quad = \int_{S_1} \big(c'(x) - c(x)\big)\big\langle u'(\xi(x)) - u'(x), \gamma(x)\big\rangle \\ &\qquad\quad \times \big\langle u(\xi(x)) - u'(\xi(x)) - u(x) + u'(x), \gamma(x)\big\rangle \,\mathrm{d}\Gamma \\ &\quad \leqslant K_2 \big\|c'(x) - c(x)\big\|_{L_\infty(S_1)} \|u'\|_{[H^1(\Omega)]^3} \|u-u'\|_{[H^1(\Omega)]^3}, \end{aligned} \tag{6.5}$$

where $K_2$ is independent of the data mentioned. Let us assume that $u'$ is a solution to the bolt problem. After applying Korn's inequality we get

$$\begin{aligned} K_3 \|u'\|^2_{[H^1(\Omega)]^3} &\leqslant A(u', u') + a'(u', u') = L(u') \\ &\leqslant K_4 \,\big(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3}\big) \|u'\|_{[H^1(\Omega)]^3}, \end{aligned} \tag{6.6}$$

where the constants $K_3$, $K_4$ are also independent of the given data. Combining the inequalities (6.4)–(6.6) we get the desired estimate (6.1). □

R e m a r k 6.1. If we fixed $F, P$ in the estimate (6.1), we could handle it as a continuous dependence of the solution on $c(x)$. Theorem 3.1 gives another type of such a dependence but that dependence cannot be derived from the estimate (6.1). If we consider the sequence $c_n(x)$ in Example 3.1 we can see that $c_n(x)$ does not converge to $c(x)$ in the essential norm so we cannot apply the estimate.

Let $V$ be the subspace of $[H^1(\Omega)]^3$ defined in Chapter 2 and let $V_\xi$ be a subspace of $V$ defined in the following way:

$$V_\xi = \Big\{ u \in V \mid \big\langle u(\xi(x)) - u(x), \gamma(x)\big\rangle = 0 \text{ on } S_1 \Big\},$$

where $\xi\colon S_1 \mapsto S_2$ is a given transformation.

**Theorem 6.2.** *Let the assumptions of Theorem* 6.1 *be fulfilled and let* $\xi(x)\colon S_1 \mapsto S_2$, $c(x)\colon S_1 \mapsto R$ *be given. Moreover, let there exist a positive constant* $K$ *such that* $c(x) \geqslant K$ *for each* $x \in S_1$. *Let* $\lambda_n$ *be a sequence of positive numbers which converges to infinity and let* $u_n$ *be the sequence of the solutions to the bolt problems corresponding to* $\xi(x)$, $\lambda_n c(x)$. *Then* $u_n$ *converges to* $u$ *in* $[H^1(\Omega)]^3$, *which is the minimum of the following functional on* $V_\xi$*:*

$$\mathscr{L}_0 = \frac{1}{2}\,A(u,u) - L(u), \quad u \in V_\xi,$$

*where* $A(.,.)$, $L(.)$ *are defined by* (2.10).

P r o o f. Let $a_n(.,.)$ be the bilinear forms which are defined by (3.1), where $c_n(x)$ are equal to $\lambda_n c(x)$. Due to Korn's inequality

$$\exists K_1 > 0,\ \forall n: \quad K_1\,\|u_n\|^2_{[H^1(\Omega)]^3} \leqslant A(u_n,u_n) + a_n(u_n,u_n) = L(u_n), \tag{6.7}$$

which implies

$$\exists K_2 > 0, \quad \|u_n\|_{[H^1(\Omega)]^3} \leqslant K_2. \tag{6.8}$$

Applying (6.7), (6.8) we get the relation

$$\exists K_3 > 0, \quad a_n(u_n,u_n) \leqslant K_3, \tag{6.9}$$

which implies that $\big\|\big\langle u_n(\xi(x)) - u_n(x), \gamma(x)\big\rangle\big\|_{L_2(S_1)}$ converges to 0. The inequality (6.8) implies that there exists a subsequence of $u_n$, which weakly converges to $u^*$. Because of the Kondrachov theorem [5] and some results from the measure theory [6] we get that $u^*$ belongs to $V_\xi$. Let us consider the sequence of equalities

$$A(u_n,v) + a_n(u_n,v) = L(v), \quad v \in V_\xi. \tag{6.10}$$

Because $v \in V_\xi$ then $a_n(u_n,v) = 0$ for all $n$, which implies that $u^*$ is a minimum of $\mathscr{L}_0(.)$ on $V_\xi$. Consider the terms

$$\begin{aligned} &A(u_n, u_n - u^*) + a_n(u_n, u_n - u^*) - L(u_n - u^*),\\ &A(u^*, u_n - u^*) + a_n(u^*, u_n - u^*) - L(u_n - u^*). \end{aligned} \tag{6.11}$$

The first term equals 0 for all $n$ and the other one converges to 0, because $u_n$ weakly converges to $u^*$ and $a_n(u^*, u_n - u^*) = 0$. Subtracting the terms (6.11) and applying Korn's inequality we get

$$K_n\,\|u_n - u^*\|^2_{[H^1(\Omega)]^3} \leqslant A(u_n - u^*, u_n - u^*) + a_n(u_n - u^*, u_n - u^*).$$

The right hand side of this inequality converges to 0, which is the desired result. $\square$

R e m a r k 6.2. This theorem says that if we gradually replace the material of bolts by a harder one, then this process has got its limit point, which is a solution to the elasticity problem with constrains.

## 7. Another variational problem arising in the modelling of bolt systems

We have been dealing with the linear problems so far but let us notice the situation in Fig. 1. It is standard practice in geomechanics that the bolts are fixed by bearing plates at the ends, which are located on the wall of the underground opening supported by these bolts. These bearing plates rest against the wall without the possibility of penetrating into the rock mass. So any contractions of bolts are impossible. On the other hand, there can exist boundary conditions which cause such contractions in the formulation of the bolt problem given above. For this reason, it is necessary to reformulate the problem.
Let us introduce other two forms

$$\bar{a}(u,v) = \int\limits_{S_1} c(x)\Big[\big\langle u(\xi(x)) - u(x), \gamma(x)\big\rangle\Big]^+ \big\langle v(\xi(x)) - v(x), \gamma(x)\big\rangle \,\mathrm{d}\Gamma,$$

$$\bar{\bar{a}}(u,v) = \int\limits_{S_2} c(x)\Big[\big\langle u(\xi(x)) - u(x), \gamma(x)\big\rangle\Big]^+ \Big[\big\langle v(\xi(x)) - v(x), \gamma(x)\big\rangle\Big]^+ \,\mathrm{d}\Gamma,$$

where the symbol $[\ ]^+$ is defined in the following way:

$$\big[f(x)\big]^+ = \begin{cases} f(x) & \text{if } f(x) \geqslant 0, \\ 0 & \text{if } f(x) < 0. \end{cases}$$

For the restrictions imposed on the bolts, the functional of the total potential energy has to be defined in the following way:

$$\overline{\mathscr{L}}(u) = \frac{1}{2}\,A(u,u) + \frac{1}{2}\,\bar{\bar{a}}(u,u) - L(u), \quad u \in V, \tag{7.1}$$

where the forms $A(.,.)$, $L(.)$ and the space $V$ are defined in Chapter 2.

**Definition 7.1.** An element $u \in V$ will be called a solution to the problem $\overline{\mathscr{P}}$ if $\overline{\mathscr{L}}(u) \leqslant \overline{\mathscr{L}}(v)$ for all $v \in V$.

Under the assumptions mentioned above, the functional $\overline{\mathscr{L}}(.)$ is coercive, differentiable and convex, which results in the existence of the unique solution to the problem $\overline{\mathscr{P}}$. We refer the reader to [8], [9]. Moreover, the existence of a solution to the problem $\overline{\mathscr{P}}$ is equivalent to the existence of a solution to the variational equality (see e.g. [8])

$$A(u,v) + \bar{a}(u,v) = L(v) \quad \forall v \in V. \tag{7.2}$$

Let us notice that the Gateaux differential $D\varphi(u,v)$ at the point $u$, where $\varphi(u) = \frac{1}{2}\,\bar{\bar{a}}(u,u)$, is equal to $\bar{a}(u,v)$.

There is a question whether some of the theorems given above can be proved for the problem $\overline{\mathscr{P}}$. Let us mention that the forms $\bar{a}(.,.)$, $\bar{\bar{a}}(.,.)$ are not bilinear, which results in the non-linearity of the problem $\overline{\mathscr{P}}$.

**Theorem 7.1.** *Let the assumptions of Theorem* 2.1 *be fulfilled and let $\xi_n, c_n$ b-converge to $\xi, c$. Then the sequence $u_n$ of the solutions to the problem $\overline{\mathscr{P}}_n$ corresponding to $\xi_n, c_n$ converges to the solution $u$ to the problem $\overline{\mathscr{P}}$ corresponding to $\xi, c$ in the space $[H^1(\Omega)]^3$.*

P r o o f. We will only give a sketch of the proof because the method is similar to the proof of Theorem 3.1. We will only notice the parts in which these proofs differ from each other.

Let us notice that we can define the convergence of $\bar{a}_n(.,.)$ to $\bar{a}(.,.)$ in the same way by replacing the forms $a_n(.,.)$, $a(.,.)$ in Definition 3.2 by $\bar{a}_n(.,.)$, $\bar{a}(.,.)$. Then we can prove a similar version of Lemma 3.1 for this convergence, which is based on the fact that the trace operator $T\colon\ H^1(\Omega) \mapsto L_2(S)$ is compact, where $S$ is the surface of $\Omega$. This result is a simple consequence of the compact embedding theorem. The proof of this theorem is parallel to the one of Theorem 3.1, we only have to replace $a_n(.,.)$, $a(.,.)$ by $\bar{a}_n(.,.)$, $\bar{a}(.,.)$. The only exception is the relation (3.13) which has to be replaced by

$$\begin{aligned} A(u_n - u, u_n - u) + \bar{a}_n(u_n, u_n - u) - \bar{a}(u_n, u_n - u) \\ + \bar{a}(u_n, u_n - u) - \bar{a}(u, u_n - u) = 0, \end{aligned} \tag{7.3}$$

where $\bar{a}_n(u_n, u_n - u) - \bar{a}(u_n, u_n - u)$ converges to 0, because of the new version of Lemma 3.1. Because of the compactness of the trace operator $T$ (see (3.3)), $\bar{a}(u_n, u_n - u)$, $\bar{a}(u, u_n - u)$ converge to 0, too. The rest of the proof coincides with the part of the proof of Theorem 3.1 which follows from the equation (7.3). □

The following two theorems are the versions of Theorem 6.1 and Theorem 6.2 for the problem $\overline{\mathscr{P}}$.

**Theorem 7.2.** *Let the assumptions of Theorem* 2.1 *be fulfilled. Moreover, let there be* $\xi(x)\colon S_1 \mapsto S_1$, $c(x)\colon S_1 \mapsto R$, $c'(x)\colon S_1 \mapsto R$ *which characterize two different bolt systems and satisfy the usual conditions. Let* $u, u'$ *be solutions to the problems* $\overline{\mathscr{P}}, \overline{\mathscr{P}'}$ *corresponding to the data* $(F, P, \xi, c)$ *and* $(F, P, \xi, c')$. *Then there exists a constant* $K$ *independent of* $F, P, \xi, c, c'$ *such that the following inequality holds:*

$$\|u - u'\|_{[H^1(\Omega)]^3} \leqslant K\|c(x) - c'(x)\|_{L_\infty(S_1)} \left(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3}\right). \tag{7.4}$$

P r o o f. The proof of this theorem is parallel to the one of Theorem 6.1. We only have to replace $a(.,.)$, $a'(.,.)$ by $\bar{a}(.,.)$, $\bar{a}'(.,.)$. Let us notice the following term which corresponds to the term (6.4)

$$\begin{aligned} K_1\|u - u'\|^2_{[H^1(\Omega)]^3} &\leqslant A(u - u', u - u') + \bar{a}(u, u - u') - \bar{a}(u', u - u') \\ &= \bar{a}'(u', u - u') - \bar{a}(u', u - u'), \end{aligned} \tag{7.5}$$

where the constant $K_1$ is independent of the given data and the inequality in that term is a consequence of Korn's inequality $K_1\,\|u - u'\|^2_{[H^1(\Omega)]^3} \leqslant A(u - u', u - u')$. To prove the first inequality in the relation (7.5), it is necessary to verify $\bar{a}(u, u - u') - \bar{a}(u', u - u') \geqslant 0$.

$$\begin{aligned} &\bar{a}(u, u - u') - \bar{a}(u', u - u') \\ &\quad = \int_{S_1} c(x)\Big(\big[\langle u(\xi(x)) - u(x), \gamma(x)\rangle\big]^+ - \big[\langle u'(\xi(x)) - u'(x), \gamma(x)\rangle\big]^+\Big) \\ &\qquad\qquad \times \Big(\langle u(\xi(x)) - u(x), \gamma(x)\rangle - \langle u'(\xi(x)) - u'(x), \gamma(x)\rangle\Big)\,\mathrm{d}\Gamma. \end{aligned} \tag{7.6}$$

The right hand side is non-negative because of the fact that

$$\left([a]^+ - [b]^+\right)(a - b) \geqslant 0$$

for all real numbers. Let us estimate the right hand side of (7.5):

$$\begin{aligned} &\bar{a}'(u', u - u') - \bar{a}(u', u - u') \\ &\quad = \int_{S_1} \big(c'(x) - c(x)\big)\big[\langle u'(\xi(x)) - u'(x), \gamma(x)\rangle\big]^+ \\ &\qquad\qquad \times \langle u(\xi(x)) - u'(\xi(x)) - u(x) + u'(x), \gamma(x)\rangle\,\mathrm{d}\Gamma \\ &\quad \leqslant K_2\,\|c'(x) - c(x)\|_{L_\infty(S_1)}\,\|u'\|_{[H^1(\Omega)]^3}\,\|u - u'\|_{[H^1(\Omega)]^3}, \end{aligned} \tag{7.7}$$

where the constant $K_2$ is independent of the given data for the same reasons as those given in the proof of Theorem 6.1. The rest of the proof coincides with the corresponding part of the proof of Theorem 6.1. □

The reformulation of Theorem 6.2 needs some modifications. Let $V$ be the space defined in Chapter 2. Let $K_\xi$ be a subset of $V$ defined in the following way:

$$K_\xi = \Big\{u \in V \mid \big\langle u(\xi(x)) - u(x), \gamma(x)\big\rangle \leqslant 0 \text{ on } S_1\Big\}.$$

It is evident that $K_\xi$ is closed and convex. The inequality in the definition of $K_\xi$ reflects the fact that the bolts are infinitely stiff but they can be pushed out of the rock mass.

**Theorem 7.3.** *Let the assumptions of Theorem 7.2 be fulfilled and let $\xi(x)\colon S_1 \mapsto S_1$, $c(x)\colon S_1 \mapsto R$ be given. Moreover, let there exist a positive constant $K$ such that $c(x) \geqslant K$ for each $x \in S_1$. Let $\lambda_n$ be a sequence of positive numbers which converges to infinity and let $u_n$ be the sequence of the solutions to the problem $\overline{\mathscr{P}}$ corresponding to $\xi(x)$, $\lambda_n c(x)$. Then $u_n$ converges to $u$ in $[H^1(\Omega)]^3$, which is the minimum of*

$$\mathscr{L}_0 = \frac{1}{2}\, A(u,u) - L(u)$$

*on $K_\xi$.*

P r o o f. The proof is similar to the proof of Theorem 6.2 so we will briefly give the main ideas. The sequence $u_n$ satisfies the sequence of equations

$$A(u_n, u_n) + \bar{a}_n(u_n, u_n) = L(u_n). \tag{7.8}$$

Let us mention that $\bar{a}(u_n, u_n) \geqslant 0$. Then applying Korn's inequality, we get

$$\|u_n\|_{[H^1(\Omega)]^3} \leqslant K_1, \tag{7.9}$$

where $K_1$ is a constant. Combining (7.8) and (7.9) we obtain that there exists a positive constant $K_2$ such that

$$\bar{a}_n(u_n, u_n) \leqslant K_2. \tag{7.10}$$

This fact together with the definitions of $\bar{a}_n(.,.)$ gives that

$$\Big\| \big[\big\langle u_n(\xi(x)) - u_n(x), \gamma(x)\big\rangle\big]^+ \Big\|_{L_2(S_1)} \to 0. \tag{7.11}$$

According to (7.9), (7.11) there is a subsequence of $u_n$ that converges to $u^*$ in $[L_2(S_1 \cup S_2)]^3$, and $u^*$ belongs to $K_\xi$. Consider the sequence of equalities

$$A(u_n, u_n - v) + \bar{a}_n(u_n, u_n - v) = L(u_n - v), \tag{7.12}$$

where $v$ is an arbitrary element from $K_\xi$, which results in the fact that $\bar{a}_n(u_n, u_n - v) \geqslant 0$ for all $n$. Then the sequence of equalities (7.12) leads to the sequence of inequalities

$$A(u_n, u_n - v) \leqslant L(u_n - v). \tag{7.13}$$

If we notice that the functional $\varphi(u) = A(u,u)$ is weakly lower semi-continuous (see e.g. [8]), then (7.13) leads to the inequality

$$A(u^*, u^* - v) \leqslant L(u^* - v), \tag{7.14}$$

which implies that $u^*$ is a minimum of $\mathscr{L}_0$ on $K_\xi$. Consider the two sequences

$$\begin{gathered} A(u_n, u_n - u^*) + \bar{a}_n(u_n, u_n - u^*) - L(u_n - u^*), \\ A(u^*, u_n - u^*) - L(u_n - u^*). \end{gathered} \tag{7.15}$$

The first sequence identically equals 0 while the other one converges to 0. Let us notice that $\bar{a}_n(u_n, u_n - u^*) \geqslant 0$, hence after subtracting these sequences and applying Korn's inequality, we get the desired result. □

## 8. Conclusion

Geomechanical problems are specific in comparison with mechanical engineering ones. It is very difficult for the engineer to obtain the input data for individual geomechanical problems. On the other hand mathematical modelling in this area is important from the following point of view: In mechanical engineering the designer can make a prototype of the detail to test, but in geomechanics it is impossible to make any prototype of the underground construction. So calculations are a very important way how to deal with these problems. We cannot expect a high exactness from the calculations, but we rather expect that they provide us with certain information which reveals the main features of the behaviour of the rock mass in the surrounding of the underground construction. The model of the bolt support discussed in this paper is based on the hypotheses given in Chapter 2. In Chapter 3 an asymptotic result, which makes the finite element approximation easier, is verified.

The numerical code based on the model of rock bolt systems developed above was written and inserted in GEM 22, which is the numerical code developed in the Institute of Geonics for solving geomechanical problems.

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