SINGULAR PERTURBATIONS IN OPTIMAL CONTROL PROBLEM WITH APPLICATION TO NONLINEAR STRUCTURAL ANALYSIS

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Summary. This paper concerns an optimal control problem of elliptic singular perturbations in variational inequalities (with controls appearing in coefficients, right hand sides and convex sets of states as well). The existence of an optimal control is verified. Applications to the optimal control of an elasto-plastic plate with a small rigidity and with an obstacle are presented. For elasto-plastic plates with a moving part of the boundary a primal finite element model is applied and a convergence result is obtained.

Keywords: Optimal control problem, singular perturbations in variational inequalities, convex set, elasto-plastic plate, small rigidity, obstacle.

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INTRODUCTION

Singular perturbations play a special role as an adequate mathematical tool for describing several important physical phenomena such as propagation of waves in media in the presence of small energy dissipation or dispersions, appearance of boundary or interior layers in fluid and gas dynamics as well as in the elasto-plasticity theory, semiclassical assymptotic approximations in quantum mechanics, phenomena in the semiconductor-device theory etc. We shall deal with singular perturbations appearing in coefficients, right hand sides and in the form of convex sets of states for optimal control problems governed by elliptic variational inequalities. We investigate some properties of the solutions. The existence theorem (for the singularly perturbed optimal control) will be applied to the perturbed optimal control of elasto-plastic plates with small thickness and to membranes (the membrane approximation to the plate obstacle problem as a special example of singular perturbations for elliptic variational inequalities—Sec. 1). In Sec. 2 we shall deal with discretization of the abstract problem. In Sec. 3 a particular realization of the general scheme to the initially-stressed elasto-plastic plate problem is performed. In Sec. 4 a finite element approximation to the optimization problem is done.

Singular perturbations in variational inequalities were considered by Huet [6], Lions [10], [11], Greenlee [4] and Eckhaus, Moet [3], while the optimal control problems were considered by Lions [10]. The main concern was there the existence of solutions with some weak convergence theorems.

Before touching the main topic we introduce the notation. Let $H(\Omega)$ be a normed linear space. Following Mosco ([13]), we introduce the convergence of a sequence of subsets:

Definition 1. A sequence of subsets $\{C_n(\Omega)\}_n$ of a normed space $H(\Omega)$ converges to a set $C(\Omega) \subset H(\Omega)$ if $C(\Omega)$ contains all weak limits of sequences $\{v_{n_k}\}_k$, $v_{n_k} \in C_{n_k}(\Omega)$, where $\{C_{n_k}(\Omega)\}_k$ are arbitrary subsequences of $\{C_n(\Omega)\}_n$, and if every element $v \in C(\Omega)$ is the (strong) limit of a sequence $\{v_n\}_n$, $v_n \in C_n(\Omega)$.

We shall write $C(\Omega) = \operatorname*{Lim}_{n\to+\infty} C_n(\Omega)$.

We employ the following notation: by "$\to$" and "$\rightharpoonup$" the strong and weak convergence in the appropriate spaces will be denoted. As usual, $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{R}$ the real axis. For two Banach spaces X, Y, $\mathcal{L}(X,Y)$ denotes the space of all linear operators from X to Y.

1. Existence theorems

Let the control space $U(\Omega)$ be a reflexive Banach space with a norm $\|\cdot\|_{U(\Omega)}$. Let $U_{\mathrm{ad}}(\Omega) \subset U(\Omega)$ be the set of admissible controls in $U(\Omega)$. Let $\mathscr{X}(\Omega)$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathscr{X}(\Omega)}$. Furthermore, let $V(\Omega)$ and $W(\Omega)$ be two reflexive Banach spaces with norms $\|\cdot\|_{V(\Omega)}$, $\|\cdot\|_{W(\Omega)}$, being compactly imbedded into $\mathscr{X}(\Omega)$ by imbeddings $\mathscr{I}_{V(\Omega)}$, $\mathscr{I}_{W(\Omega)}$, respectively, such that the ranges $\mathscr{I}_H(H)$ are dense in $\mathscr{X}(\Omega)$ for $H = V(\Omega)$, $W(\Omega)$. Let us denote by $V^*(\Omega)$ and $W^*(\Omega)$ the dual spaces of $V(\Omega)$ and $W(\Omega)$ and by $\|\cdot\|_{V^*(\Omega)}$, $\|\cdot\|_{W^*(\Omega)}$ their norms with respect to given duality pairings $\langle\cdot,\cdot\rangle_{V(\Omega)}$, $\langle\cdot,\cdot\rangle_{W(\Omega)}$, where, by convention, $\langle x,y\rangle_H \equiv \langle \mathscr{I}_H^{*-1}x, \mathscr{I}_H y\rangle_{\mathscr{X}(\Omega)}$ for $H = V(\Omega)$, $W(\Omega)$, $y \in H$ and $x \in \mathscr{I}_H^*(\mathscr{X}(\Omega))$ which is dense in H^* and will be identified with $X(\Omega)$. For a Banach space $\mathscr{H}$ and two nonnegative constants λ, Λ we denote by $\mathscr{E}_{\mathscr{H}}(\lambda,\Lambda)$ the set of all operators D from $\mathscr{H}$ into $\mathscr{H}^*$ for which the inequalities

$$\begin{aligned} &\lambda\|v-w\|_{\mathscr{H}}^2 \leqslant \langle Dv - Dw, v-w\rangle_{\mathscr{H}} \text{ and} \\ &\|Dv - Dw\|_{\mathscr{H}^*} \leqslant \Lambda\|v-w\|_{\mathscr{H}} \text{ for all } v \in \mathscr{H} \end{aligned} \tag{1.1}$$

hold. We assume that

(N0) $$V(\Omega) \hookrightarrow W(\Omega),\ V(\Omega) \text{ is dense in } W(\Omega) \text{ and } U_{\rm ad}(\Omega) \subset U(\Omega) \text{ is compact in } U(\Omega).$$

We introduce the systems $\{\mathscr{K}(e,\Omega)\}_{e\in U_{\rm ad}(\Omega)}$, $\{\hat{\mathscr{K}}(e,\Omega)\}_{e\in U_{\rm ad}(\Omega)}$ of nonempty convex closed sets $\mathscr{K}(e,\Omega) \subset V(\Omega)$, $\hat{\mathscr{K}}(e,\Omega) \subset W(\Omega)$, $e \in U_{\rm ad}(\Omega)$, and the systems of operators $\{\mathscr{A}(e);\ e \in U_{\rm ad}(\Omega)\}$ acting from $V(\Omega)$ into $V^*(\Omega)$) and $\{\mathscr{B}(e);\ e \in U_{\rm ad}(\Omega)\}$ acting from $W(\Omega)$ into $W^*(\Omega)$ satisfying the following assumptions:
$(H\mathscr{A})$
$$\begin{cases} 1^\circ\ e_n \to e_0 \text{ in } U(\Omega) \Longrightarrow \mathscr{K}(e_0,\Omega) = \operatorname*{Lim}_{n\to+\infty} \mathscr{K}(e_n,\Omega), \\ 2^\circ\ \{\mathscr{A}(e);\ e \in U_{\rm ad}(\Omega)\} \subset \mathscr{E}_{V(\Omega)}(0, c_{\mathscr{A}}), \\ 3^\circ\ e_n \to e_0 \text{ in } U(\Omega) \Longrightarrow \mathscr{A}(e_n)v \to \mathscr{A}(e_0)v \text{ in } V^*(\Omega) \text{ for all } v \in V^*(\Omega)), \\ 4^\circ\ \text{there is } \alpha_{\mathscr{A}} > 0 \text{ such that for all } e \in U_{\rm ad}(\Omega) \text{ and all } v,w \in V(\Omega) \\ \quad \text{the inequality} \\ \quad \langle \mathscr{A}(e)v - \mathscr{A}(e)w, v-w\rangle_{V(\Omega)} + \|v-w\|^2_{W(\Omega)} \geqslant \alpha_{\mathscr{A}}\|v-w\|^2_{V(\Omega)} \text{ holds,} \end{cases}$$

and
$(H\mathscr{B})$
$$\begin{cases} 1^\circ\ \operatorname{cl}\mathscr{K}(e,\Omega) = \hat{\mathscr{K}}(e,\Omega), e \in U_{\rm ad}(\Omega) \text{ (closure in } W(\Omega)), \\ 2^\circ\ e_n \to e_0 \text{ in } U(\Omega) \Longrightarrow \hat{\mathscr{K}}(e_0,\Omega) = \operatorname*{Lim}_{n\to+\infty} \hat{\mathscr{K}}(e_n,\Omega), \\ 3^\circ\ \{\mathscr{B}(e);\ e \in U_{\rm ad}(\Omega)\} \subset \mathscr{E}_{W(\Omega)}(\alpha_{\mathscr{B}}, c_{\mathscr{B}}) \text{ with } \alpha_{\mathscr{B}} > 0, \\ 4^\circ\ e_n \to e_0 \text{ in } U(\Omega) \Longrightarrow \mathscr{B}(e_n)v \to \mathscr{B}(e_0)v \text{ in } W^*(\Omega) \text{ for all } v \in W(\Omega)). \end{cases}$$

Note that $W^*(\Omega) \hookrightarrow V^*(\Omega)$ continuously, and one has the transposition formula

$$\langle F, v\rangle_{V(\Omega)} = \langle F, v\rangle_{W(\Omega)} \text{ for any } v \in V(\Omega) \text{ and for any } F \in W^*(\Omega).$$

We assume, moreover, that

$(B0)$ $\quad f \in W^*(\Omega)$ and $B\colon U(\Omega) \to W^*(\Omega)$ is a continuous operator.

For every $\varepsilon > 0$ and for every $e \in U_{\rm ad}(\Omega)$ there exists a unique state function $u_\varepsilon(e) \in \mathscr{K}(e,\Omega)$ such that

(1.2) $$\langle \varepsilon\mathscr{A}(e)u_\varepsilon(e) + \mathscr{B}(e)u_\varepsilon(e), v - u_\varepsilon(e)\rangle_{V(\Omega)} \geqslant \langle f + Be, v - u_\varepsilon(e)\rangle_{W(\Omega)} \quad \text{for all } v \in \mathscr{K}(e,\Omega).$$

Indeed, thanks to the general theory of variational inequalities ([1], [12], [14]), it is enough to prove that there is $c_\varepsilon > 0$ such that

$$(1.3)\qquad \langle \varepsilon(\mathscr{A}(e)v - \mathscr{A}(e)w), v - w\rangle_{V(\Omega)} + \langle \mathscr{B}(e)v - \mathscr{B}(e)w, v - w\rangle_{W(\Omega)} \geqslant c_\varepsilon \|v - w\|^2_{V(\Omega)},\ v, w \in V(\Omega)$$

and this immediately follows from $((H\mathscr{A}), 2°, 4°)$, $((H\mathscr{B}), 3°)$ (e.g. by contradiction).

Thanks to $((H\mathscr{B}), 3°)$, for any $e \in U_{\mathrm{ad}}(\Omega)$

$$(1.4)\qquad \begin{gathered} \exists!\, u(e) \in \hat{\mathscr{K}}(e, \Omega) \text{ such that } \forall v \in \hat{\mathscr{K}}(e, \Omega) \\ \langle \mathscr{B}(e)u(e), v - u(e)\rangle_{W(\Omega)} \geqslant \langle f + Be, v - u(e)\rangle_{W(\Omega)}. \end{gathered}$$

Let us consider a functional $\mathscr{L}\colon U(\Omega) \times W(\Omega) \to \mathbb{R}^+ \equiv \{a \in \mathbb{R};\ a \geqslant 0\}$ for which the following condition holds:

$$(E0)\qquad \begin{cases} 1°\ \{v_n\}_n \subset W(\Omega),\ v \in W(\Omega),\ v_n \to v \text{ in } W(\Omega) \Longrightarrow \\ \quad \mathscr{L}(e, v) = \lim\limits_{n \to +\infty} \mathscr{L}(e, v_n), \\ 2°\ \{v_n\}_n \subset W(\Omega),\ v \in W(\Omega),\ \{e_n\}_n \subset U_{\mathrm{ad}}(\Omega),\ e \in U_{\mathrm{ad}}(\Omega), \\ \quad e_n \to e \text{ in } U(\Omega), v_n \rightharpoonup v \text{ (weakly) in } W(\Omega) \Longrightarrow \\ \quad \mathscr{L}(e, v) \leqslant \liminf\limits_{n \to +\infty} \mathscr{L}(e_n, v_n). \end{cases}$$

We introduce the functional J_ε by

$$(1.5)\qquad J_\varepsilon(e) = \mathscr{L}(e, u_\varepsilon(e)),\ e \in U_{\mathrm{ad}}(\Omega),$$

where $u_\varepsilon(e)$ is the uniquely determined solution of (1.3), $e \in U_{\mathrm{ad}}(\Omega)$. We shall solve the following optimization problem $(\mathscr{P}_\varepsilon)$:

Find a control $e_\varepsilon \in U_{\mathrm{ad}}(\Omega)$ such that

$$(\mathscr{P}_\varepsilon)\qquad J_\varepsilon(e_\varepsilon) = \inf_{e \in U_{\mathrm{ad}}(\Omega)} J_\varepsilon(e).$$

We say that e_ε is an optimal control of the problem $(\mathscr{P}_\varepsilon)$.

Theorem 1. *Let the assumptions $(N0)$, $(H\mathscr{A})$, $(H\mathscr{B})$, $(B0)$ and $(E0)$ be satisfied. Then there exists at least one solution to $(\mathscr{P}_\varepsilon)$ for any $\varepsilon > 0$.*

Proof. Due to the compactness of $U_{\mathrm{ad}}(\Omega)$ in $U(\Omega)$, there exists a sequence $\{e^n_\varepsilon\}_n \subset U_{\mathrm{ad}}(\Omega)$ such that

$$(1.6)\qquad \lim_{n \to +\infty} e^n_\varepsilon = e^0_\varepsilon \text{ in } U(\Omega),\ e^0_\varepsilon \in U_{\mathrm{ad}}(\Omega) \text{ and } \lim_{n \to +\infty} J_\varepsilon(e^n_\varepsilon) = \inf_{e \in U_{\mathrm{ad}}(\Omega)} J_\varepsilon(e).$$

Denoting $u_\varepsilon(e_\varepsilon^n) := u_\varepsilon^n \in \mathscr{K}(e_\varepsilon^n, \Omega)$ we obtain the inequality

$$(1.7) \qquad \langle \varepsilon \mathscr{A}(e_\varepsilon^n) u_\varepsilon^n + \mathscr{B}(e_\varepsilon^n) u_\varepsilon^n, v - u_\varepsilon^n \rangle_{V(\Omega)} \geqslant \langle f + Be_\varepsilon^n, v - u_\varepsilon^n \rangle_{W(\Omega)}$$

for all $v \in \mathscr{K}(e_\varepsilon^n, \Omega)$. We take an arbitrary $v_0 \in \mathscr{K}(e_\varepsilon^0, \Omega)$ and (by $(H\mathscr{A}), 1°$) a sequence $\{v_n\}_n \in \prod_{n \in N} \mathscr{K}(e_\varepsilon^n, \Omega)$ such that $v_n \overset{V(\Omega)}{\longrightarrow} v_0$. Putting $v = v_n$ in (1.7), adding $\varepsilon \langle \mathscr{A}(e_\varepsilon^n) v_n, u_\varepsilon^n - v_n \rangle_{V(\Omega)} + \langle \mathscr{B}(e_\varepsilon^n) v_n, u_\varepsilon^n - v_n \rangle_{W(\Omega)}$ to its both sides and multiplying the resulting inequality by -1, we obtain

$$\begin{aligned}(1.8)\ \langle \varepsilon(\mathscr{A}(e_\varepsilon^n) u_\varepsilon^n &- \mathscr{A}(e_\varepsilon^n) v_n), u_\varepsilon^n - v_n \rangle_{V(\Omega)} + \langle \mathscr{B}(e_\varepsilon^n) u_\varepsilon^n - \mathscr{B}(e_\varepsilon^n) v_n, u_\varepsilon^n - v_n \rangle_{W(\Omega)} \\ &\leqslant \langle \varepsilon \mathscr{A}(e_\varepsilon^n) v_n, v_n - u_\varepsilon^n \rangle_{V(\Omega)} + \langle \mathscr{B}(e_\varepsilon^n) v_n, v_n - u_\varepsilon^n \rangle_{W(\Omega)} \\ &\quad + \langle f + Be_\varepsilon^n, u_\varepsilon^n - v_n \rangle_{W(\Omega)}, \quad n \in \mathbb{N}.\end{aligned}$$

From (1.3), (1.8), $((H\mathscr{A}), 2°, 3°)$, $((H\mathscr{B}), 4°)$ and $(B0)$ it follows that

$$(1.9) \qquad \|u_\varepsilon^n\|_{V(\Omega)} \leqslant c(\varepsilon),\ n \in \mathbb{N} \text{ for fixed } \varepsilon > 0.$$

This yields the existence of a subsequence $\{u_\varepsilon^{n_k}\}_k$ and of an element $u_\varepsilon^0 \in V(\Omega)$ such that

$$(1.10) \qquad u_\varepsilon^{n_k} \rightharpoonup u_\varepsilon^0 \text{ in } V(\Omega).$$

As $u_\varepsilon^n \in \mathscr{K}(e_\varepsilon^n, \Omega)$, the assumption $((H\mathscr{A}), 1°)$ yields

$$(1.11) \qquad u_\varepsilon^0 \in \mathscr{K}(e_\varepsilon^0, \Omega).$$

By $((H\mathscr{A}), 1°)$ there exists a sequence $\{\Theta_k\}_k$, $\Theta_k \in \mathscr{K}(e_\varepsilon^{n_k}, \Omega)$, such that $\Theta_k \to u_\varepsilon^\circ$ in $V(\Omega)$. Inserting $v := \Theta_k$ in (1.7), adding $\langle \varepsilon \mathscr{A}(e_\varepsilon^{n_k}) \Theta_k + \mathscr{B}(e_\varepsilon^{n_k}) \Theta_k, u_\varepsilon^{n_k} - \Theta_k \rangle$ to its both sides and multiplying the resulting inequality by -1, we obtain

$$\begin{aligned}(1.12) \qquad \limsup_{k \to +\infty} & \langle (\varepsilon \mathscr{A}(e_\varepsilon^{n_k}) + \mathscr{B}(e_\varepsilon^{n_k}))(u_\varepsilon^{n_k} - \Theta_k), u_\varepsilon^{n_k} - \Theta_k \rangle_{V(\Omega)} \\ &\leqslant \limsup_{k \to +\infty} \left| \langle \varepsilon \mathscr{A}(e_\varepsilon^{n_k}) \Theta_k, \Theta_k - u_\varepsilon^{n_k} \rangle_{V(\Omega)} \right| \\ &\quad + \limsup_{k \to +\infty} \left| \langle \mathscr{B}(e_\varepsilon^{n_k}) \Theta_k, \Theta_k - u_\varepsilon^{n_k} \rangle_{W(\Omega)} \right| \\ &\quad + \limsup_{k \to +\infty} \left| \langle f + Be_\varepsilon^{n_k}, u_\varepsilon^{n_k} - \Theta_k \rangle_{W(\Omega)} \right| = 0.\end{aligned}$$

The last equality follows from $(B0)$ and from the facts

$$(1.13)\qquad \left(e_n \overset{U(\Omega)}{\longrightarrow} e \text{ and } v^n \overset{V(\Omega)}{\longrightarrow} v \text{ for } n \to +\infty\right) \Longrightarrow$$
$$\|\mathscr{A}(e_n)v^n - \mathscr{A}(e)v\|_{V^*(\Omega)}$$
$$\leqslant c_{\mathscr{A}}\|v^n - v\|_{V(\Omega)} + \|\mathscr{A}(e_n)v - \mathscr{A}(e)v\|_{V^*(\Omega)} \to 0 \text{ for } n \to +\infty,$$
$$(1.14)\qquad \left(e_n \overset{U(\Omega)}{\longrightarrow} e \text{ and } w^n \overset{W(\Omega)}{\longrightarrow} w \text{ for } n \to +\infty\right) \Longrightarrow$$
$$\|\mathscr{B}(e_n)w^n - \mathscr{B}(e)w\|_{W^*(\Omega)}$$
$$\leqslant c_{\mathscr{B}}\|w^n - w\|_{W(\Omega)} + \|\mathscr{B}(e_n)w - \mathscr{B}(e)w\|_{W^*(\Omega)} \to 0 \text{ for } n \to +\infty$$

which are consequences of $((H\mathscr{A}), 2°, 3°)$, $((H\mathscr{B}), 3°, 4°)$, respectively. Due to the uniform monotonicity of $[\varepsilon\mathscr{A}(e_\varepsilon^{n_k}) + \mathscr{B}(e_\varepsilon^{n_k})]$ (cf. (1.3)) we obtain the strong convergence

$$(1.15)\qquad u_\varepsilon^{n_k} \overset{V(\Omega)}{\longrightarrow} u_\varepsilon^0 \quad \text{for } k \to +\infty.$$

Moreover, (1.15) together with (1.13) and (1.14) yields

$$(1.16)\qquad \mathscr{A}(e_\varepsilon^{n_k})u_\varepsilon^{n_k} \overset{V^*(\Omega)}{\longrightarrow} \mathscr{A}(e_\varepsilon^0)u_\varepsilon^0, \quad \mathscr{B}(e_\varepsilon^{n_k})u_\varepsilon^{n_k} \overset{W^*(\Omega)}{\longrightarrow} \mathscr{B}(e_\varepsilon^0)u_\varepsilon^0 \quad \text{for } k \to +\infty.$$

Given a $v \in \mathscr{K}(e_\varepsilon^0, \Omega)$, by the assumption $((H\mathscr{A}), 1°)$ there exists a sequence $\{v^k\}_k, v^k \in \mathscr{K}(e_\varepsilon^{n_k}, \Omega)$, $v^k \to v$ in $V(\Omega)$. Limiting (1.7) with $v = v^k$, we have

$$(1.17)\qquad \langle \varepsilon\mathscr{A}(e_\varepsilon^0)u_\varepsilon^0, v - u_\varepsilon^0\rangle_{V(\Omega)} + \langle \mathscr{B}(e_\varepsilon^0)u_\varepsilon^0, v - u_\varepsilon^0\rangle_{W(\Omega)} \geqslant \langle f + Be_\varepsilon^0, v - u_\varepsilon^0\rangle_{W(\Omega)}$$

and, as $v \in \mathscr{K}(e_\varepsilon^0, \Omega)$ is chosen arbitrarily, we get

$$(1.18)\qquad u_\varepsilon^0 \equiv u_\varepsilon(e_\varepsilon^0).$$

Then $((E0), 2°)$ and (1.10) yield

$$(1.19)\qquad \mathscr{L}(e_\varepsilon^0, u_\varepsilon(e_\varepsilon^0)) \leqslant \liminf_{n\to+\infty} \mathscr{L}(e_\varepsilon^n, u_\varepsilon(e_\varepsilon^n)) = \inf_{e\in U_{\mathrm{ad}}(\Omega)} \mathscr{L}(e, u_\varepsilon(e)).$$

Hence $\mathscr{L}(e_\varepsilon^0, u_\varepsilon(e_\varepsilon^0)) = \inf\{\mathscr{L}(e, u_\varepsilon(e));\ e \in U_{\mathrm{ad}}(\Omega)\}$, which completes the proof. $\square$

Limit state function and limit cost function. We define the limit state function for any $e \in U_{\mathrm{ad}}(\Omega)$ by the variational inequality

$$(1.20)\qquad \begin{aligned} &\text{Find } u_0(e) \in \hat{\mathscr{K}}(e, \Omega) \text{ such that} \\ &\langle \mathscr{B}(e)u_0(e), v - u_0(e)\rangle_{W(\Omega)} \geqslant \langle f + Be, v - u_0(e)\rangle_{W(\Omega)} \ \forall v \in \hat{\mathscr{K}}(e, \Omega) \end{aligned}$$

and the limit cost function

$$J_0(e) = \mathscr{L}(e, u_0(e)). \tag{1.21}$$

In this case one has the limit control problem $(\mathscr{P}_0)$ defined as follows:

$$(\mathscr{P}_0) \qquad \text{Find } e_0 \in \operatorname{Arg\,inf}\{J_0(e);\, e \in U_{\rm ad}(\Omega)\}.$$

Theorem 2. *Let the assumptions $(N0)$, $(H\mathscr{B})$, $(B0)$ and $(E0)$ be satisfied. Then there exists at least one solution to $(\mathscr{P}_0)$.*

The proof is analogous to that of Theorem 1. □

There arises a natural question concerning the type of relation between solutions to $(\mathscr{P}_0)$ and $(\mathscr{P}_\varepsilon)$ if $\varepsilon \to 0_+$. We prove the following theorem:

Theorem 3. *Let the assumptions $(N0)$, $(H\mathscr{A})$, $(H\mathscr{B})$, $(B0)$ and $(E0)$ be satisfied. Let e_{ε_n}, e_0 be solutions of the problem $(\mathscr{P}_{\varepsilon_n})$, $(\mathscr{P}_0)$, respectively. Then there exists a sequence $\{\varepsilon_n\}_n$, $\varepsilon_n \to 0$ for $n \to +\infty$, such that*

$$\left\{\begin{aligned} &e_{\varepsilon_n} \to e_0 \text{ in } U(\Omega), \quad u_{\varepsilon_n}(e_{\varepsilon_n}) \to u_0(e_0) \text{ in } W(\Omega),\\ &J_{\varepsilon_n}(e_{\varepsilon_n}) = \inf_{e\in U_{\rm ad}(\Omega)} J_{\varepsilon_n}(e) \to J_0(e_0) = \inf_{e\in U_{\rm ad}(\Omega)} J_0(e). \end{aligned}\right. \tag{1.22}$$

Proof. Due to the compactness of $U_{\rm ad}(\Omega)$ there exists a sequence $\{e_{\varepsilon_n}\}_n \subset U_{\rm ad}(\Omega)$ such that $e_{\varepsilon_n} \to e_0$ in $U(\Omega)$. The "state function" $u_{\varepsilon_n}(e_{\varepsilon_n}) \in \mathscr{K}(e_{\varepsilon_n}, \Omega)$ is a solution of the state variational inequality

$$\begin{aligned} &\langle \varepsilon_n \mathscr{A}(e_{\varepsilon_n}) u_{\varepsilon_n}(e_{\varepsilon_n}) + \mathscr{B}(e_{\varepsilon_n}) u_{\varepsilon_n}(e_{\varepsilon_n}), v - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{V(\Omega)}\\ &\geqslant \langle f + Be_{\varepsilon_n}, v - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{W(\Omega)} \text{ for any } v \in \mathscr{K}(e_{\varepsilon_n}, \Omega)\\ &\qquad \text{for given } e_{\varepsilon_n} \in U_{\rm ad}(\Omega),\ \varepsilon_n > 0,\ n \in \mathbb{N}. \end{aligned} \tag{1.23}$$

We take an arbitrary $v_0 \in \mathscr{K}(e^0_\varepsilon, \Omega)$ and a sequence $\{v_n\}_n \in \prod_{n\in\mathbb{N}} \mathscr{K}(e_{\varepsilon_n}, \Omega)$ such that $v_n \overset{V(\Omega)}{\longrightarrow} v_0$. In the inequality (1.23) we take the fixed $v = v_n$, add $\varepsilon_n\langle \mathscr{A}(e_{\varepsilon_n})v_n, u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\rangle_{V(\Omega)} + \langle \mathscr{B}(e_{\varepsilon_n})v_n, u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\rangle_{W(\Omega)}$ to both sides of (1.23), multiply the resulting inequality by -1 and use $((H\mathscr{A}), 2°)$ and $((H\mathscr{B}), 3°)$. It follows that

$$\begin{aligned} &\varepsilon_n\big(\langle \mathscr{A}(e_{\varepsilon_n})u_{\varepsilon_n}(e_{\varepsilon_n}) - \mathscr{A}(e_{\varepsilon_n})v_n, u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\rangle_{V(\Omega)} + \|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|^2_{W(\Omega)}\big)\\ &\quad + (\alpha_{\mathscr{B}} - \varepsilon_n)\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|^2_{W(\Omega)}\\ &\leqslant \langle f + Be_{\varepsilon_n}, u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\rangle_{W(\Omega)} + \langle(\varepsilon_n\mathscr{A}(e_{\varepsilon_n}) + \mathscr{B}(e_{\varepsilon_n}))\, v_n, v_n - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{V(\Omega)}. \end{aligned}$$

Setting $\varepsilon_n \leqslant \alpha_{\mathscr{B}}/2$ and applying $((H\mathscr{A}),(H\mathscr{B}))$ we get

$$
\begin{aligned}
&(\varepsilon_n\alpha_{\mathscr{A}})\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|^2_{V(\Omega)} + \tfrac{1}{2}\alpha_{\mathscr{B}}\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|^2_{W(\Omega)} \\
&\qquad \leqslant c_1\|f + Be_{\varepsilon_n}\|_{W^*(\Omega)}\,\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|_{W(\Omega)} \\
&\qquad\qquad + c_2\varepsilon_n\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|_{V(\Omega)}\,\|v_n\|_{V(\Omega)} + c_3\|u_{\varepsilon_n}(e_{\varepsilon_n}) - v_n\|_{W(\Omega)}\,\|v_n\|_{W(\Omega)},
\end{aligned}
$$

where c_1, c_2, c_3 are constants which do not depend on n. Hence we conclude that

$$
\begin{aligned}
\text{(1.24)} \qquad &\|u_{\varepsilon_n}(e_{\varepsilon_n})\|_{W(\Omega)} \leqslant c, \qquad \sqrt{\varepsilon_n}\|u_{\varepsilon_n}(e_{\varepsilon_n})\|_{V(\Omega)} \leqslant c \\
&\qquad \Longrightarrow \sqrt{\varepsilon_n}\|\mathscr{A}(e_{\varepsilon_n})u_{\varepsilon_n}(e_{\varepsilon_n})\|_{V^*(\Omega)} \leqslant \tilde{c}
\end{aligned}
$$

for some $c, \tilde{c}$ independent of n. We can therefore extract a subsequence $\left\{u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}})\right\}_k$ such that

$$
\begin{aligned}
\text{(1.25)} \qquad &u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \rightharpoonup \tilde{w} \ \text{ in } W(\Omega) \ \text{ for } k \to +\infty \quad \text{ and } \quad \tilde{w} \in \hat{\mathscr{K}}(e_0,\Omega), \\
&\sqrt{\varepsilon_n}u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \rightharpoonup \tilde{v} \ \text{ in } V(\Omega) \ \text{ for } k \to +\infty
\end{aligned}
$$

exploiting the assumption $((H\mathscr{B}),2°)$, too. Moreover, there is a sequence $\{\tilde{w}_k\}_k$ such that $\tilde{w}_k \in \mathscr{K}(e_{\varepsilon_{n_k}},\Omega)$ and $\tilde{w}_k \xrightarrow{W(\Omega)} \tilde{w}$. We put $v = \tilde{w}_k$ into (1.23) formulated for the index n_k, add $\langle \varepsilon_{n_k}\mathscr{A}(e_{\varepsilon_n})\tilde{w}_k + \mathscr{B}(e_{\varepsilon_{n_k}})\tilde{w}_k, u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) - \tilde{w}_k\rangle_{V(\Omega)}$ to its both sides, multiply the resulting inequality by -1 and employ $((H\mathscr{A}),2°)$, $((H\mathscr{B}),3°)$, $(B0)$, (1.13) and (1.14) again. As the right hand side of the resulting inequality tends to 0 (cf. (1.12)), we obtain

$$
\begin{aligned}
\text{(1.26)} \qquad &\limsup_{k\to+\infty} \alpha_{\mathscr{B}}\|\tilde{w}_k - u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}})\|^2_{W(\Omega)} = 0 \implies u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \xrightarrow{W(\Omega)} \tilde{w}, \\
&\limsup_{k\to+\infty} \left\langle \varepsilon_{n_k}\mathscr{A}(e_{\varepsilon_{n_k}})\left(u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) - \tilde{w}_k\right), u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) - \tilde{w}_k\right\rangle_{V(\Omega)} \\
&\qquad = \limsup_{k\to+\infty} \left\langle \varepsilon_{n_k}\mathscr{A}(e_{\varepsilon_{n_k}})u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}), u_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}})\right\rangle_{V(\Omega)} = 0.
\end{aligned}
$$

We take $w \in \hat{\mathscr{K}}(e_0,\Omega)$ arbitrary. We can find $\{w_k\}_k \subset V(\Omega)$ such that $w_k \in \mathscr{K}(e_{\varepsilon_{n_k}},\Omega)$ and $w_k \xrightarrow{W(\Omega)} w$. To prove

$$
\text{(1.27)} \qquad \tilde{w} = u_0(e_0)
$$

we return to (1.23) for the index n_k and put $v = w_k$ there. Due to $(B0)$, (1.25) and (1.26) it is easy to see that for $k \to +\infty$ we obtain

$$
\text{(1.28)} \qquad \langle \mathscr{B}(e_0)u_0(e_0), w - u_0(e_0)\rangle_{W(\Omega)} \geqslant \langle f + Be_0, v - u_0(e_0)\rangle_{W(\Omega)}
$$

and (1.27) is valid. Moreover, the method of the proof shows that the convergence

$$(1.29) \qquad u_\varepsilon(e) \overset{W(\Omega)}{\longrightarrow} u_0(e) \ \forall e \in U_{\mathrm{ad}}(\Omega)$$

holds. Indeed, if it were not true, there would be a sequence $\varepsilon_k \to 0$ and a constant $\ell > 0$ independent of k such that

$$(1.30) \qquad \|u_{\varepsilon_k}(e) - u_0(e)\|_{W(\Omega)} \geqslant \ell \ \forall k \in \mathbb{N}.$$

Putting an arbitrary fixed $v \in \mathscr{K}(e, \Omega)$ into the appropriate variational inequalities, we arrive at

$$(1.31) \qquad \begin{aligned} &\|u_{\varepsilon_k}(e)\|_{W(\Omega)} \leqslant c, \qquad \sqrt{\varepsilon_k}\|u_{\varepsilon_k}(e)\|_{V(\Omega)} \leqslant c \\ &\Longrightarrow \sqrt{\varepsilon_k}\|\mathscr{A}(e)u_{\varepsilon_k}(e)\|_{V^*(\Omega)} \leqslant \tilde{c}, \end{aligned}$$

where c, $\tilde{c}$ do not depend on k. The existing $W(\Omega)$-weak limit of a suitable subsequence $\{u_{\varepsilon_{k_n}}(e)\}_n \subset \{u_{\varepsilon_k}(e)\}_k$ must be $u_0(e)$ due to $((H\mathscr{B}), 1°)$ and due to quite analogous arguments to those used in deriving (1.27) through (1.26) and (1.28). This is a contradiction to (1.30).

Now, from (1.29), from the fact that $J_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \leqslant J_{\varepsilon_{n_k}}(e)$ for all $e \in U_{\mathrm{ad}}(\Omega)$ and all k, and from $((E0), 1°)$, we get

$$(1.32) \qquad \begin{aligned} &\limsup_{k\to+\infty} J_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \leqslant J_0(e) \ \forall e \in U_{\mathrm{ad}}(\Omega) \\ \Longrightarrow \quad &\limsup_{k\to+\infty} J_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \leqslant \inf_{e\in U_{\mathrm{ad}}(\Omega)} J_0(e) \leqslant J_0(e_0). \end{aligned}$$

Furthermore, we observe that $((E0), 2°)$ (1.26) and (1.27) imply $\liminf\limits_{k\to+\infty} J_{\varepsilon_{n_k}}(e_{\varepsilon_{n_k}}) \geqslant \mathscr{L}(e_0, u_0(e_0)) = J_0(e_0)$. Comparing this result with (1.32) we see that necessarily

$$(1.33) \qquad \inf_{e\in U_{\mathrm{ad}}(\Omega)} J_0(e) = J_0(e_0).$$

Theorem 3 is proved. □

2. Scheme of discretization

Let us assume that $U_{\rm ad}(\Omega) \subset U(\Omega)$ is compact. We describe the discretization of problem $(\mathscr{P})$ and prove the convergence of the sequence of finite-dimensional solutions as h, the discretization parameter, tends to zero. With any $h \in (0,1)$ we associate:

$$
(V\mathscr{A})_h \quad \begin{cases} 1^\circ \text{ finite-dimensional subspaces } V_h(\Omega) \subset V(\Omega),\ U^h(\Omega) \subset U(\Omega), \\ 2^\circ \text{ closed convex subsets } \mathscr{K}_h(e_h,\Omega) \subset V_h(\Omega) \text{ (certain approximations} \\ \quad \text{of } \mathscr{K}(e,\Omega)), \\ 3^\circ \text{ closed convex subsets } U^h_{\rm ad}(\Omega) \subset U^h(\Omega) \text{ (certain approximations} \\ \quad \text{of } U_{\rm ad}(\Omega)), \\ 4^\circ \text{ operators } \mathscr{A}_h(e_h)\colon V_h(\Omega) \to V_h^*(\Omega),\ e_h \in U^h_{\rm ad}(\Omega) \text{ (approximations} \\ \quad \text{of the operators in } (H\mathscr{A})), \\ 5^\circ\ \mathscr{L}_h\colon U^h(\Omega) \times V^h(\Omega) \to \mathbb{R} \text{ convex lower semicontinuous proper} \\ \quad \text{functionals (approximations of the cost functional } \mathscr{L}). \end{cases}
$$

Moreover, with any $h \in (0,1)$ we associate:

$$
(W\mathscr{B})_h \quad \begin{cases} 1^\circ \text{ finite-dimensional subspaces } W_h(\Omega) \subset W(\Omega)\ (V_h(\Omega) \subset W_h(\Omega)), \\ 2^\circ \text{ closed convex subsets } \hat{\mathscr{K}}_h(e_h,\Omega) \subset W(\Omega) \text{ (approximations} \\ \quad \text{of } \hat{\mathscr{K}}(e,\Omega)), \\ 3^\circ \text{ operators } \mathscr{B}_h(e_h)\colon W_h(\Omega) \to W_h^*(\Omega),\ e_h \in U^h_{\rm ad}(\Omega) \\ \quad \text{(approximations of the operators in } (H\mathscr{B})), \\ 4^\circ\ f_h \in W_h^*(\Omega),\ B_h \in \mathcal{L}(U^h(\Omega), W_h^*(\Omega)) \text{ (approximations} \\ \quad \text{of } f \text{ and } B). \end{cases}
$$

The approximations of the state equations (1.4) and (1.2) are now defined by means of the Ritz-Galerkin procedure:

$$
(2.1) \qquad \begin{cases} u_{\varepsilon h}(e_h) \in \mathscr{K}_h(e_h,\Omega) \\ \langle \varepsilon\mathscr{A}_h(e_h)u_{\varepsilon h}(e_h) + \mathscr{B}_h(e_h)u_{\varepsilon h}(e_h), v_h - u_{\varepsilon h}(e_h)\rangle_{V(\Omega)} \\ \geqslant \langle f_h + B_h e_h, v_h - u_{\varepsilon h}(e_h)\rangle_{V(\Omega)}, \\ \text{for any } v_h \in \mathscr{K}_h(e_h,\Omega) \text{ and } e_h \in U^h_{\rm ad}(\Omega) \end{cases}
$$

and

$$
(2.2) \qquad \begin{cases} u_{0h}(e_h) \in \hat{\mathscr{K}}_h(e_h,\Omega), \\ \langle \mathscr{B}_h(e_h)u_{0h}(e_h), v_h - u_{0h}(e_h)\rangle_{W(\Omega)} \\ \geqslant \langle f_h + B_h e_h, v_h - u_{0h}(e_h)\rangle_{W(\Omega)} \\ \text{for any } v_h \in \hat{\mathscr{K}}_h(e_h,\Omega) \text{ and } e_h \in U^h_{\rm ad}(\Omega). \end{cases}
$$

The families $\{\mathscr{K}_h(e_h,\Omega)\}_h$ and $\left\{\hat{\mathscr{K}}_h(e_h,\Omega)\right\}_h$ are supposed to satisfy the following two conditions:

$$(L\mathscr{A})_h \quad \begin{cases} 1^\circ\ h_n \to 0,\ e_{h_n} \stackrel{U(\Omega)}{\longrightarrow} e \text{ such that } e_{h_n} \in U_{\mathrm{ad}}^{h_n}(\Omega)\ \forall n \in \mathbb{N} \Longrightarrow \text{ for any} \\ \quad \text{bounded sequence } \{v_{h_n}\}_n \text{ in } V(\Omega) \text{ such that } v_{h_n} \in \mathscr{K}_{h_n}(e_{h_n},\Omega), \text{ all} \\ \quad \text{its weak cluster points belong to } \mathscr{K}(e,\Omega), \\ 2^\circ \text{ there are } \Lambda_{\mathscr{K}(e,\Omega)} \subset V(\Omega),\ \mathrm{cl}\,\Lambda_{\mathscr{K}(e,\Omega)} = \mathscr{K}(e,\Omega) \text{ such that for any} \\ \quad h_n \to 0 \text{ and any } e_{h_n} \stackrel{U(\Omega)}{\longrightarrow} e \text{ there is } \mathscr{R}_{e_{h_n}e}\colon \Lambda_{\mathscr{K}(e,\Omega)} \to \mathscr{K}_h(e_{h_n},\Omega) \\ \quad \text{such that for all } v \in \Lambda_{\mathscr{K}(e,\Omega)} \text{ we have } \lim_{n\to+\infty} \mathscr{R}_{e_{h_n}e}v = v \text{ strongly in} \\ \quad V(\Omega) \end{cases}$$

and

$$(L\mathscr{B})_h \quad \begin{cases} 1^\circ\ h_n \to 0,\ e_{h_n} \stackrel{U(\Omega)}{\longrightarrow} e \text{ such that } e_{h_n} \in U_{\mathrm{ad}}^{h_n}(\Omega)\ \forall n \in \mathbb{N} \Longrightarrow \text{ for any} \\ \quad \text{bounded sequence } \{v_{h_n}\}_n \text{ in } W(\Omega) \text{ such that } v_{h_n} \in \hat{\mathscr{K}}_{h_n}(e_{h_n},\Omega), \\ \quad \text{all its weak cluster points belong to } \hat{\mathscr{K}}(e,\Omega), \\ 2^\circ \text{ there are } \Lambda_{\hat{\mathscr{K}}(e,\Omega)} \subset W(\Omega),\ \mathrm{cl}\,\Lambda_{\hat{\mathscr{K}}(e,\Omega)} = \hat{\mathscr{K}}(e,\Omega) \text{ such that for any} \\ \quad h_n \to 0 \text{ and any } e_{h_n} \stackrel{U(\Omega)}{\longrightarrow} e \text{ there is } \Upsilon_{e_{h_n}e}\colon \Lambda_{\hat{\mathscr{K}}(e,\Omega)} \to \hat{\mathscr{K}}_h(e_{h_n},\Omega) \\ \quad \text{such that for all } w \in \Lambda_{\hat{\mathscr{K}}(e,\Omega)} \text{ we have } \lim_{n\to+\infty} \Upsilon_{e_{h_n}e}w = w \text{ strongly} \\ \quad \text{in } W(\Omega). \end{cases}$$

Let us note that we do not necessarily have $\mathscr{K}_h(e_h,\Omega) \subset \mathscr{K}(e,\Omega)$, $(\hat{\mathscr{K}}_h(e_h,\Omega) \subset \hat{\mathscr{K}}(e,\Omega))$ and $U_{\mathrm{ad}}^h(\Omega) \subset U_{\mathrm{ad}}(\Omega)$. If, however, this is true for any $h \in (0,1)$, we say that we have an internal approximation of $\mathscr{K}(e,\Omega)$, $\hat{\mathscr{K}}(e,\Omega)$, $U_{\mathrm{ad}}(\Omega)$, respectively.

For the existence theorem to the problems (2.1) and (2.2) and for the analysis of the relation between (1.4), (2.1) and the relation between (1.2), (2.2) we shall need the following hypotheses:

$$(H\mathscr{A})_h \quad \begin{cases} 1^\circ \text{ there is } \hat{c}_{\mathscr{A}} \text{ such that } \mathscr{A}_h(e_h) \in \mathscr{E}_{V_h(\Omega)}(0,\hat{c}_{\mathscr{A}}) \text{ for any } h \in (0,1) \text{ and} \\ \quad \text{any } e_h \in U_{\mathrm{ad}}^h(\Omega), \\ 2^\circ\ \mathscr{A}_{h_n}(e_{h_n})v_{h_n} \to \mathscr{A}(e)v \text{ in } V^*(\Omega) \text{ for } n \to +\infty, \text{ if } h_n \to 0,\ e_{h_n} \to e \\ \quad \text{in } U_{\mathrm{ad}}(\Omega) \text{ and } v_{h_n} \rightharpoonup v \text{ in } V(\Omega) \text{ for } n \to +\infty, \\ 3^\circ \text{ there is } \hat{\alpha}_{\mathscr{A}} > 0 \text{ such that for all } h \in (0,1),\ e_h \in U_{\mathrm{ad}}^h(\Omega) \text{ and} \\ \quad v_h, w_h \in V_h(\Omega),\ \langle \mathscr{A}_h(e_h)v_h - \mathscr{A}_h(e_h)w_h, v_h - w_h\rangle_{V(\Omega)} + \|v_h - \\ \quad w_h\|^2_{W(\Omega)} \geqslant \hat{\alpha}_{\mathscr{A}}\|v_h - w_h\|^2_{V(\Omega)}. \end{cases}$$

Furthermore, we suppose that the following hypotheses concerning $\mathscr{B}_h(e_h)$ hold:

$$(H\mathscr{B})_h \quad \begin{cases} 1^\circ \text{ there are } \hat{\alpha}_{\mathscr{B}} \text{ and } \hat{c}_{\mathscr{B}} \text{ such that } \mathscr{B}_h(e_h) \in \mathscr{E}_{W_h(\Omega)}(\hat{\alpha}_{\mathscr{B}}, \hat{c}_{\mathscr{B}}) \text{ for any} \\ \quad h \in (0,1) \text{ and any } e_h \in U^h_{\mathrm{ad}}(\Omega), \\ 2^\circ\ \mathscr{B}_{h_n}(e_{h_n})v_{h_n} \to \mathscr{B}(e)v \text{ in } W^*(\Omega) \text{ for } n \to +\infty, \text{ if } h_n \to 0,\ e_{h_n} \to e \\ \quad \text{in } U_{\mathrm{ad}}(\Omega) \text{ and } v_{h_n} \rightharpoonup v \text{ in } W(\Omega) \text{ for } n \to +\infty, \\ 3^\circ \text{ there is } c > 0 \text{ such that } \|f_h\|_{W_h^*(\Omega)} \leqslant c \text{ for any } h \in (0,1) \text{ and} \\ \quad \text{any } f_h \in W_h^*(\Omega) \text{ and } (h_n \to 0\ \&\ \{v_n\}_n \in \prod_{n\in\mathbb{N}} V_{h_n}(\Omega) \text{ such that} \\ \quad v_n \overset{V(\Omega)}{\rightharpoonup} v) \Rightarrow \langle f_n, v_n\rangle_{V_{h_n}(\Omega)} \to \langle f, v\rangle_{V(\Omega)}, \\ 4^\circ\ e_h \in U^h_{\mathrm{ad}}(\Omega) \text{ and } \|e_h\|_{U(\Omega)} \leqslant c_0 \implies \text{there is } c \in \mathbb{R}_+ \text{ such that} \\ \quad \|B_h e_h\|_{W(\Omega)} \leqslant c \text{ for any } h \in (0,1), \\ 5^\circ\ h_n \to 0,\ v_{h_n} \in V_{h_n}(\Omega),\ v_{h_n} \to v \text{ in } W(\Omega),\ e_{h_n} \to e \text{ in } U(\Omega) \text{ for} \\ \quad n \to +\infty \implies \langle B_{h_n} e_{h_n}, v_{h_n}\rangle_{W(\Omega)} \to \langle Be, v\rangle_{W(\Omega)} \text{ for } n \to +\infty. \end{cases}$$

Morever, we assume that

$$(E0)_h \quad \begin{cases} 1^\circ\ v_h^n \in W_h(\Omega) \text{ and } v_h^n \to v_h \text{ in } W(\Omega) \implies \mathscr{L}_h(e_h, v_h) = \lim_{n\to+\infty} \mathscr{L}_h(e_h, v_h^n), \\ 2^\circ\ e_h^n \in U^h_{\mathrm{ad}}(\Omega),\ e_h^n \to e_h \text{ in } U(\Omega),\ v_h^n \in V_h(\Omega),\ v_h^n \to v_h \text{ in } V(\Omega) \implies \\ \quad \mathscr{L}_h(e_h, v_h) \leqslant \liminf_{n\to+\infty} \mathscr{L}_h(e_h^n, v_h^n). \end{cases}$$

For every $\varepsilon > 0$, $h > 0$ and for every $e_h \in U^h_{\mathrm{ad}}(\Omega)$ there exists a unique $u_{\varepsilon h}(e_h) \in \mathscr{K}_h(e_h, \Omega)$ of the variational inequality (2.1). Indeed, due to $((H\mathscr{A})_h, 1^\circ)$ and $((H\mathscr{B})_h, 1^\circ)$ there exists a constant $c_{\mathscr{A}\mathscr{B}} > 0$ such that for any $\varepsilon > 0$

$$\langle \varepsilon(\mathscr{A}_h(e_h)v_h - \mathscr{A}_h(e_h)w_h), v_h - w_h\rangle_{V(\Omega)} + \langle \mathscr{B}_h(e_h)v_h - \mathscr{B}_h(e_h)w_h, v_h - w_h\rangle_{W(\Omega)} \geqslant c_{\mathscr{A}\mathscr{B}}\|v_h - w_h\|^2_{V(\Omega)}.$$

For a set M and a function $\ell\colon M \to \mathbb{R}$ we denote by $\operatorname{Arg\,min}_M \ell$ the set of minimizers of ℓ on M. The discrete version of $(\mathscr{P}_\varepsilon)$ then reads as follows:

$(\mathscr{P}_{\varepsilon h})$

Find $e_{\varepsilon h} \in \operatorname*{Arg\,min}_{e_h\in U^h_{\mathrm{ad}}(\Omega)} \mathscr{L}_h(e_h, u_{\varepsilon h}(e_h)) \equiv \operatorname*{Arg\,min}_{e_h\in U^h_{\mathrm{ad}}(\Omega)} J_{\varepsilon h}(e_h)$ with $u_{\varepsilon h}(e_h)$ as above

and the discrete version of the limit control problem $(\mathscr{P}_0)$ reads:

$(\mathscr{P}_{0h})$

Find $e_{0h} \in \operatorname*{Arg\,min}_{e_h\in U^h_{\mathrm{ad}}(\Omega)} \mathscr{L}_h(e_h, u_{0h}(e_h)) \equiv \operatorname*{Arg\,min}_{e_h\in U^h_{\mathrm{ad}}(\Omega)} J_{0h}(e_h)$ with $u_{0h}(e_h)$ as above.

By $[e_{\varepsilon h}, u_{\varepsilon h}(e_{\varepsilon h})], [e_{0h}, u_{0h}(e_{0h})]$ we denote an optimal pair for $(\mathscr{P}_{\varepsilon h}), (\mathscr{P}_{0h})$, respectively. The following lemmas hold:

Lemma 1. *For every $h > 0$ and for every $\varepsilon > 0$ there exists at least one optimal pair $[e_{\varepsilon h}, u_{\varepsilon h}(e_{\varepsilon h})]$ for the problem $(\mathscr{P}_{\varepsilon h})$.*

The proof is quite analogous to that of Theorem 1. □

Lemma 2. *Under the above introduced hypotheses $(H\mathscr{A})_h$ and $(L\mathscr{A})_h$, let $\{e_{h_n}\}_n \in \prod_{n\in\mathbb{N}} U_{\mathrm{ad}}^{h_n}(\Omega)$ be such that $h_n \to 0$ and $e_{h_n} \to e$ in $U(\Omega)$ for $n \to +\infty$. Then $u_{\varepsilon h_n}(e_{h_n}) \to u_\varepsilon(e_\varepsilon)$ in $V(\Omega)$ for any fixed $\varepsilon > 0$.*

Proof. We take $v \in \Lambda_{\mathscr{K}(e,\Omega)}$ and put $\mathscr{R}_{e_{h_n}e} v$ into (2.1) for the corresponding h_n—cf. 2° of $(L\mathscr{A})_h$. Then, employing the standard procedure repeated several times in Sec. 1, where the respective assumptions of $(H\mathscr{A})$ and $(H\mathscr{B})$ are replaced by $(H\mathscr{A})_h$ and $(H\mathscr{B})_h$, we arrive at the estimate

$$\|u_{\varepsilon h_n}(e_{h_n})\|_{V(\Omega)} \leqslant \hat{c}_{\mathscr{A}\mathscr{B}}(\varepsilon),\ n \in \mathbb{N},$$

valid for $\varepsilon > 0$ with a positive constant $\hat{c}_{\mathscr{A}\mathscr{B}}$ independent of $n \in \mathbb{N}$. Thus there exists a subsequence $\{u_{\varepsilon h_{n_k}}(e_{h_{n_k}})\}_k$ of $\{u_{\varepsilon h_n}(e_{h_n})\}_n$ and an element $u_\varepsilon(e) \in V(\Omega)$ such that

$$u_{\varepsilon h_{n_k}}(e_{h_{n_k}}) \rightharpoonup u_\varepsilon \text{ for } k \to +\infty \text{ for any fixed } \varepsilon > 0.$$

Moreover, we have $u_\varepsilon \in \mathscr{K}(e,\Omega)$ (due to $((L\mathscr{A})_h, 1°)$). Using condition $((L\mathscr{A})_h, 2°)$ again for some $z \in \Lambda_{\mathscr{K}(e,\Omega)}$, we obtain the existence of a sequence $\{\mathscr{R}_{e_{h_{n_j}}e} z\}_j \in \prod_{j\in\mathbb{N}} \mathscr{K}_{h_{n_j}}(e_{h_{n_j}}, \Omega)$ such that $\lim_{j\to+\infty} \mathscr{R}_{e_{h_{n_j}}e} z = z$ strongly in $V(\Omega)$ as $e_{h_{n_j}} \to e$ in $U(\Omega)$. Then we proceed like in the proof of Theorem 1 (from (1.11) to (1.16), here the sequence $\{\mathscr{R}_{e_{h_n}e} z\}_n$ plays the role of $\{\Theta_k\}$). In this way we get

$$\left\{\begin{aligned} &u_{\varepsilon h_n}(e_{h_n}) \xrightarrow{V(\Omega)} u_\varepsilon(e), \\ &\mathscr{A}_{h_n}(e_{h_n}) u_{\varepsilon h_n}(e_{h_n}) \xrightarrow{V^*(\Omega)} \mathscr{A}(e) u_\varepsilon(e), \\ &\mathscr{B}_{h_n}(e_{h_n}) u_{\varepsilon h_n}(e_{h_n}) \xrightarrow{W^*(\Omega)} \mathscr{B}(e) u_\varepsilon(e). \end{aligned}\right. \tag{2.3}$$

□

In order to study the relation of optimal pairs to $(\mathscr{P}_{\varepsilon h})$ and $(\mathscr{P}_\varepsilon)$ we need the following additional assumption:

$$(HU)_h \quad \begin{cases} 1^\circ \text{ The family } \{U^h_{\mathrm{ad}}(\Omega);\, h \in (0,1)\} \text{ is compact in the following sense:} \\ \quad \text{for any set } M := \{e_h;\, e_h \in U^h_{\mathrm{ad}}(\Omega),\ h \in N \subset (0,1), 0 \in \operatorname{cl} N\} \text{ there} \\ \quad \text{is a sequence } h_n \to 0 \text{ and an element } e \in U_{\mathrm{ad}}(\Omega) \text{ such that } e_{h_n} \to e \\ \quad \text{in } U(\Omega). \\ 2^\circ \text{ For any } e \in U_{\mathrm{ad}}(\Omega) \text{ there exists a sequence } \{h_n\}_n \subset \mathbb{R} \text{ such that} \\ \quad h_n \to 0 \text{ and } \{e_{h_n}\}_n \in \prod_{n\in\mathbb{N}} U^{h_n}_{\mathrm{ad}}(\Omega) \text{ such that } e_{h_n} \to e \text{ in } U(\Omega). \\ 3^\circ\ h_n \to 0_+,\ e_{h_n} \in U^{h_n}_{\mathrm{ad}}(\Omega),\ n \in \mathbb{N},\ e_{h_n} \xrightarrow{U(\Omega)} e,\ v_{h_n} \xrightarrow{V(\Omega)} v \text{ for } n \to \\ \quad +\infty \implies \mathscr{L}_{h_n}(e_{h_n}, v_{h_n}) \to \mathscr{L}(e,v). \end{cases}$$

Then we have

Theorem 4. *Let $\varepsilon > 0$ be fixed and let $((H\mathscr{A})_h, (H\mathscr{B})_h)$ and $((L\mathscr{A})_h, (HU)_h)$ be satisfied. Let $[e_{\varepsilon h}, u_{\varepsilon h}(e_{\varepsilon h})]$ be an optimal pair of $(\mathscr{P}_{\varepsilon h})$, $e_{\varepsilon h} \in U^h_{\mathrm{ad}}(\Omega)$, $h \in (0,1)$, $\varepsilon > 0$. Then there exists a sequence $\{h_n\}_n$, $h_n \to 0$ for $n \to +\infty$ and a pair of elements $[e_\varepsilon, u_\varepsilon(e_\varepsilon)] \in U_{\mathrm{ad}}(\Omega) \times \mathscr{K}(e_\varepsilon, \Omega)$ such that $[e_{\varepsilon h_n}, u_{\varepsilon h_n}(e_{\varepsilon h_n})] \to [e_\varepsilon, u_\varepsilon(e_\varepsilon)]$ in $U(\Omega) \times V(\Omega)$ for $n \to +\infty$.*

Proof. The assumption $((HU)_h, 1^\circ)$ yields the existence of a sequence $\{e_{\varepsilon h_n}\}_n$ and $e_\varepsilon \in U_{\mathrm{ad}}(\Omega)$ such that $e_{\varepsilon h_n} \to e_\varepsilon$ in $U(\Omega)$. By virtue of Lemma 2 we have $u_{\varepsilon h_n}(e_{\varepsilon h_n}) \to u_\varepsilon(e_\varepsilon)$ in $V(\Omega)$. Then due to $((L\mathscr{A})_h, 2^\circ)$ one has $u_\varepsilon(e_\varepsilon) \in \mathscr{K}(e_\varepsilon, \Omega)$, i.e. $e_\varepsilon \in U_{\mathrm{ad}}(\Omega)$. The definition of $(\mathscr{P}_{\varepsilon h})$ gives

$$\mathscr{L}_{h_n}(e_{\varepsilon h_n}, u_{\varepsilon h_n}(e_{\varepsilon h_n})) \leqslant \mathscr{L}_{h_n}(e_{h_n}, u_{\varepsilon h_n}(e_{h_n}))\ \forall e_{h_n} \in U^{h_n}_{\mathrm{ad}}(\Omega). \tag{2.4}$$

Let $\tilde{e} \in U_{\mathrm{ad}}(\Omega)$ be given. Due to $((HU)_h, 2^\circ)$ one can find sequences $\{h_n\}_n$, $h_n \to 0_+$ and $\{\tilde{e}_{h_n}\}_n \in \prod_{n\in\mathbb{N}} U^{h_n}_{\mathrm{ad}}(\Omega)$ such that $\tilde{e}_{h_n} \to \tilde{e}$ in $U(\Omega)$. Thus $u_{\varepsilon h_n}(\tilde{e}_{h_n}) \to u_\varepsilon(\tilde{e})$ in $V(\Omega)$ and with the use of (2.4) and $((HU)_h, 3^\circ)$ we obtain

$$\mathscr{L}(e_\varepsilon, u_\varepsilon(e_\varepsilon)) \leqslant \mathscr{L}(\tilde{e}, u_\varepsilon(\tilde{e}))\ \forall \tilde{e} \in U_{\mathrm{ad}}(\Omega)$$

and the proof is complete. □

The problem $(\mathscr{P}_0)$ can be treated quite analogously and an appropriate variant of Theorem 4 for this case is the following

Theorem 5. *Let $(H\mathscr{B})_h$ and $(HU)_h$ for $\varepsilon = 0$ be satisfied. Let $[e_{0h}, u_{0h}(e_{0h})]$ be an optimal pair of $(\mathscr{P}_{0h})$, $e_{0h} \in U^h_{\mathrm{ad}}(\Omega)$, $h \in (0,1)$. Then there exists a sequence $h_n \to 0_+$ and a sequence of pairs $\{[e_{0h_n}, u_{0h_n}(e_{0h_n})]\}_n$ and a pair of elements $[e_0, u(e_0)] \in U_{\mathrm{ad}}(\Omega) \times \hat{\mathscr{K}}(e_0, \Omega)$ such that $[e_{0h_n}, u_{0h_n}(e_{0h_n})] \to [e_0, u(e_0)]$ in $U(\Omega) \times W(\Omega)$ for $h_n \to 0_+$.*

3. Initially stressed elasto-plastic plates

In many practical applications, plates are in a state of initial membrane stress. When subsequently subjected to transverse pressure loads, their structural behaviour and response can be entirely different from plates which are free from such internal stresses.

Let us consider an elasto-plastic plate having small flexural rigidity and being referred to a fixed orthogonal Cartesian coordinate system. The middle surface of the plate is indentified in its undeformed state with an open bounded domain Ω in $\mathbb{R}^2$. The plate has a small thickness $2h_p$ (may be geometrically characterized as "thin" domain in $\mathbb{R}^3$) and its middle plain coincides with the $O_{x_1x_2}$-plain of the coordinate system $O_{x_1x_2x_3}$. Let the boundary of Ω (denoted by $\partial\Omega$) be Lipschitz. We will consider the physical situation in which the transverse displacement of an elasto-plastic plate is constrained by the presence of a variable obstacle (rigid frictionless surface located at a distance $\mathscr{S} \equiv \mathscr{S}(x_1, x_2)$ under the middle plain of the plate). Thus the function $v \equiv v(x_1, x_2)$ describing the admissible transverse displacement must satisfy the relation $v \geqslant \mathscr{S} + h_p$ which is assumed to be non-positive. The transverse reactive force $\mu \equiv \mu(v)$ and the displacement v are supposed to satisfy the usual contact condition of the Signorini type

$$
\begin{aligned}
&v - (\mathscr{S} + h_p) \geqslant 0, \quad \mu \geqslant 0,\\
&(v - (\mathscr{S} + h_p))\mu = 0 \quad \text{in } \Omega.
\end{aligned}
$$

The distribution of a transversal load $q \equiv q(x_1, x_2)$ and a rigid frictionless obstacle (stiff punch) $\mathscr{S}$ may be viewed as design variables. To simplify notation, they are denoted as a design vector $e \equiv [q, \mathscr{S}] \in C(\bar{\Omega}) \times H^2(\Omega) \equiv U(\Omega)$. We define $U_{\mathrm{ad}}(\Omega) \equiv U^q_{\mathrm{ad}}(\Omega) \times U^{\mathscr{S}}_{\mathrm{ad}}(\Omega)$, where

$$
(3.1) \qquad \begin{cases}
U^q_{\mathrm{ad}}(\Omega) := \Big\{ q \in W^1_\infty(\Omega);\ 0 \leqslant q \leqslant c_{1q}, |\partial q/\partial x_i| \leqslant c_i, \\
\qquad\qquad i = 1,2 \text{ on } \Omega,\ \displaystyle\int_\Omega q \, \mathrm{d}\Omega = c_{2q} \Big\}, \\
U^{\mathscr{S}}_{\mathrm{ad}}(\Omega) := \{ \mathscr{S} \in H^{2+\eta}(\Omega);\ -c_{1\mathscr{S}} \leqslant \mathscr{S} \leqslant 0, \\
\qquad\qquad \|\mathscr{S}\|_{H^{2+\eta}(\Omega)} \leqslant c_{2\mathscr{S}} \text{ on } \Omega,\ \mathscr{S}(\partial\Omega) = 0 \},
\end{cases}
$$

where η, c_{ij}, $i = 1,2$, $j = q$, $\mathscr{S}$, and c_i, $i = 1,2$, are given positive constants such that the respective $U^j_{\mathrm{ad}}(\Omega)$ is nonempty. Let the plate be simply supported at $\partial\Omega$. Therefore we assume $V(\Omega) = H^2(\Omega) \cap \mathring{H}^1(\Omega)$ and $W(\Omega) = \mathring{H}^1(\Omega)$. The set of kinematically admissible virtual displacements is defined by

$$
(3.2) \qquad \mathscr{K}(e, \Omega) := \{ v \in V(\Omega);\ v \geqslant \mathscr{S} + h_p \text{ on } \Omega \}, \text{ where } \mathscr{S} \in U^{\mathscr{S}}_{\mathrm{ad}}(\Omega).
$$

Define the virtual work of external loads by

$$(3.3) \qquad \langle L(e), v\rangle_{V(\Omega)} = (\langle f + Be, v\rangle_{V(\Omega)}) = \langle q, v\rangle_{V(\Omega)}, \ v \in V(\Omega).$$

(Thus $W^*(\Omega)$ can be called the space of loads.) The operator $\mathscr{A}$ corresponds here to the bending (when no initial membrane pre-stress occurs) of the elasto-plastic plate (see [9], Chapter 4). First we recall some basic relations from the deformation theory of elasto-plastic plates. Let $\varrho \in C_1([0,+\infty])$ be a material function fulfiling the conditions

$$(3.4) \qquad 0 < \varphi_0 \leqslant \varrho(\xi) \leqslant \omega_0, \quad 0 < \psi_0 \leqslant d[\xi\varrho(\xi^2)]/d\xi \leqslant \nu_0 \ \forall \xi \geqslant 0,$$

where φ_0, ψ_0, ν_0, ω_0 are certain constants. We define functions

$$\mathscr{F}\colon t \longmapsto \int_{-h_p}^{h_p} \varrho(z^2 t) z^2 \, \mathrm{d}z, \ t \geqslant 0,$$

$$\begin{aligned} H_{\mathscr{A}}\colon [v,w] \longmapsto & \frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 w}{\partial x_1^2} + \frac{\alpha}{2}\left(\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1^2}\right) \\ & + \frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 w}{\partial x_2^2} + \beta^2 \frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2}, \end{aligned}$$

where α and β are some real constants such that $H_{\mathscr{A}}$ is positive definite in the second derivatives of functions on Ω. We set

$$(3.5) \qquad a(v,w) = 2\int_{\Omega} \mathscr{F}(H_{\mathscr{A}}(v,v)) H_{\mathscr{A}}(v,w) \, \mathrm{d}\Omega$$

and the operator $\mathscr{A}$ is now defined by the relation

$$(3.6) \qquad \langle \mathscr{A} v, w\rangle_{V(\Omega)} = a(v,w), \ v,w \in V(\Omega).$$

Clearly, $\left|\langle \mathscr{A} v, w\rangle_{V(\Omega)}\right| \leqslant \mathrm{const}\, \|v\|_{V(\Omega)} \|w\|_{V(\Omega)}$, $v,w \in V(\Omega)$.

A membrane is a thin plate offering no resistence to bending and acting only in a tension. For investigation of the equilibrium position of an elasto-plastic membrane occupying the domain Ω attached to rigid support $\mathscr{S}$ and submitted to the action of forces q we introduce the material function $\varrho \in C^1([0,+\infty])$. We assume that ϱ satisfies (3.4). Moreover, we introduce functions

$$\mathscr{N}\colon t \mapsto 2\varrho(t) h_m, \ t \geqslant 0 \text{ (where } h_m \text{ is the thickness of the membrane)},$$

$$H_{\mathscr{B}}(v,w) := \frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}$$

and a function b on $W(\Omega) \times W(\Omega)$ by

$$b(v,w) = 2 \int_{\Omega} \mathscr{N}(H_{\mathscr{B}}(v,v)) H_{\mathscr{B}}(v,w)\,\mathrm{d}\Omega. \tag{3.7}$$

Then the operator $\mathscr{B}\colon W(\Omega) \to W^*(\Omega)$ is defined by

$$\langle \mathscr{B}v, w\rangle_{W(\Omega)} = b(v,w),\ \ v,w \in W(\Omega). \tag{3.8}$$

Under the above assumptions, we have

Lemma 3. *For every $e \in U_{\mathrm{ad}}(\Omega)$ the set $\mathscr{K}(e,\Omega)$ is non-empty, closed and convex. The system $\{\mathscr{K}(e,\Omega)\}$ fulfils the condition $((H\mathscr{A}), 1°)$.*

Proof. The form of $\mathscr{K}(e,\Omega)$ follows directly from its definition. If $v_n \in \mathscr{K}(e_n,\Omega)$, $\mathscr{S}_n \to \mathscr{S}$ in $H^2(\Omega)$ and $v_n \rightharpoonup v$ in $V(\Omega)$, then $v_n \to v$ and $\mathscr{S}_n \to \mathscr{S}$ in $C^0(\bar{\Omega})$ and the inequality for the limit remains valid. □

Lemma 4. *The operators $\mathscr{A}$ and $\mathscr{B}$ fulfil the respective assumptions in $(H\mathscr{A})$ and $(H\mathscr{B})$.*

Proof. Clearly, $\mathscr{F}(t) \in \langle \frac{2}{3}\varphi_0 h_p^3, \frac{2}{3}\omega_0 h_p^3\rangle$ and $(\partial\mathscr{F}/\partial t)(t) \in \langle \frac{2}{3}\psi_0 h_p^3, \frac{2}{3}\nu_0 h_p^3\rangle$ and $\int_{\Omega} H_{\mathscr{A}}(v,v)\,\mathrm{d}\Omega \geqslant \widehat{c}_{\mathscr{A}}\|v\|_{V(\Omega)}$ on $V(\Omega)$. As $[v,z] \mapsto \int_{\Omega} H_{\mathscr{A}}(v,z)\,\mathrm{d}\Omega$ can be taken as a scalar product on $V(\Omega)$, we can apply Theorem 1.1 in Chapter III, Sec. 1 of [9] to obtain

$$\begin{gathered} \|\mathscr{A}v - \mathscr{A}z\|_{V^*(\Omega)} \leqslant M_{2\mathscr{A}}\|v-z\|_{V(\Omega)} \text{ and} \\ \langle \mathscr{A}v - \mathscr{A}z, v-z\rangle_{V(\Omega)} \geqslant M_{1\mathscr{A}}\|v-z\|^2_{V(\Omega)}\ \forall v,z \in \Omega \end{gathered} \tag{3.9}$$

with positive constants $M_{1\mathscr{A}}, M_{2,\mathscr{A}}$ independent of $v,z \in V(\Omega)$. Using the same procedure for the operator $\mathscr{B}$ we obtain for it relations similar to (3.9) for the spaces $W(\Omega)$, $W^*(\Omega)$ and with constants $M_{1\mathscr{B}}$ and $M_{2\mathscr{B}}$. Thus

$$\mathscr{A} \equiv \mathscr{A}(e) \in \mathscr{E}_{V(\Omega)}(M_{1\mathscr{A}}, M_{2\mathscr{A}}) \text{ and } \mathscr{B} \equiv \mathscr{B}(e) \in \mathscr{E}_{W(\Omega)}(M_{1\mathscr{B}}, M_{2\mathscr{B}}) \tag{3.10}$$

(both being independent of $e \in U_{\mathrm{ad}}(\Omega)$). The rest of the proof is trivial. □

Let us consider the cost function to the optimal control problem in the form

$$\mathscr{L}\colon [e,v] \mapsto \int_{\Omega} [v - z_d]^2\,\mathrm{d}\Omega \tag{3.11}$$

with a given $z_d \in L_2(\Omega)$. It is obvious that all assumptions in $(E0)$ are fulfilled.

Remark 1. The desired surface of the punch (the shape of an obstacle) and the desired distribution of the external forces are given by the distribution z_d of the deflection and we consider a control parameter as a subject to constraints, i.e. $e \in U_{\mathrm{ad}}(\Omega)$ such that the system response $u_\varepsilon(e)$ is a minimum deviation of z_d in the defined sense.

From Lemmas 3 and 4 and the above mentioned arguments it follows that all assumptions of Theorem 3 are satisfied. Hence there exists at least one solution of the optimization problems $(\mathscr{P}_\varepsilon)$, $(\mathscr{P}_0)$, respectively. Particularly, there is a sequence $\varepsilon_n \to 0$ and the sequence $\{[\mathscr{S}_{\varepsilon_n}, q_n]\}_n$ of optimal solutions to $(\mathscr{P}_{\varepsilon_n})$, respectively, and there is an optimal solution $[\mathscr{S}_0, q_0]$ to $(\mathscr{P}_0)$ such that

$$(3.12) \qquad \mathscr{S}_{\varepsilon_n} \xrightarrow{H^2(\Omega)} \mathscr{S}_0, \quad q_{\varepsilon_n} \xrightarrow{C^0(\bar{\Omega})} q_0 \quad \text{and} \quad u_{\varepsilon_n}(e_{\varepsilon_n}) \xrightarrow{\mathring{H}^1(\Omega)} u_0(e_0).$$

Remark 2. Let $\mathscr{K}(e,\Omega) \equiv \mathscr{K}(\Omega)$, $e \in \mathscr{K}(\Omega)$ and $\frac{1}{\sqrt{\varepsilon_n}}\|L(e_{\varepsilon_n}) - L(e_0)\|_{H^{-1}(\Omega)}$ happen to tend to 0 for $\varepsilon_n \to 0$. Then one has

$$(3.13) \qquad \|u_{\varepsilon_n}(e_{\varepsilon_n}) - u_0(e_0)\|_{\mathring{H}^1(\Omega)} = O(\sqrt{\varepsilon_n}), \ \varepsilon_n \to 0, \text{ and } u_{\varepsilon_n} \xrightarrow{H^2(\Omega)} u_0(e_0).$$

Indeed, this follows easily by putting $v = u_0(e_0)$ into the variational inequality $\langle \varepsilon_n \mathscr{A} u_{\varepsilon_n}(e_{\varepsilon_n}), v - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{H^2(\Omega)} + \langle \mathscr{B} u_{\varepsilon_n}(e_{\varepsilon_n}) - L(e_{\varepsilon_n}), v - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{\mathring{H}^1(\Omega)} \geqslant 0$ and $v = u_{\varepsilon_n}(e_{\varepsilon_n})$ into the variational inequality $\langle \mathscr{B} u_0(e_0) - L(e_0), v - u_0(e_0)\rangle_{\mathring{H}^1(\Omega)} \geqslant 0$. The relation $u_0(e_0) \in H^2(\Omega)$ is proved in [2]. Due to (3.10) we obtain for each $n \in \mathbb{N}$

$$\begin{aligned}(3.14) \quad M_{1\mathscr{A}}\|u_{\varepsilon_n}(e_{\varepsilon_n}) - u_0(e_0)\|^2_{H^2(\Omega)} &+ \frac{M_{1\mathscr{B}}}{\varepsilon_n}\|u_{\varepsilon_n}(e_{\varepsilon_n}) - u_0(e_0)\|^2_{\mathring{H}^1(\Omega)} \\ &\leqslant \langle \mathscr{A} u_0(e_0), u_0(e_0) - u_{\varepsilon_n}(e_{\varepsilon_n})\rangle_{H^2(\Omega)} \\ &\quad + \frac{1}{\varepsilon_n}\|L(e_{\varepsilon_n}) - L(e_0)\|_{H^{-1}(\Omega)}\|u_{\varepsilon_n}(e_{\varepsilon_n}) - u_0(e_0)\|_{\mathring{H}^1(\Omega)}.\end{aligned}$$

The relations (3.10) and (3.14) immediately yield $u_{\varepsilon_n}(e_{\varepsilon_n}) \to u_0(e_0)$ in $H^2(\Omega)$. Using (3.14) again, we get the assertion.

Remark 3. The choice $V(\Omega) = \mathring{H}^2(\Omega)$ is related to the so called clamped problem. If the remaining assumptions of this section are preserved, then the results remain valid for this case with the exception of those in Remark 2.

4. Approximation by finite elements

We shall propose an approximate procedure to the problem treated in the preceding section. We confine ourselves to particular domains being paralellograms. Let the plate be supported at a part $\partial\Omega_u$ of the boundary $\partial\Omega$ and unilaterally supported at the remaining part $\partial\Omega_c$, both with a positive measure. We introduce sets

$$\begin{aligned} \mathscr{V}(\Omega) &:= \{v \in C^\infty(\bar{\Omega})\,;\ v = 0 \text{ on } \partial\Omega_u\}, \\ K(\partial\Omega) &:= \{v \in \mathscr{V}(\Omega)\,;\ v \geqslant 0 \text{ on } \partial\Omega_c\}. \end{aligned}$$

Let $V(\Omega)$ be the closure of $\mathscr{V}(\Omega)$ and $\mathscr{K}(\partial\Omega)$ the closure of $K(\partial\Omega)$ in the space $H^2(\Omega)$. It is easily seen that $\mathscr{K}(\partial\Omega)$, playing here the role of $\mathscr{K}(\Omega)$ from Sec. 1, is a convex set in $H^2(\Omega)$. It corresponds to the unilateral obstacle for the deflection. On the other hand, $W(\Omega)$ is assumed to be the closure of $\mathscr{V}(\Omega)$ in the space $H^1(\Omega)$ and the closure (in $H^1(\Omega)$) of $\mathscr{K}(\partial\Omega)$ is $\hat{\mathscr{K}}(\partial\Omega)$ which plays here the role of $\hat{\mathscr{K}}(\Omega)$ used earlier. The distribution of the external transverse forces $q \equiv e$ will be sought here in the set $U_{\mathrm{ad}}(\Omega) \equiv U^q_{\mathrm{ad}}(\Omega)$ which is defined in (3.1).

Let $\mathscr{T}_h$ denote a uniform partition of Ω into a finite number of small (open) paralellograms $\mathscr{O}_i$ by means of two systems of equidistant straight lines parallel to the sides of Ω. We assume $\bar{\Omega} = \bigcup_{i=1}^{m(h)} \overline{\mathscr{O}_i}$, $\mathscr{O}_i \cap \mathscr{O}_j = \emptyset$ for $i \neq j$ and denote $h = \operatorname{diam} \mathscr{O}_i$. Moreover, we assume that $\mathscr{T}_h$ is consistent with the partition of the boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_c$. We denote by N_h the set of nodes and set $N_{*h} \equiv N_h \cap \partial\Omega_c$. We introduce the spaces $Q_k(\mathscr{O}_i)$ of bilinear ($k = 1$) or bicubic ($k = 3$) polynomials defined on the paralellogram $\mathscr{O}_i$, $i = 1, \dots, m(h)$. If $\mathscr{O}_i$ is not rectangular, then the spaces $Q_k(\mathscr{O}_i)$ can be defined via the affine mapping

$$F\colon [y_1, y_2] \mapsto [y_1 + y_2 \cos\alpha, y_2 \sin\alpha] \tag{4.1}$$

(with α being the angle of the sides) and thus we can assume the rectangularity of $\mathscr{O}_i$ without any loss of generality. We denote

$$\begin{aligned} U^h_{\mathrm{ad}}(\Omega) &:= \{q \in U_{\mathrm{ad}}(\Omega)\,;\ q|_{\mathscr{O}_i} \in Q_1(\mathscr{O}_i),\ i = 1, \dots, m(h)\}, \\ V_h(\Omega) &:= \{v \in V(\Omega)\,;\ v|_{\mathscr{O}_i} \in Q_3(\mathscr{O}_i),\ i = 1, \dots, m(h)\}, \\ W_h(\Omega) &:= \{v \in W(\Omega)\,;\ v|_{\mathscr{O}_i} \in Q_1(\mathscr{O}_i),\ i = 1, \dots, m(h)\}, \\ \mathscr{K}_h(\partial\Omega) &:= \{v \in V_h(\Omega)\,;\ v(s) \geqslant 0,\ s \in N_{*h}\}, \\ \hat{\mathscr{K}}_h(\partial\Omega) &:= \{v \in W_h(\Omega)\,;\ v(s) \geqslant 0,\ s \in N_{*h}\}. \end{aligned}$$

We observe that $\mathscr{K}_h(\partial\Omega)$, $\hat{\mathscr{K}}_h(\partial\Omega)$ are closed, convex and nonempty subsets of $V_h(\Omega)$, $W_h(\Omega)$, respectively. Clearly, $\mathscr{K}_h(\partial\Omega) \not\subset \mathscr{K}(\partial\Omega)$ and $\hat{\mathscr{K}}_h(\partial\Omega) \subset \hat{\mathscr{K}}(\partial\Omega)$. We consider the following discrete variants of a from (3.5), b from (3.7) and $L(e)$ from (3.3):

$$a(v_h, w_h) := 2 \int_\Omega \mathscr{F}(H_{\mathscr{A}}(v_h, v_h)) H_{\mathscr{A}}(v_h, w_h)\, \mathrm{d}\Omega, \ v_h, w_h \in V_h(\Omega),$$

$$b(v_h, w_h) := 2 \int_\Omega \mathscr{F}(H_{\mathscr{B}}(v_h, v_h)) H_{\mathscr{B}}(v_h, w_h)\, \mathrm{d}\Omega, \ v_h, w_h \in W_h(\Omega),$$

$$\langle L(e_h), v_h \rangle_{W(\Omega)} := \int_\Omega q_h v_h \,\mathrm{d}\Omega, \ e_h \in U^h_{\mathrm{ad}}(\Omega) \text{ and } v_h \in V_h(\Omega).$$

We will not use numerical integration, hence

$$\langle \mathscr{A}_h v_h, w_h \rangle_{V(\Omega)} := \langle \mathscr{A} v_h, w_h \rangle_{V(\Omega)}, \langle \mathscr{B}_h v_h, w_h \rangle_{W(\Omega)} := \langle \mathscr{B} v_h, w_h \rangle_{W(\Omega)},$$
$$\langle L_h(e_h), v_h \rangle_{V(\Omega)} := \langle L(e_h), v_h \rangle_{V(\Omega)}, \mathscr{L}_h(e_h, v_h) := \mathscr{L}(e_h, v_h)$$

for all arguments from the respective spaces. The form of $(\mathscr{P}_{\varepsilon h})$ and $(\mathscr{P}_{0h})$ for our case is now defined. In what follows, we shall consider only families $\{\mathscr{T}_{h_n};\, n \in \mathbb{N}\}$ of partitions for which the partition $\mathscr{T}_{h_{n+1}}$ refines the partition $\mathscr{T}_{h_n}$, $n \in \mathbb{N}$. The family $\{\mathscr{T}_h;\, h \in M\}$ for a set $M \subset \mathbb{R}^+$ will be called regular if

$$\text{(4.2)} \qquad (\exists C > 0)\,(\forall h \in M)\ \frac{h}{\varrho(h)} \leqslant C, \ \text{ where } \varrho\colon h \mapsto \min_{\mathscr{O}_i \in \mathscr{T}_h} \max_{\substack{\mathscr{C}_i \subset \mathscr{O}_i \\ \mathscr{C}_i \text{circle}}} \operatorname{diam} \mathscr{C}_i.$$

The introduced family $\{\mathscr{T}_{h_n};\, n \in \mathbb{N}\}$ is evidently regular.

Lemma 5. *Let a positive sequence h_n satisfies $h_n \to 0$ for $n \to +\infty$, and let a sequence $\{e_{h_n}\}_n \in \prod_{n\in\mathbb{N}} U^{h_n}_{\mathrm{ad}}(\Omega)$ converges to a function e in $U(\Omega)$ for $n \to +\infty$ and some $\varepsilon > 0$. Let $u_{\varepsilon h_n}(e_{h_n})$ be the solution of the appropriate version of (2.1), $n \in N$, and let $u_\varepsilon(e)$ be the solution of the appropriate version of (1.2). Then*

$$\text{(4.3)} \qquad \|u_{\varepsilon h_n}(e_{h_n}) - u_\varepsilon(e)\|_{V(\Omega)} \to 0 \ \textit{for } n \to +\infty \textit{ and fixed } \varepsilon > 0$$

holds for any regular family of partitions $\{\mathscr{T}_{h_n}\}$.

Proof. The existence of the solutions under investigation follows from (3.10). Due to (3.10) again and due to the particular form of $\mathscr{K}_{h_n}(\partial\Omega)$ which is a convex cone with the vertex at 0 there is some $u_{*\varepsilon}$ such that $u_{\varepsilon h_n}(e_{h_n}) \rightharpoonup u_{*\varepsilon}$ in $V(\Omega)$. We shall show $u_{*\varepsilon} \in \mathscr{K}(\partial\Omega)$: Due to the imbedding $H^2(\Omega) \hookrightarrow C^0(\bar\Omega)$ the convergence

$$\text{(4.4)} \qquad u_{\varepsilon h_n}(e_{h_n}) \overset{C^0(\bar\Omega)}{\longrightarrow} u_{*\varepsilon}$$

holds. Hence $u_{*\varepsilon} \geqslant 0$ on $\partial\Omega_c$. Now, we prove $u_{*\varepsilon} = u_\varepsilon(e)$ (i.e. $u_{*\varepsilon}$ fulfils (1.2)). It is well-known that for any $v \in \mathscr{K}(\partial\Omega)$ there are $v_n \in K(\partial\Omega)$ such that $v_n \overset{H^2(\Omega)}{\longrightarrow} v$ for $n \to +\infty$. Let $\mathscr{R}_{h_n}$ be the projector of $K(\partial\Omega)$ onto $V_{h_n}(\Omega)$ (for a given function it orders its V_h-interpolants over the partition $\mathscr{T}_h$). Then $\mathscr{R}_{h_n} v_n \in \mathscr{K}_{h_n}(\partial\Omega)$, because the nodal parameters involve all values $v_n(s)$, $s \in N_{*h}$. Moreover, by [8] $\|v_n - \mathscr{R}_{h_n} v_{h_n}\|_{V(\Omega)} \to 0$ holds for any regular family of partitions $\{\mathscr{T}_h\}$. Hence the condition $((L\mathscr{A})_h)$ is fulfilled with a projector independent of e_{h_n} and e. Now, by the standard procedure used in Secs. 1 and 2 we get

$$\mathscr{A} u_{\varepsilon h_n}(e_{h_n}) \overset{V^*(\Omega)}{\rightharpoonup} \mathscr{A} u_{*\varepsilon}, \quad \mathscr{B} u_{\varepsilon h_n}(e_{h_n}) \overset{W^*(\Omega)}{\rightharpoonup} \mathscr{B} u_{*\varepsilon},$$

which together with $L(e_{\varepsilon h_n}) \overset{W^*(\Omega)}{\longrightarrow} L(e)$ yields (via an appropriate procedure of the preceding sections) that $u_{*\varepsilon} = u_\varepsilon(e)$ and the strong convergence (4.3) occurs. □

Lemma 6. *The problems $(\mathscr{P}_{\varepsilon h})$ and $(\mathscr{P}_{0h})$ possess at least one solution, respectively, for any $h > 0$ and $\varepsilon > 0$.*

The proof is analogous to those in Sec.1. □

Remark 4. A standard procedure from the finite element theory (cf. [7] or [8]) yields that $\overline{\bigcup_{h_n} U_{\mathrm{ad}}^{h_n}(\Omega)} = U_{\mathrm{ad}}(\Omega)$.

Theorem 6. *For any fixed $\varepsilon \geqslant 0$ there is a sequence $\{e_{\varepsilon h_n}\}_n$ of solutions of the approximate problems $(\mathscr{P}_{\varepsilon h_n})$ such that*

$$e_{\varepsilon h_n} \overset{U(\Omega)}{\longrightarrow} e_\varepsilon \quad \text{and} \quad u_{\varepsilon h_n}(e_{\varepsilon h_n}) \overset{V(\Omega)}{\longrightarrow} u_\varepsilon(e_\varepsilon), \tag{4.5}$$

where $e \in U_{\mathrm{ad}}(\Omega)$ is a solution of the optimization problem $(\mathscr{P}_\varepsilon)$. If the solution of $(\mathscr{P}_\varepsilon)$ is unique, then the relation (4.5) with the indices h instead of h_n holds for $h \to 0$.

Proof. Consider a fixed $\varepsilon \geqslant 0$ and a function $\eta_\varepsilon \in U_{\mathrm{ad}}(\Omega)$. By virtue of Remark 4 there exists a sequence $\{\eta_{\varepsilon h_n}\} \subset \prod_{n\in\mathbb{N}} U_{\mathrm{ad}}^{h_n}(\Omega)$ such that $\lim_{n\to+\infty} \|\eta_{\varepsilon h_n} - \eta_\varepsilon\|_{C(\overline{\Omega})} = 0$. The compactness of $U_{\mathrm{ad}}(\Omega)$ in $C_0(\overline{\Omega})$ implies the existence of $\{e_{\varepsilon h_{n_k}}\} \subset \{e_{\varepsilon h_n}\}$ such that $e_{\varepsilon h_{n_k}} \to e_\varepsilon$ which belongs to $U_{\mathrm{ad}}(\Omega)$. From the definition of the problem $(\mathscr{P}_{\varepsilon h_n})$ we conclude that $J_\varepsilon(e_{\varepsilon h_n}) \leqslant J_\varepsilon(\eta_{\varepsilon h_n})$. Applying Lemma 5 and $(E0), 1°$ we arrive at the relation $J_\varepsilon(e_\varepsilon) \leqslant J_\varepsilon(\eta_\varepsilon)$. For any sequence $\{e_{\varepsilon h_n}\}_{n\in\mathbb{N}}$ there is a convergent subsequence with the above described property. If the solution $(\mathscr{P}_\varepsilon)$ is unique, we obtain (by an easy contradiction proof) that (4.5) holds for $h \to 0$. □

5. Conclusion

The above presented approach was applied to problems related to nonlinear models of elastoplastic plates. It can be employed e.g. in some kind of problems for elastic von Karman plates or for axisymmetric elastoplastic shells, too. There is also a lot of dynamic problems which can be treated in the framework of singular perturbations. However, they need a modified approach.

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