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FINITE ELEMENT VARIATIONAL CRIMES IN THE CASE OF SEMIREGULAR ELEMENTS

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Summary. The finite element method for a strongly elliptic mixed boundary value problem is analyzed in the domain Ω whose boundary $\partial\Omega$ is formed by two circles Γ_1, Γ_2 with the same center S_0 and radii R_1, $R_2 = R_1 + \varrho$, where $\varrho \ll R_1$. On one circle the homogeneous Dirichlet boundary condition and on the other one the nonhomogeneous Neumann boundary condition are prescribed. Both possibilities for $u = 0$ are considered. The standard finite elements satisfying the minimum angle condition are in this case inconvenient; thus triangles obeying only the maximum angle condition and narrow quadrilaterals are used. The restrictions of test functions on triangles are linear functions while on quadrilaterals they are four-node isoparametric functions. Both the effect of numerical integration and that of approximation of the boundary are analyzed. The rate of convergence $O(h)$ in the norm of the Sobolev space H^1 is proved under the following conditions: 1. the data are sufficiently smooth; 2. the lengths b_M and h_M of the smallest and largest sides, respectively, of every element M ($M = T, K$) satisfy the relations $C_1 h_M^2 \leqslant b_M \leqslant C_2 h_M^2$ where T stands for a triangle and K for a quadrilateral.

Keywords: finite element method, elliptic problems, semiregular elements, maximum angle condition, variational crimes

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1. Formulation of the problem

We shall consider the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2}$$

$$\sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} n_i(\Omega) = q \quad \text{on } \Gamma_2, \tag{3}$$

where Ω is a two-dimensional bounded domain with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being the circles with radii R_1 and $R_2 = R_1 + \varrho$, respectively. We assume that the circles Γ_1, Γ_2 have the same center S_0 and that

$$R_1 \gg \varrho. \tag{4}$$

The symbols $n_i(G)$ $(i = 1, 2)$ denote the components of the unit outward normal to ∂G.

A weak solution of problem (1)–(3) is a solution of the following variational problem (which can be obtained from (1)–(3) by means of Green's theorem in a standard way).

1. Problem. Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let

$$V = \{v \in H^1(\Omega)\colon v = 0 \quad \text{on } \Gamma_1\}, \tag{5}$$

$$a(w, v) = \sum_{i=1}^{2} \iint_\Omega k_i(x) \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}\, \mathrm{d}x_1 \mathrm{d}x_2, \tag{6}$$

$$L(v) = L^\Omega(v) + L^\Gamma(v) = \iint_\Omega vf\, \mathrm{d}x_1 \mathrm{d}x_2 + \int_{\Gamma_2} vq\, \mathrm{d}s\,, \tag{7}$$

where

$$k_i \in W^{1,\infty}(\Omega), \quad f \in W^{1,\infty}(\Omega), \quad q = Q|_{\Gamma_2}, \quad Q \in C^2(\overline{U}), \tag{8}$$

$$k_i(x) \geqslant \mu_0 > 0, \tag{9}$$

U being a neighbourhood of Γ_2 (i.e., a domain containing Γ_2). Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{10}$$

Assumptions (8)–(9) guarantee that the symmetric bilinear form (6) is bounded and strongly coercive and that the linear form (7) is continuous. (Of course, this also holds when $f \in L_2(\Omega)$ and $q \in L_2(\Gamma_2)$. We assume (8) because of numerical integration.)

2. Lemma. *Let a solution $u \in V$ of Problem 1 satisfy $u \in H^2(\Omega)$. Then relation (1) holds almost everywhere in Ω and relation (3) holds almost everywhere on Γ_2.*

The proof is omitted. Also the following lemma is well-known:

3. Lemma. *If (9) holds then Problem 1 has a unique solution.*

We shall solve Problem 1 approximately by the finite element method. To this end let us approximate Γ_2 by a regular polygon Γ_{2h} with vertices $Q_1, \ldots, Q_n$ such that every segment Q_iQ_{i+1} has no common point with Γ_1. Let the vertices $P_1, \ldots, P_n$ of the polygon Γ_{1h} approximating Γ_1 be obtained in the following way: P_i is the intersection of the segment S_0Q_i with Γ_1. The symbol Ω_h will denote the polygonal domain with the boundary $\partial\Omega_h$.

We divide each segment P_iQ_i by the points $A_1^i, A_2^i, \ldots, A_{m-1}^i$ into m parts of the same length in such a way that we have formally $A_0^i = P_i$, $A_m^i = Q_i$. The points A_j^i are the vertices of quadrilaterals into which the domain Ω_h is divided. In order to simplify our considerations we divide every quadrilateral $A_{m-1}^i A_{m-1}^{i+1} Q_i Q_{i+1}$ into two triangles. This simplification will be removed in Theorem 31.

We admit to use narrow quadrilaterals and narrow triangles. This means that we shall have

$$\frac{\varrho}{m} \ll h \tag{11}$$

in our considerations, where h is the length of the greatest segment in the division of Ω_h. The corresponding division consisting of closed quadrilaterals $\overline{K}$ and closed triangles $\overline{T}$ will be denoted by $\mathscr{D}_h$.

We shall assume that $k_i \in W^{1,\infty}(\widetilde{\Omega})$, $f \in W^{1,\infty}(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is such that $\Omega_h \subset \widetilde{\Omega}$ for sufficiently small h. When we consider the functions k_i and f in Ω_h we shall use symbols $\widetilde{k}_i$ and $\widetilde{f}$. In the opposite case the original symbols k_i and f will be used.

The discrete problem is now formulated in an almost standard way. (The expression "almost" concerns the approximation of the term $L^\Gamma(v)$ which will need some space.) We define spaces

$$\begin{aligned} X_h = \{v \in C(\overline{\Omega}_h) \colon v\big|_K = {} & \text{ a four-node isoparametric function } \quad \forall \overline{K} \in \mathscr{D}_h, \\ v\big|_T = {} & \text{ a linear polynomial } \quad \forall \overline{T} \in \mathscr{D}_h\} \end{aligned} \tag{12}$$

and

$$V_h = \{v \in X_h : v = 0 \text{ on } \Gamma_{1h}\}. \tag{13}$$

We set

$$\widetilde{a}_h(v,w) = \sum_{i=1}^{2} \iint_{\Omega_h} \widetilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2 \quad \forall v,w \in H^1(\Omega_h) \tag{14}$$

and

$$\widetilde{L}_h^{\Omega}(v) = \iint_{\Omega_h} v\widetilde{f}\,\mathrm{d}x_1\mathrm{d}x_2 \quad \forall v \in X_h. \tag{15}$$

To define $\widetilde{L}_h^{\Gamma}(v)$ is more complicated. We start with a suitable expression of $L^{\Gamma}(\overline{v})$, where $\overline{v}$ is the natural extension of $v \in V_h$ (in more detail see Notation 23). In connection with these considerations we shall use the symbols x, y instead of x_1, x_2. According to the definition and properties of the line integral we can write

$$L^{\Gamma}(\overline{v}) = \int_{\Gamma_2} q\overline{v}\,\mathrm{d}s = \sum_{k=1}^{4} \int_{\Gamma_2^{(k)}} q\overline{v}\,\mathrm{d}s$$

where $\Gamma_2^{(k)}$ is a quarter of the circle Γ_2 with the endpoints B_k, B_{k+1}, where $B_1 = [-\sqrt{2}R_2/2, \sqrt{2}R_2/2]$, $B_2 = [\sqrt{2}R_2/2, \sqrt{2}R_2/2]$, $B_3 = [\sqrt{2}R_2/2, -\sqrt{2}R_2/2]$, $B_4 = [-\sqrt{2}R_2/2, -\sqrt{2}R_2/2]$ and $B_5 \equiv B_1$. Let the points $Q_1, \ldots, Q_n$ be chosen in such a way that $n = 4N$ and $B_1 = Q_1$, $B_2 = Q_{N+1}$, $B_3 = Q_{2N+1}$, $B_4 = Q_{3N+1}$. Let us denote

$$x_1 := -\sqrt{2}R_2/2,\ x_{N+1} := \sqrt{2}R_2/2,\ y_1 := -\sqrt{2}R_2/2,\ y_{N+1} := \sqrt{2}R_2/2,$$
$$x_r := x_1 + (r-1)(x_{N+1} - x_1)/N,\ y_r := y_1 + (r-1)(y_{N+1} - y_1)/N\ (r = 1, \ldots, N+1),$$
$$g(t) := \sqrt{R_2^2 - t^2}.$$

Then we can write

$$L^{\Gamma(1)}(\overline{v}) := \int_{\Gamma_2^{(1)}} q\overline{v}\,\mathrm{d}s = \sum_{r=1}^{N} \int_{x_r}^{x_{r+1}} q(x, g(x))\overline{v}(x, g(x))\sqrt{1 + [g'(x)]^2}\,\mathrm{d}x,$$
$$L^{\Gamma(2)}(\overline{v}) := \int_{\Gamma_2^{(2)}} q\overline{v}\,\mathrm{d}s = \sum_{r=1}^{N} \int_{y_r}^{y_{r+1}} q(g(y), y)\overline{v}(g(y), y)\sqrt{1 + [g'(y)]^2}\,\mathrm{d}y,$$
$$L^{\Gamma(3)}(\overline{v}) := \int_{\Gamma_2^{(3)}} q\overline{v}\,\mathrm{d}s = \sum_{r=1}^{N} \int_{x_r}^{x_{r+1}} q(x, -g(x))\overline{v}(x, -g(x))\sqrt{1 + [g'(x)]^2}\,\mathrm{d}x,$$
$$L^{\Gamma(4)}(\overline{v}) := \int_{\Gamma_2^{(4)}} q\overline{v}\,\mathrm{d}s = \sum_{r=1}^{N} \int_{y_r}^{y_{r+1}} q(-g(y), y)\overline{v}(-g(y), y)\sqrt{1 + [g'(y)]^2}\,\mathrm{d}y.$$

Let ξ_r, η_r be the local coordinate system oriented in the same way as the system x, y, with the origin at the point Q_r and with such an axis ξ_r that its nonnegative part contains the segment Q_rQ_{r+1}. Let α_r be the angle made by the axis ξ_r with the axis x. Then in the case of $\Gamma_2^{(1)}$

$$\begin{aligned} x &= x(\xi_r, \eta_r) := x_r + \xi_r \cos\alpha_r - \eta_r \sin\alpha_r, \\ y &= y(\xi_r, \eta_r) := g(x_r) + \xi_r \sin\alpha_r + \eta_r \cos\alpha_r \end{aligned} \tag{16}$$

is the orthogonal transformation between the systems x, y and ξ_r, η_r.

Let us denote

$$q_{n,r}(\xi_r, \eta_r) := q(x(\xi_r, \eta_r), y(\xi_r, \eta_r)), \tag{17}$$

$$v_{n,r}(\xi_r, \eta_r) := v(x(\xi_r, \eta_r), y(\xi_r, \eta_r)) \tag{18}$$

and let

$$\eta_r = \varphi_r(\xi_r), \quad \xi_r \in [0, l_r]$$

be the analytic expression of the arc

$$y = g(x), \quad x \in [x_r, x_{r+1}]$$

in the system ξ_r, η_r. Then, according to the theorem on invariance of the line integral with respect to an orthogonal transformation,

$$\begin{aligned} &\int_{x_r}^{x_{r+1}} q(x, g(x))\overline{v}(x, g(x))\sqrt{1 + [g'(x)]^2}\,\mathrm{d}x \\ &= \int_0^{l_r} q_{n,r}(\xi_r, \varphi_r(\xi_r))\overline{v}_{n,r}(\xi_r, \varphi_r(\xi_r))\sqrt{1 + [\varphi_r'(\xi_r)]^2}\,\mathrm{d}\xi_r \end{aligned}$$

where $l_r = \mathrm{dist}(Q_r, Q_{r+1})$ and $\overline{v}_{n,r}$ is the natural extension of the function $v_{n,r}$. Let us note that in the case of the circle Γ_2 we have

$$\varphi_r(\xi_r) = -R_2 \cos\frac{\sigma_r}{2} + \sqrt{\left(R_2 \cos\frac{\sigma_r}{2}\right)^2 + l_r\xi_r - \xi_r^2} \tag{19}$$

where σ_r is the angle made by the segments S_0Q_r and S_0Q_{r+1}, S_0 being the center of the circle Γ_2.

The preceding relations give

$$\begin{aligned} L^{\Gamma(1)}(\overline{v}) &:= \int_{\Gamma_2^{(1)}} q\overline{v}\,\mathrm{d}s \\ &= \sum_{r=1}^{N} \int_0^{l_r} q_{n,r}(\xi_r, \varphi_r(\xi_r))\overline{v}_{n,r}(\xi_r, \varphi_r(\xi_r))\sqrt{1 + [\varphi_r'(\xi_r)]^2}\,\mathrm{d}\xi_r \end{aligned} \tag{20}$$

and we can define an approximation $\widetilde{L}_h^{\Gamma(1)}(v)$ of $L^{\Gamma(1)}(\overline{v})$ by

$$\widetilde{L}_h^{\Gamma(1)}(v) = \sum_{r=1}^{N} \int_0^{l_r} q_{n,r}(\xi_r, \varphi_r(\xi_r)) v_{n,r}(\xi_r, 0)\, \mathrm{d}\xi_r. \tag{21}$$

The expressions of $L^{\Gamma(2)}(\overline{v})$, $L^{\Gamma(3)}(\overline{v})$, $L^{\Gamma(4)}(\overline{v})$ and their approximations $\widetilde{L}_h^{\Gamma(2)}(v)$, $\widetilde{L}_h^{\Gamma(3)}(v)$, $\widetilde{L}_h^{\Gamma(4)}(v)$ are similar to (20) and (21), respectively. As

$$L^{\Gamma}(\overline{v}) = \sum_{k=1}^{4} L^{\Gamma(k)}(\overline{v}) \tag{22}$$

we have

$$\widetilde{L}_h^{\Gamma}(v) = \sum_{k=1}^{4} \widetilde{L}_h^{\Gamma(k)}(v). \tag{23}$$

The symbols $a_h(v,w)$, $L_h^{\Omega}(v)$ and $L_h^{\Gamma}(v)$, where $v, w \in X_h$, will denote the approximations of $\widetilde{a}_h(v,w)$, $\widetilde{L}_h^{\Omega}(v)$ and $\widetilde{L}_h^{\Gamma}(v)$, respectively, when using numerical integration. For all $v, w \in X_h$ we have

$$\begin{aligned} a_h(v,w) = & \sum_{\overline{T} \in \mathcal{D}_h} \sum_{i=1}^{2} \sum_{j=1}^{N_T} 2\omega_{T_0,j} \widetilde{k}_i(x_{T,j}) \left.\frac{\partial v}{\partial x_i}\right|_T \left.\frac{\partial w}{\partial x_i}\right|_T \operatorname{mes}_2 T \\ & + \sum_{\overline{K} \in \mathcal{D}_h} \sum_{i=1}^{2} \sum_{j=1}^{N_K} \omega_{K_0,j} \widetilde{k}_i(x_{K,j}) \frac{\partial v}{\partial x_i}(x_{K,j}) \frac{\partial w}{\partial x_i}(x_{K,j}) |J_K(\xi_{1j}, \xi_{2j})| \end{aligned} \tag{24}$$

where $x_{T,j}$ and $x_{K,j}$ are the integration points on a triangle $\overline{T}$ and quadrilateral $\overline{K}$, respectively, and $\omega_{T_0,j}$ and $\omega_{K_0,j}$ are the corresponding coefficients of the given integration formulas (prescribed on the reference triangle $\overline{T}_0$ and reference square $\overline{K}_0$, respectively). The symbol $J_K(\xi_1, \xi_2)$ denotes the Jacobian of transformation (33) which maps the reference square $\overline{K}_0$ one-to-one onto $\overline{K}$. The points $[\xi_{1j}, \xi_{2j}]$ are integration points prescribed on $\overline{K}_0$ and

$$x_{K,j} = [x_1^K(\xi_{1j}, \xi_{2j}), x_2^K(\xi_{1j}, \xi_{2j})].$$

As to $x_{T,j}$ (and $\omega_{T_0,j}$) we mention the simplest possibilities: $N_T = 1$, $2\omega_{T_0,j} = 1$, $x_{T,j} = P_0^T$ (the center of gravity of T); $N_T = 3$, $2\omega_{T_0,j} = \frac{1}{3}$, $x_{T,j} = P_j^T$ (the vertices of T) – both formulas are of the first degree of precision ($d = 1$). If $N_T = 3$, $2\omega_{T_0,j} = \frac{1}{3}$ and $x_{T,j} = Q_j^T$ (the midpoints of the sides) then $d = 2$.

Similarly, for all $v, w \in X_h$ we have

$$
\begin{aligned}
(25) \qquad L_h^{\Omega}(v) = & \sum_{\overline{T} \in \mathscr{D}_h} \sum_{j=1}^{N_T} 2\omega_{T_0,j} v(x_{T,j}) \widetilde{f}(x_{T,j}) \operatorname{mes}_2 T \\
& + \sum_{\overline{K} \in \mathscr{D}_h} \sum_{j=1}^{N_K} \omega_{K_0,j} v(x_{K,j}) \widetilde{f}(x_{K,j}) |J_K(\xi_{1j}, \xi_{2j})|.
\end{aligned}
$$

Finally,

$$
(26) \qquad L_h^{\Gamma(1)}(v) = \sum_{r=1}^{N} \sum_{j=1}^{N_r} l_r \beta_{r,j} q_{n,r}(s_{r,j}, \varphi_r(s_{r,j})) v_{n,r}(s_{r,j}, 0)
$$

where $s_{r,j}$ are integration points on the segment $[0, l_r]$ and $\beta_{r,j}$ the corresponding coefficients of the given integration formula. (For $d = 1$ we have either $N_r = 1$, $\beta_{r,1} = 1$, $s_{r,1} = l_r/2$, or $N_r = 2$, $\beta_{r,j} = \frac{1}{2}$, $s_{r,1} = 0$, $s_{r,2} = l_r$; for $d = 2$ we have $N_r = 3$, $\beta_{r,1} = \beta_{r,3} = \frac{1}{6}$, $\beta_{r,2} = \frac{4}{6}$, $s_{r,1} = 0$, $s_{r,2} = l_r/2$, $s_{r,3} = l_r$.)

Now we can define the approximate problem:

4. Problem. Find $u_h \in V_h$ such that

$$
(27) \qquad a_h(u_h, v) = L_h(v) \quad \forall v \in V_h.
$$

2. An abstract error estimate

5. Definition. Let $u \in H^2(\Omega)$. We define $Q_h u \in X_h$ by

$$
\begin{aligned}
Q_h u|_{\overline{K} \in \mathscr{D}_h} &= Q_K u = \text{the four-node isoparametric interpolant of } u, \\
Q_h u|_{\overline{T} \in \mathscr{D}_h} &= I_T u = \text{the linear interpolant of } u.
\end{aligned}
$$

6. Lemma. *Let Γ_0 be the circle with a center S_0 and radius $R_0 = R_1 - \varrho$. Let $\widetilde{\Omega}$ be a bounded domain such that $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_2$. There exists a linear and bounded extension operator $E\colon H^2(\Omega) \to H^2(\widetilde{\Omega})$ such that the constant C appearing in the inequality*

$$
\|E(v)\|_{2,\widetilde{\Omega}} \leqslant C\|v\|_{2,\Omega} \quad \forall v \in H^2(\Omega)
$$

does not depend on R_1/ϱ.

Lemma 6 follows from the considerations introduced in [6, pp. 20–22].

7. Theorem. *Let $u \in H^2(\Omega)$, $\widetilde{u} := E(u)$ and let the condition*

$$\|v\|^2_{1,\Omega_h} \leqslant C a_h(v,v) \quad \forall v \in V_h,\ \forall h < h_0 \tag{28}$$

be satisfied, where h_0 is sufficiently small. Then Problem 4 has a unique solution $u_h \in V_h$ and we have

$$\begin{aligned} \|\widetilde{u} - u_h\|_{1,\Omega_h} \leqslant C\Bigg(\|Q_h u - \widetilde{u}\|_{1,\Omega_h} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|a_h(Q_h u, w) - \widetilde{a}_h(Q_h u, w)|}{\|w\|_{1,\Omega_h}} \\ + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}^{\Omega}_h(w) - L^{\Omega}_h(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}^{\Gamma}_h(w) - L^{\Gamma}_h(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)|}{\|w\|_{1,\Omega_h}} \Bigg). \end{aligned} \tag{29}$$

P r o o f. Inequality (28) and the Lax-Milgram lemma guarantee that Problem 4 has a unique solution $u_h \in V_h$.

Now we prove estimate (29). Let us denote

$$v := Q_h u - u_h. \tag{30}$$

Then by (28) and (27) we have

$$\begin{aligned} \|Q_h u - u_h\|^2_{1,\Omega_h} &\leqslant C a_h(Q_h u - u_h, v) = C\{a_h(Q_h u, v) - L_h(v)\} \\ &= C\{a_h(Q_h u, v) - \widetilde{a}_h(Q_h u, v) - L_h(v) + \widetilde{L}_h(v) - \widetilde{L}_h(v) \\ &\quad + \widetilde{a}_h(\widetilde{u}, v) + \widetilde{a}_h(Q_h u - \widetilde{u}, v)\}. \end{aligned}$$

This estimate, the triangular inequality, the boundedness of $\widetilde{a}_h(Q_h u - \widetilde{u}, v)$ and (30) imply (29). □

Our first aim is to prove that condition (28) is satisfied. This will be done in Section 4 where we also estimate the second, third and fourth terms appearing on the right-hand side of (29). These terms express the error of numerical integration.

The estimate of the first term, which expresses the interpolation error, is introduced in Section 3. This estimate follows from the known interpolation theorems. The fifth term, which expresses the error due to the approximation of the boundary, will be estimated in the Section 5.

3. The interpolation error

Now we shall estimate the first term appearing on the right-hand side of (29).

8. Theorem. *We have*

$$\|Q_h u - \tilde{u}\|_{1,\Omega_h} \leqslant Ch\|u\|_{1,\Omega}$$

where the constant C is independent of h, u and the division $\mathscr{D}_h$.

The proof follows from the definition of $Q_h u$, Lemma 6 and the following two lemmas.

9. Lemma. *Let $\overline{K}$ be a narrow quadrilateral with parallel long sides. Let $u \in H^2(K)$. Then we have*

$$\|u - Q_K u\|_{0,K} \leqslant \left(C_5 + \frac{C_{12}\varepsilon_K}{h_K \sin\beta_K}\right) h_K^2 |u|_{2,K},$$

$$|u - Q_K u|_{1,K} \leqslant \left(C_{11} + \frac{C_{16}}{\sin\alpha_K}\right) \frac{h_K}{\sin\beta_K} |u|_{2,K}$$

where $Q_K u$ is the four-node isoparametric interpolant of u on $\overline{K}$, h_K is the length of the greatest side of $\overline{K}$, α_K and β_K ($\alpha_K \leqslant \beta_K$) are the angles made by the greatest side with the two short sides and ε_K is the length of the short side at α_K. In the case $\varepsilon_K \leqslant h_K/12$ the constants C_5, C_{11}, C_{12}, C_{16} satisfy

$$C_5 = 55.019093,\ C_{11} = 12.801823,\ C_{12} = 21.658241,\ C_{16} = C_{12}C_{15} = 19.47235264.$$

For the proof see [8].

10. Lemma. *Let $u \in H^2(T)$ and let $I_T u$ be the linear polynomial satisfying $(I_T u)(P_i^T) = u(P_i^T)$ $(i = 1, 2, 3)$ where P_1^T, P_2^T, P_3^T are the vertices of $\overline{T}$. Then*

$$\|u - I_T u\|_{1,T} \leqslant \frac{C}{\sin\gamma_T} h_T \|u\|_{2,T}$$

where γ_T is the maximum angle of T and the constant C does not depend on $\overline{T}$ and u.

Lemma 10 is a special case of the interpolation theorem for linear interpolations introduced in [4]. (In [4] the spaces $W^{2,p}(T)$ $(p > 1)$ are considered instead of the spaces $H^2(T)$. The result of [4] generalizes in the case of linear interpolations the results introduced in both [1] and [3].)

First we shall analyze the numerical integration on quadrilaterals. Let $\overline{K}$ be a quadrilateral whose greatest side lies on the axis x_1 and let it have the vertices

$$P_1(h,0),\ P_2(0,0),\ P_3(\delta\cos\beta,\delta\sin\beta),\ P_4(h-\varepsilon\cos\alpha,\varepsilon\sin\alpha)$$

where $\varepsilon = \operatorname{dist}(P_1,P_4)$, $\delta = \operatorname{dist}(P_2,P_3)$ and α and β are the angles at P_1 and P_2, respectively. As each quadrilateral belonging to $\mathscr{D}_h$ has parallel long sides we have

$$b := \frac{\varrho}{m} = \varepsilon\sin\alpha = \delta\sin\beta. \tag{31}$$

Let $\overline{K}_0$ be the reference square lying in the coordinate system ξ_1,ξ_2 and having the vertices $P_1^*(1,0)$, $P_2^*(0,0)$, $P_3^*(0,1)$, $P_4^*(1,1)$. If we denote

$$\varepsilon_3 = \delta\cos\beta,\quad \varepsilon_4 = \varepsilon\cos\alpha,\quad \varepsilon^* = \varepsilon_3+\varepsilon_4$$

then the one-to-one mapping of $\overline{K}_0$ onto $\overline{K}$ has the form

$$x_1 = h\xi_1 + \varepsilon_3\xi_2 - \varepsilon^*\xi_1\xi_2,\quad x_2 = b\xi_2. \tag{32}$$

If the side P_1P_2 makes an angle φ with the axis x_1 and the vertex P_2 has coordinates x_{10}, x_{20} then (32) is substituted by the mapping

$$\begin{aligned} x_1 &= x_1^K(\xi_1,\xi_2) = x_{10} + (h\xi_1+\varepsilon_3\xi_2-\varepsilon^*\xi_1\xi_2)\cos\varphi - b\xi_2\sin\varphi,\\ x_2 &= x_2^K(\xi_1,\xi_2) = x_{20} + (h\xi_1+\varepsilon_3\xi_2-\varepsilon^*\xi_1\xi_2)\sin\varphi + b\xi_2\cos\varphi. \end{aligned} \tag{33}$$

Both transformations (32) and (33) have the same Jacobian

$$J_K = (h-\varepsilon^*\xi_2)b. \tag{34}$$

It should be noted that for $n \gg 1$ we have

$$\varepsilon_i \approx \frac{1}{2n}\Big(2\pi\big(R_1+\Delta+\frac{\varrho}{m}\big) - 2\pi(R_1+\Delta)\Big) = \frac{\pi\varrho}{nm}\quad (i=3,4;\ 0\leqslant\Delta\leqslant\varrho(1-1/m)).$$

Further

$$h \approx \frac{2\pi R_1}{n}.$$

The last two relations imply in this case

$$\varepsilon_i = \sigma_i b,\quad \sigma_i \leqslant Ch\quad (i=3,4). \tag{35}$$

Let us denote

$$(1) := 2, \quad (2) := 1, \quad \kappa_{ij} = (-1)^{i+j}.$$

Then we can write (omitting the subscript K at J)

$$\frac{\partial \xi_i}{\partial x_j} = \kappa_{ij} \frac{1}{J} \frac{\partial x_{(j)}}{\partial \xi_{(i)}} \quad (i, j = 1, 2)$$

and the theorem on transformation of an integral yields

$$E_K\left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}\right) = E_{K_0}\left(\sum_{i,r,s=1}^{2} \widetilde{k}_i^* \chi_{irs} \frac{\partial v^*}{\partial \xi_r} \frac{\partial w^*}{\partial \xi_s}\right) \tag{36}$$

where

$$E_K(F) := \iint_K F(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 - \sum_{j=1}^{N_K} \omega_{K_0,j} F(x_{K,j}) |J_K(\xi_{1j}, \xi_{2j})|, \tag{37}$$

$$F^*(\xi_1, \xi_2) := F(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2)),$$

$$E_{K_0}(F) := \iint_{K_0} F^*(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 - \sum_{j=1}^{N_K} \omega_{K_0,j} F^*(\xi_{1j}, \xi_{2j}), \tag{38}$$

$$\chi_{irs} = \kappa_{ir} \kappa_{is} \frac{1}{J} \frac{\partial x_{(i)}}{\partial \xi_{(r)}} \frac{\partial x_{(i)}}{\partial \xi_{(s)}}. \tag{39}$$

11. Lemma. *We have*

$$\left\| \frac{\partial \widetilde{k}_i^*}{\partial \xi_1} \right\|_{0,\infty,K_0} \leqslant Ch|\widetilde{k}_i|_{1,\infty,K}, \quad \left\| \frac{\partial \widetilde{k}_i^*}{\partial \xi_2} \right\|_{0,\infty,K_0} \leqslant Cb|\widetilde{k}_i|_{1,\infty,K}, \tag{40}$$

$$\left\| \frac{\partial v^*}{\partial \xi_i} \right\|_{0,\infty,K_0} \leqslant C \left\| \frac{\partial v^*}{\partial \xi_i} \right\|_{0,K_0} \quad (i = 1, 2), \tag{41}$$

$$\left\| \frac{\partial v^*}{\partial \xi_1} \right\|_{0,K_0} \leqslant C\sqrt{\frac{h}{b}} |v|_{1,K}, \quad \left\| \frac{\partial v^*}{\partial \xi_2} \right\|_{0,K_0} \leqslant C\sqrt{\frac{b}{h}} |v|_{1,K}. \tag{42}$$

The proof of (40) and (42) follows immediately from transformation (33) and relations (34), (35). As to estimate (41), it is well-known (see, e.g., the proof of [7, Lemma 11.5]).

12. Lemma. *For all bilinear polynomials v^*, w^* and $\psi \in W^{1,\infty}(K_0)$ we have*

$$\left| E_{K_0}\left(\psi \frac{\partial v^*}{\partial \xi_i} \frac{\partial w^*}{\partial \xi_j}\right) \right| \leqslant C \left\| \frac{\partial v^*}{\partial \xi_i} \right\|_{0,K_0} \left\| \frac{\partial w^*}{\partial \xi_j} \right\|_{0,K_0} |\psi|_{1,\infty,K_0} \tag{43}$$

provided

$$E_{K_0}(p) = 0 \quad \forall p \in \mathscr{P}_2, \tag{44}$$

where $\mathscr{P}_k$ denotes the set of polynomials of degree not greater than k.

The proof is an immediate consequence of the Bramble-Hilbert lemma and relation (41).

13. Theorem. *Let (44) hold. Then we have*

$$\left| E_K \left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \right| \leqslant Ch \max_{i=1,2} \|\widetilde{k}_i\|_{1,\infty,K} |v|_{1,K} |w|_{1,K} \quad \forall v, w \in X_h. \tag{45}$$

Proof. We start with the special case (32). According to (34), (35), (39) and (40),

$$|\tilde{k}_1^* \chi_{111}|_{1,\infty,K_0} = \left| \tilde{k}_1^* \frac{1}{J} \left(\frac{\partial x_2}{\partial \xi_2} \right)^2 \right|_{1,\infty,K_0} = \left| \tilde{k}_1^* \frac{b^2}{bh(1-\varepsilon^* \xi_2/h)} \right|_{1,\infty,K_0} \tag{46}$$

$$\leqslant C \frac{b}{h} |\tilde{k}_1^*|_{1,\infty,K_0} + \frac{b}{h} \|\tilde{k}_1^*\|_{0,\infty,K_0} \left\| \frac{\partial}{\partial \xi_2} \left(\frac{1}{1-\varepsilon^*\xi_2/h} \right) \right\|_{0,\infty,K_0}$$

$$\leqslant Cb \|\widetilde{k}_1\|_{1,\infty,K},$$

$$\chi_{112} = \chi_{121} = \chi_{122} \equiv 0, \tag{47}$$

$$|\tilde{k}_2^* \chi_{211}|_{1,\infty,K_0} = \left| \tilde{k}_2^* \frac{1}{J} \left(\frac{\partial x_1}{\partial \xi_2} \right)^2 \right|_{1,\infty,K_0} = \frac{(\varepsilon^*)^2}{bh} \left| \tilde{k}_2^* \frac{(\varepsilon_3/\varepsilon^* - \xi_1)^2}{1-\varepsilon^*\xi_2/h} \right|_{1,\infty,K_0} \tag{48}$$

$$\leqslant Cbh \|\widetilde{k}_2\|_{1,\infty,K},$$

$$|\tilde{k}_2^* \chi_{212}|_{1,\infty,K_0} = \left| \tilde{k}_2^* \frac{1}{J} \frac{\partial x_1}{\partial \xi_2} \frac{\partial x_1}{\partial \xi_1} \right|_{1,\infty,K_0} \tag{49}$$

$$= \left| \tilde{k}_2^* \frac{(h-\varepsilon^*\xi_2)(\varepsilon_3 - \varepsilon^*\xi_1)}{(h-\varepsilon^*\xi_2)b} \right|_{1,\infty,K_0} \leqslant Ch \|\widetilde{k}_2\|_{1,\infty,K},$$

$$|\tilde{k}_2^* \chi_{221}|_{1,\infty,K_0} = \left| \tilde{k}_2^* \frac{1}{J} \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2} \right|_{1,\infty,K_0} \leqslant Ch \|\widetilde{k}_2\|_{1,\infty,K}, \tag{50}$$

$$|\tilde{k}_2^* \chi_{222}|_{1,\infty,K_0} = \left| \tilde{k}_2^* \frac{1}{J} \left(\frac{\partial x_1}{\partial \xi_1} \right)^2 \right|_{1,\infty,K_0} \tag{51}$$

$$= \frac{h}{b} \left| \tilde{k}_2^* \left(1 - \frac{\varepsilon^*}{h} \xi_2 \right) \right|_{1,\infty,K_0} \leqslant C \frac{h^2}{b} \|\widetilde{k}_2\|_{1,\infty,K}.$$

Combining (46)–(51) with (36) and (43) we obtain (45) by means of (42).

As the Jacobian J of both transformations (32) and (33) is the same the proof in the general case (33) is very similar; thus we omit it. □

14. Remark. In the cases when relation (35) is not satisfied (however, the long sides are parallel) the assertion of Theorem 13 can be proved provided

$$E_{K_0}(p) = 0 \quad \forall p \in \mathscr{P}_4.$$

15. Remark. The case of a quadrilateral K with parallel long sides is a special case of quadrilaterals K satisfying the condition

$$|\varepsilon \sin \alpha - \delta \sin \beta| \leqslant Cbh. \tag{52}$$

It can be proved that the results of Theorem 13 and Remark 14 can be extended to the case (52).

The effect of numerical integration in the case of narrow triangles must be analyzed more carefully than in the case of regular triangles. Let $\overline{T}$ be an arbitrary triangle lying in the plane x_1, x_2 and let $\overline{T}_0$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ lying in the plane ξ_1, ξ_2. Let

$$x_1 = x_1(\xi_1, \xi_2), \quad x_2 = x_2(\xi_1, \xi_2) \tag{53}$$

be the linear transformation which maps $\overline{T}_0$ one-to-one onto $\overline{T}$ (for its form see, for example, [7, Theorem 9.1]) and let $\xi_1 = \xi_1(x_1, x_2)$, $\xi_2 = \xi_2(x_1, x_2)$ be its inverse.

16. Lemma. *Let $v \in C(\overline{T})$ and let*

$$v^*(\xi_1, \xi_2) = v(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2)).$$

Then we have

$$\left\| \sum_{r=1}^{2} \frac{\partial v^*}{\partial \xi_r} \frac{\partial \xi_r}{\partial x_i} \right\|_{0,T_0} \leqslant C|J|^{-1/2}|v|_{1,T} \tag{54}$$

where J is the Jacobian of (53).

P r o o f. The symbol δ_{ij} will denote the Kronecker delta. We have ($\partial \xi_r / \partial x_i$ are constants)

$$\begin{aligned} \left\| \sum_{r=1}^{2} \frac{\partial v^*}{\partial \xi_r} \frac{\partial \xi_r}{\partial x_i} \right\|_{0,T_0}^2 &= |J|^{-1} \iint_T \left(\sum_{r=1}^{2} \left(\frac{\partial v}{\partial x_1} \frac{\partial x_1}{\partial \xi_r} + \frac{\partial v}{\partial x_2} \frac{\partial x_2}{\partial \xi_r} \right) \frac{\partial \xi_r}{\partial x_i} \right)^2 \mathrm{d}x_1 \mathrm{d}x_2 \\ &= |J|^{-1} \iint_T \left(\frac{\partial v}{\partial x_1} \delta_{1i} + \frac{\partial v}{\partial x_2} \delta_{2i} \right)^2 \mathrm{d}x_1 \mathrm{d}x_2 \leqslant C|J|^{-1}|v|_{1,T}^2, \end{aligned}$$

which gives (54). □

The error functionals E_T and E_{T_0} on a triangle $\overline{T}$ and the reference triangle $\overline{T}_0$, respectively, are defined in a similar way as E_K and E_{K_0} (see (37) and (38)).

17. Theorem. *Let*

$$E_{T_0}(p) = 0 \quad \forall p \in \mathscr{P}_0. \tag{55}$$

Then we have

$$\left| E_T\left(\sum_{i=1}^{2} \tilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\right)\right| \leqslant Ch \max_{i=1,2} |\tilde{k}_i|_{1,\infty,T}|v|_{1,T}|w|_{1,T} \quad \forall v,\, w \in X_h. \tag{56}$$

Proof. We have

$$\left| E_T\left(\tilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\right)\right| = |J|\cdot\left| E_{T_0}\left(\tilde{k}_i^*\left(\frac{\partial v}{\partial x_i}\right)^*\left(\frac{\partial w}{\partial x_i}\right)^*\right)\right| \tag{57}$$

$$= |J|\cdot\left| E_{T_0}\left(\tilde{k}_i^*\left(\sum_{r=1}^{2}\frac{\partial v^*}{\partial \xi_r}\frac{\partial \xi_r}{\partial x_i}\right)\left(\sum_{s=1}^{2}\frac{\partial w^*}{\partial \xi_s}\frac{\partial \xi_s}{\partial x_i}\right)\right)\right| = |J|F(\tilde{k}_i^*)$$

where the notation F is used for fixed v^*, w^* and fixed $\overline{T}$ (i.e., fixed linear functions $\xi_i(x_1, x_2)$). Using the assumption $\tilde{k}_i \in W^{1,\infty}(\tilde{\Omega})$ and [7, Lemma 11.5] we obtain

$$|F(\tilde{k}_i^*)| \leqslant \|\tilde{k}_i^*\|_{0,\infty,T_0}\left\|\sum_{r=1}^{2}\frac{\partial v^*}{\partial \xi_r}\frac{\partial \xi_r}{\partial x_i}\right\|_{0,\infty,T_0}\left\|\sum_{s=1}^{2}\frac{\partial w^*}{\partial \xi_s}\frac{\partial \xi_s}{\partial x_i}\right\|_{0,\infty,T_0}$$

$$\leqslant C\|\tilde{k}_i^*\|_{1,\infty,T_0}\left\|\sum_{r=1}^{2}\frac{\partial v^*}{\partial \xi_r}\frac{\partial \xi_r}{\partial x_i}\right\|_{0,T_0}\left\|\sum_{s=1}^{2}\frac{\partial w^*}{\partial \xi_s}\frac{\partial \xi_s}{\partial x_i}\right\|_{0,T_0}.$$

Since $v, w \in X_h$ we have $v^*\big|_{T_0}, w^*\big|_{T_0} \in \mathscr{P}_1$ and assumption (55) yields

$$F(\tilde{k}_i^*) = 0 \quad \forall \tilde{k}_i^* \in \mathscr{P}_0.$$

Hence the Bramble-Hilbert lemma together with Lemma 16 and relation

$$|\tilde{k}_i^*|_{1,\infty,T_0} \leqslant Ch|\tilde{k}_i|_{1,\infty,T}$$

imply

$$|F(\tilde{k}_i^*)| \leqslant C|J|^{-1}h|\tilde{k}_i|_{1,\infty,T}|v|_{1,T}|w|_{1,T}.$$

This result and (57) give (56). □

Till now the analysis of the effect of numerical integration has been done only for triangles satisfying the minimum angle condition. Theorem 17 holds for arbitrary triangles with straight sides (not only for triangles satisfying the maximum angle condition).

For $v, w \in V_h$ we have

$$a_h(v,w) = \widetilde{a}_h(v,w) - \{\widetilde{a}_h(v,w) - a_h(v,w)\},$$

$$\widetilde{a}_h(v,w) - a_h(v,w) = \sum_{\overline{K}\in\mathscr{D}_h} E_K\left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\right) + \sum_{\overline{T}\in\mathscr{D}_h} E_T\left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\right).$$

Using these relations we obtain from Theorems 13 and 17 (details are similar as in the proof of [7, Theorem 11.8]; we use also an inequality of the type [7, (29.1)] which together with (9) implies $\|v\|^2_{1,\Omega_h} \leqslant C\tilde{a}_h(v,v)$):

18. Corollary. *Condition* (28) *is satisfied.*

19. Theorem. *Let*

$$E_{K_0}(p) = 0 \quad \forall p \in \mathscr{P}_2, \qquad E_{T_0}(p) = 0 \quad \forall p \in \mathscr{P}_0.$$

Then we have for $u \in H^2(\Omega)$

$$\sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|a_h(Q_h u, w) - \widetilde{a}_h(Q_h u, w)|}{\|w\|_{1,\Omega_h}} \leqslant Ch \max_{i=1,2} \|\widetilde{k}_i\|_{1,\infty,\widetilde{\Omega}} \|u\|_{2,\Omega} \tag{58}$$

where the constant C *does not depend on* u, $\widetilde{k}_i$, *and* h.

P r o o f. Relation (58) follows from Theorems 13, 17 and Lemmas 9, 10. Details are the same as in the proof of [7, Theorem 11.12]. □

20. Theorem. *Let*

$$E_{K_0}(p) = 0 \quad \forall p \in \mathscr{P}_2 \text{ (or } \forall p \in \mathscr{Q}_1),$$
$$E_{T_0}(p) = 0 \quad \forall p \in \mathscr{P}_0$$

where $\mathscr{Q}_1$ *is the set of all bilinear polynomials. Then we have*

$$\sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} \leqslant Ch\|\widetilde{f}\|_{1,\infty,\widetilde{\Omega}} \sqrt{\mathrm{mes}_2\,\Omega}, \tag{59}$$

where the constant C does not depend on $\widetilde{f}$ and h.

Proof. The following chain of inequalities is based on standard arguments and the preceding results (for simplicity we write f instead of $\widetilde{f}$):

$$
\begin{aligned}
|E_K(wf)| &= |E_{K_0}(f^*w^*J_K)| \leqslant C|f^*w^*J_K|_{0,\infty,K_0} \\
&\leqslant C\|f^*J_K\|_{1,\infty,K_0}|w^*|_{0,\infty,K_0} \leqslant C|f^*J_K|_{1,\infty,K_0}\|w^*\|_{0,K_0} \\
&\leqslant C(|f^*|_{1,\infty,K_0}|J_K|_{0,\infty,K_0} + |f^*|_{0,\infty,K_0}|J_K|_{1,\infty,K_0})\|w^*\|_{0,K_0} \\
&\leqslant C(h_K|f|_{1,\infty,K}h_Kb_K + |f|_{0,\infty,K}b_K^2h_K)(b_Kh_K)^{-1/2}\|w\|_{0,K} \\
&\leqslant Ch_K\sqrt{\operatorname{mes}_2 K}\|f\|_{1,\infty,K}\|w\|_{0,K}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|E_T(wf)| &= |J_T|\cdot|E_{T_0}(f^*w^*)| \leqslant C|J_T|\cdot|f^*w^*|_{0,\infty,T_0} \\
&\leqslant C|J_T|\cdot\|w^*f^*\|_{1,\infty,T_0} \leqslant C\operatorname{mes}_2 T|w^*f^*|_{1,\infty,T_0} \\
&\leqslant C\operatorname{mes}_2 T(|w^*|_{1,\infty,T_0}|f^*|_{0,\infty,T_0} + |w^*|_{0,\infty,T_0}|f^*|_{1,\infty,T_0}) \\
&\leqslant C\operatorname{mes}_2 T(|w^*|_{1,T_0}|f|_{0,\infty,T} + |w^*|_{0,T_0}h_T|f|_{1,\infty,T}) \\
&\leqslant Ch_T\sqrt{\operatorname{mes}_2 T}\|f\|_{1,\infty,T}\|w\|_{1,T}.
\end{aligned}
$$

Summing and using the Cauchy inequality we obtain (59) because

$$|\widetilde{L}_h^\Omega(w) - L_h^\Omega(w)| \leqslant \sum|E_K(wf)| + \sum|E_T(wf)|.$$

□

In order to estimate the effect of numerical integration along Γ_2 we introduce the following error functionals:

$$
\begin{aligned}
E_r(F) &:= \int_0^{l_r} F(\xi_r)\,\mathrm{d}\xi_r - \sum_{j=1}^{N_r} l_r\beta_{r,j}F(s_{r,j}), \\
E_0(F^*) &:= \int_0^1 F^*(t)\,\mathrm{d}t - \sum_{j=1}^{N_r}\beta_{r,j}F^*(t_j)
\end{aligned}
$$

where

$$F^*(t) := F(l_rt), \quad t \in I \equiv [0,1].$$

Hence

$$E_r(F) = l_rE_0(F^*). \tag{60}$$

When considering the line integrals we need also the trace inequalities which are introduced in the following lemma.

21. Lemma. *We have*

$$\|v\|_{0,\partial\Omega} \leqslant \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega), \tag{61}$$

$$\|v\|_{0,\partial\Omega_h} \leqslant \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega_h} \quad \forall v \in H^1(\Omega_h) \tag{62}$$

where the constant C does not depend on v, h and ϱ.

The proofs of (61) and (62) are similar to [5, pp. 15–16]).

22. Theorem. *Let*

$$E_0(p) = 0 \quad \forall p \in \mathscr{P}_2.$$

Then we have

$$\sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|\widetilde{L}_h^\Gamma(w) - L_h^\Gamma(w)|}{\|w\|_{1,\Omega_h}} \leqslant \frac{C}{\sqrt{\varrho}} h^2 M_2(q)\sqrt{\operatorname{mes}_1 \Gamma_2} \tag{63}$$

where the constant C does not depend on q, ϱ and h and where

$$M_2(q) = 5\max\left(2, \frac{2}{R_2}\right) \max_{(x,y)\in\Gamma_2}\left(\left|\frac{\partial^2 Q}{\partial x^2}\right|, \left|\frac{\partial^2 Q}{\partial x\partial y}\right|, \left|\frac{\partial^2 Q}{\partial y^2}\right|, \left|\frac{\partial Q}{\partial x}\right|, \left|\frac{\partial Q}{\partial y}\right|\right). \tag{64}$$

P r o o f. We denote

$$\|w_{n,r}\|^2_{0,l_r} := \int_0^{l_r} [w_{n,r}(\xi_r, 0)]^2 \,\mathrm{d}\xi_r,$$
$$w^*_{n,r}(t) := w_{n,r}(l_r t, 0), \quad t \in I \equiv [0,1].$$

Then we have

$$\|w^*_{n,r}\|_{0,I} = l_r^{-1/2}\|w_{n,r}\|_{0,l_r}.$$

Further, we set

$$\widetilde{q}_{n,r}(\xi_r) := q_{n,r}(\xi_r, \varphi(\xi_r)), \quad \widetilde{q}^*_{n,r}(t) := \widetilde{q}_{n,r}(l_r t). \tag{65}$$

Then, according to (60), we have

$$E_r(\widetilde{q}_{n,r} w_{n,r}) = l_r E_0(\widetilde{q}^*_{n,r} w^*_{n,r}). \tag{66}$$

The following chain of inequalities is again evident:

$$|E_0(\widetilde{q}^*_{n,r} w^*_{n,r})| \leqslant C|\widetilde{q}^*_{n,r} w^*_{n,r}|_{0,\infty,I} \leqslant C\|\widetilde{q}^*_{n,r}\|_{2,\infty,I}|w^*_{n,r}|_{0,\infty,I}$$
$$\leqslant C|\widetilde{q}^*_{n,r}|_{2,\infty,I}\|w^*_{n,r}\|_{0,I} \leqslant Cl_r^2|\widetilde{q}_{n,r}|_{2,\infty,l_r} l_r^{-1/2}\|w_{n,r}\|_{0,l_r}.$$

This result together with (66) implies

$$\sum_{r=1}^{N} |E_r(\widetilde{q}_{n,r} w_{n,r})| \leqslant Ch^2 \max_{r=1,\dots,N} |\widetilde{q}_{n,r}|_{2,\infty,l_r} \sum_{r=1}^{N} l_r^{1/2} \|w_{n,r}\|_{0,l_r}. \tag{67}$$

The Cauchy inequality yields

$$\sum_{r=1}^{N} l_r^{1/2} \|w_{n,r}\|_{0,l_r} \leqslant \sqrt{\sum_{r=1}^{N} l_r} \sqrt{\sum_{r=1}^{N} \|w_{n,r}\|^2_{0,l_r}} = \sqrt{\operatorname{mes}_1 \Gamma_{2h}^{(1)}} \|w\|_{0,\Gamma_{2h}^{(1)}}. \tag{68}$$

Combining (67) and (68) together with the trace inequality (62) we obtain

$$\sum_{r=1}^{N} |E_r(\widetilde{q}_{n,r} w_{n,r})| \leqslant \frac{C}{\sqrt{\varrho}} h^2 \max_{r=1,\dots,N} |\widetilde{q}_{n,r}|_{2,\infty,l_r} \|w\|_{1,\Omega_h}. \tag{69}$$

As $q_{n,r}(\xi_r, \eta_r)$ is defined by (17) and (16), relations $(65)_1$ and $(8)_3$ imply

$$\begin{aligned}\widetilde{q}_{n,r}(\xi_r) &= q(x_r + \xi_r \cos\alpha_r - \varphi_r(\xi_r)\sin\alpha_r, g(x_r) + \xi_r \sin\alpha_r + \varphi_r(\xi_r)\cos\alpha_r) \\ &= Q(x,y)\big|_{(x,y)\in\Gamma_2^{(1)}(Q_r,Q_{r+1})},\end{aligned}$$

where $\Gamma_2^{(1)}(Q_r, Q_{r+1})$ denotes the part of $\Gamma_2^{(1)}$ with the end-points Q_r, Q_{r+1}. From the rule of differentiation of a composite function and from (19) we obtain that

$$\max_{r=1,\dots,N} |\widetilde{q}_{n,r}|_{2,\infty,l_r} \leqslant M_2(q) \tag{70}$$

where $M_2(q)$ is given by (64). Relations (69), (70) imply (63). □

5. The error of the approximation of the boundary

The estimate of the last term in (29) will be divided into several lemmas.

23. Notation. We denote

$$\tau_h = \Omega_h - \overline{\Omega}, \quad \omega_h = \Omega - \overline{\Omega}_h. \tag{71}$$

Further, let $w \in X_h$. The symbol $\overline{w}$ is called the natural extension of w and denotes the function $\overline{w}\colon \overline{\Omega}_h \cup \overline{\Omega} \to R^1$ such that $\overline{w} = w$ on Ω_h and

$$\overline{w}\big|_{\overline{T}^{\mathrm{id}} - \overline{T}} = p\big|_{\overline{T}^{\mathrm{id}} - \overline{T}}$$

where $p \in \mathscr{P}_1$ satisfies $p|_{\overline{T}} = w|_{\overline{T}}$. ($\overline{T}^{\mathrm{id}} \subset \Omega$ is the curved triangle which is approximated by $\overline{T}$.)

24. Lemma. *Let $u \in H^2(\Omega)$. Then we have for $w \in V_h$*

$$(72)\quad |\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)| \leqslant |L^\Gamma(\overline{w}) - \widetilde{L}^\Gamma_h(w)| + \left| \iint_{\omega_h} \sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(k_i \frac{\partial u}{\partial x_i}\right) \overline{w} \, \mathrm{d}x_1 \mathrm{d}x_2 \right|$$
$$+ \left| \iint_{\omega_h} \sum_{i=1}^2 k_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| + \left| \iint_{\tau_h} \left(\sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i}\right) + \widetilde{f} \right) w \, \mathrm{d}x_1 \mathrm{d}x_2 \right|.$$

P r o o f. Using the definitions of $\widetilde{a}_h(\widetilde{u}, w)$, $\widetilde{L}_h(w)$ and Green's theorem we obtain

$$\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = \iint_{\Omega_h} \sum_{i=1}^2 \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 - \widetilde{L}^\Omega_h(w) - \widetilde{L}^\Gamma_h(w)$$
$$= \int_{\Gamma_{2h}} \sum_{i=1}^2 \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} n_i(\Omega_h) w \, \mathrm{d}s - \iint_{\Omega_h} \left(\sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i}\right) + \widetilde{f} \right) w \, \mathrm{d}x_1 \mathrm{d}x_2 - \widetilde{L}^\Gamma_h(w).$$

To the right-hand side let us add zero in the form

$$-\int_{\Gamma_2} \sum_{i=1}^2 k_i \frac{\partial u}{\partial x_i} n_i(\Omega) \overline{w} \, \mathrm{d}s + L^\Gamma(\overline{w}) = 0.$$

If we denote $\Delta = \overline{T}^{\mathrm{id}} - T$ and use Lemma 2 then we can write

$$\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = -\sum_{\Delta \subset \omega_h} \int_{\partial \Delta} \sum_{i=1}^2 k_i \frac{\partial u}{\partial x_i} n_i(\Delta) \overline{w} \, \mathrm{d}s$$
$$- \iint_{\tau_h} \left(\sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i}\right) + \widetilde{f} \right) w \, \mathrm{d}x_1 \mathrm{d}x_2 + L^\Gamma(\overline{w}) - \widetilde{L}^\Gamma_h(w).$$

Transforming the first term on the right-hand side by means of Green's theorem we obtain (72). □

25. Lemma. *Let* (2) *hold. We have*

$$(73)\qquad \|v\|_{0,\omega_h} \leqslant Ch(\|v\|_{0,\Gamma_2} + h|v|_{1,\omega_h}) \quad \forall v \in H^1(\Omega),$$

$$(74)\qquad |\overline{w}|_{1,\omega_h} \leqslant Ch\sqrt{\frac{m}{\varrho}}|w|_{1,\Omega_h},$$

$$(75)\qquad \|\overline{w}\|_{0,\omega_h} \leqslant Ch(\|w\|_{0,\Gamma_{2h}} + h|\overline{w}|_{1,\omega_h}) \leqslant Ch\left(\frac{1}{\sqrt{\varrho}} + h^2\sqrt{\frac{m}{\varrho}}\right)\|w\|_{1,\Omega_h},$$

$$(76)\qquad \|w\|_{0,\tau_h} \leqslant Ch(\|w\|_{0,\Gamma_{1h}} + h|w|_{1,\tau_h}) = Ch^2|w|_{1,\tau_h}$$

where $w \in V_h$ and $\overline{w}$ is defined in Notation 23.

Proof. A) Relation (73) follows from the proof of [7, Lemma 28.3].

B) Since $\Delta = \overline{T}^{\text{id}} - T$ we have

$$|\overline{w}|^2_{1,\omega_h} = \sum_{\Delta \subset \omega_h} \operatorname{mes}_2 \Delta |(\nabla w|_T)|^2 \leqslant C \sum_{\Delta \subset \omega_h} h_T^3 |(\nabla w|_T)|^2$$
$$= C\frac{m}{\varrho} \sum_{\Delta \subset \omega_h} h_T^3 \frac{\varrho}{m} |(\nabla w|_T)|^2 \leqslant C\frac{m}{\varrho} h^2 \sum_{\Delta \subset \omega_h} |w|^2_{1,T} \leqslant C\frac{m}{\varrho} h^2 |w|^2_{1,\Omega_h}$$

because

$$\frac{\varrho}{m} h_T |(\nabla w|_T)|^2 \leqslant C|w|^2_{1,T}.$$

Hence relation (74) follows.

C) The first inequality in (75) follows from the proof of [7, Lemma 28.3] and the second from (62) and (74).

D) The inequality in (76) follows from the proof of [7, Lemma 28.3], the equality from the assumption $w \in V_h$. □

26. Lemma. *Let $u \in H^2(\Omega)$ and $f \in W^{1,\infty}(\Omega)$. Then*

$$\left| \iint_{\omega_h} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(k_i \frac{\partial u}{\partial x_i} \right) \overline{w} \,\mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant Ch^2 \left(\frac{1}{\sqrt{\varrho}} + h^2 \sqrt{\frac{m}{\varrho}} \right) \|f\|_{0,\infty,\Omega} \|w\|_{1,\Omega_h}. \tag{77}$$

Proof. Lemma 2 and the inclusion $\omega_h \subset \Omega$ yield

$$\left| \iint_{\omega_h} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(k_i \frac{\partial u}{\partial x_i} \right) \overline{w} \,\mathrm{d}x_1 \mathrm{d}x_2 \right| = \left| \iint_{\omega_h} \overline{w} f \,\mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant \|f\|_{0,\omega_h} \|\overline{w}\|_{0,\omega_h}.$$

Using the assumption $f \in W^{1,\infty}(\Omega)$, the fact that $\operatorname{mes}_2 \omega_h \leqslant Ch^2$ and estimate (75) we obtain (77). □

27. Lemma. *Let $u \in H^2(\Omega)$ and $\widetilde{k}_i \in W^{1,\infty}(\Omega)$ $(i = 1, 2)$. Then*

$$\left| \iint_{\omega_h} \sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} \,\mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant Ch^2 \frac{\sqrt{m}}{\varrho} \max_{i=1,2} \|k_i\|_{0,\infty,\Omega} \|u\|_{2,\Omega} \|w\|_{1,\Omega_h}. \tag{78}$$

If in addition

$$u \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \tag{79}$$

then

$$(80)\qquad \left|\iint_{\omega_h}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}\frac{\partial \overline{w}}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2\right| \leqslant Ch^2\sqrt{\frac{m}{\varrho}}\max_{i=1,2}\|k_i\|_{0,\infty,\Omega}|u|_{1,\infty,\Omega}\|w\|_{1,\Omega_h}.$$

P r o o f. We have

$$\left|\iint_{\omega_h}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}\frac{\partial \overline{w}}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2\right| \leqslant C\max_{i=1,2}\|k_i\|_{0,\infty,\Omega}|u|_{1,\omega_h}|\overline{w}|_{1,\omega_h}.$$

By (73) and (61),

$$|u|_{1,\omega_h} \leqslant C\frac{h}{\sqrt{\varrho}}\|u\|_{2,\Omega}.$$

This result and (74) imply (78).

Assumption (79) gives

$$|u|_{1,\omega_h} \leqslant Ch|u|_{1,\infty,\Omega}.$$

From here and (74) we obtain (80). □

28. Lemma. *Let $u \in H^2(\Omega)$ and $\widetilde{f} \in W^{1,\infty}(\widetilde{\Omega})$. Then*

$$(81)\qquad \left|\iint_{\tau_h}\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\widetilde{k}_i\frac{\partial\widetilde{u}}{\partial x_i}\right)+\widetilde{f}\right)w\,\mathrm{d}x_1\mathrm{d}x_2\right| \leqslant Ch^2(\|\widetilde{A}\widetilde{u}\|_{0,\widetilde{\Omega}}+\|\widetilde{f}\|_{0,\widetilde{\Omega}})\|w\|_{1,\Omega_h}$$

where

$$\widetilde{A}\widetilde{u} := -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\widetilde{k}_i\frac{\partial\widetilde{u}}{\partial x_i}\right).$$

P r o o f. Owing to the assumption $w \in V_h$, estimate (81) follows from (76). □

29. Lemma. *We have*

$$(82)\qquad |L^{\Gamma}(\overline{w})-\widetilde{L}_h^{\Gamma}(w)| \leqslant Ch^2\sqrt{\frac{m}{\varrho}}\|q\|_{0,\Gamma_2}\|w\|_{1,\Omega_h}.$$

P r o o f. We shall modify the proof of [2, Lemma 3.3.13]. We can write

$$|L^{\Gamma(1)}(\overline{w})-\widetilde{L}_h^{\Gamma(1)}(w)| \leqslant \sum_{r=1}^{N}|I_r|$$

where, according to (20) and (21),

$$|I_r| \leqslant \int_0^{l_r} |q_{n,r}(\xi_r, \varphi_r(\xi_r))| \cdot |w_{n,r}(\xi_r, 0) - \overline{w}_{n,r}(\xi_r, \varphi_r(\xi_r))\sqrt{1 + (\varphi_r'(\xi_r))^2}| \, \mathrm{d}\xi_r.$$

By (19) we have

$$\max_{[0,l_r]} |\varphi_r''(\xi_r)| = \frac{4R_2^2 \cos^2 \frac{\sigma_r}{2} + l_r^2}{4R_2^3 \cos^3 \frac{\sigma_r}{2}} \leqslant \frac{2}{R_2}.$$

As $\varphi_r(0) = \varphi_r(l_r) = 0$ the theorem on the error of the Lagrange interpolation gives on $[0, l_r]$

$$|\varphi_r(\xi_r)| \leqslant \frac{1}{2} \max_{[0,l_r]} |\varphi_r''(\xi_r)| \xi_r(l_r - \xi_r) \leqslant \frac{l_r^2}{4R_2} \leqslant \frac{1}{4R_2} h_{T_r}^2.$$

According to the Rolle theorem, there exists a point $\xi_r^* \in (0, l_r)$ such that $\varphi_r'(\xi_r^*) = 0$. Thus on $[0, l_r]$ we have

$$|\varphi_r'(\xi_r)| = \left| \int_{\xi_r^*}^{\xi_r} \varphi_r''(t) \, \mathrm{d}t \right| \leqslant \frac{2}{R_2} l_r \leqslant \frac{2}{R_2} h_{T_r}.$$

Using the last two estimates we easily derive the relations

$$0 \leqslant \sqrt{1 + [\varphi_r'(\xi_r)]^2} - 1 \leqslant [\varphi_r'(\xi_r)]^2/2 \leqslant C h_{T_r}^2,$$
$$|w_{n,r}(\xi_r, 0) - \overline{w}_{n,r}(\xi_r, \varphi_r(\xi_r))| \leqslant |\varphi_r(\xi_r)| \cdot |(\nabla w_{n,r}|_{T_r})| \leqslant C h_{T_r}^2 |(\nabla w_{n,r}|_{T_r})|.$$

As

$$|w_{n,r}(\xi_r, 0) - \overline{w}_{n,r}(\xi_r, \varphi_r(\xi_r))\sqrt{1 + (\varphi_r'(\xi_r))^2}|$$
$$\leqslant |w_{n,r}(\xi_r, 0) - \overline{w}_{n,r}(\xi_r, \varphi_r(\xi_r))| \sqrt{1 + (\varphi_r'(\xi_r))^2} + |w_{n,r}(\xi_r, 0)| (\sqrt{1 + (\varphi_r'(\xi_r))^2} - 1)$$

we obtain

$$\begin{aligned} |I_r| &\leqslant C h_{T_r}^2 \int_0^{l_r} |q_{n,r}(\xi_r, \varphi_r(\xi_r))| \{ |w_{n,r}(\xi_r, 0)| + |(\nabla w_{n,r}|_{T_r})| \} \, \mathrm{d}\xi_r \\ &\leqslant C h_{T_r}^2 \left(\int_0^{l_r} q_{n,r}^2(\xi_r, \varphi_r(\xi_r)) \, \mathrm{d}\xi_r \right)^{1/2} \\ &\times \left\{ h_{T_r}^{1/2} |(\nabla w_{n,r}|_{T_r})| + \left(\int_0^{l_r} w_{n,r}^2(\xi_r, 0) \, \mathrm{d}\xi_r \right)^{1/2} \right\}. \end{aligned}$$

Since

$$\int_0^{l_r} q_{n,r}^2(\xi_r, \varphi_r(\xi_r)) \, \mathrm{d}\xi_r \leqslant \int_0^{l_r} q_{n,r}^2(\xi_r, \varphi_r(\xi_r)) \sqrt{1 + (\varphi_r'(\xi_r))^2} \, \mathrm{d}\xi_r = \int_{\lambda_r} q^2 \, \mathrm{d}s,$$

$$\int_0^{l_r} w_{n,r}^2(\xi_r,0)\,\mathrm{d}\xi_r = \int_{\lambda_{rh}} w^2\,\mathrm{d}s, \quad h_{T_r}^{1/2}\sqrt{\frac{\varrho}{m}}|(\nabla w_{n,r}|_{T_r})| \leqslant C|w|_{1,T_r},$$

where $\lambda_{rh} \subset \Gamma_{2h}$, $\lambda_r \subset \Gamma_2$, we find out that

$$\sum_{r=1}^N |I_r| \leqslant Ch^2 \sum_{r=1}^N \|q\|_{0,\lambda_r} \left(\|w\|_{0,\lambda_{rh}} + \sqrt{\frac{m}{\varrho}}|w|_{1,T_r}\right).$$

This result together with (62) gives (82). □

Estimate (82) cannot be improved. Thus, if we want to obtain the rate of convergence $O(h)$ we must assume that

$$C_1 h^2 \leqslant \frac{\varrho}{m} \quad (C_1 > 0). \tag{83}$$

6. The first main result

All preceding results yield the following theorem:

30. Theorem. *Let $u \in H^2(\Omega)$, $\tilde{f} \in W^{1,\infty}(\tilde{\Omega})$, $\tilde{k}_i \in W^{1,\infty}(\tilde{\Omega})$ $(i = 1,2)$. Let assumptions $(8)_{3,4}$, (9), (83) and assumptions concerning the degrees of precision of the quadrature formulas (see Theorems 13, 17 and 22) be satisfied. Then*

$$\|\tilde{u} - u_h\|_{1,\Omega_h} \leqslant \frac{C}{\sqrt{\varrho}} h \tag{84}$$

where the constant C does not depend on u, ϱ, m, h and the division $\mathscr{D}_h$.

If in addition condition (79) is satisfied then

$$\|\tilde{u} - u_h\|_{1,\Omega_h} \leqslant Ch \tag{85}$$

where again the constant C does not depend on u, ϱ, m, h and the division $\mathscr{D}_h$.

The definition of the division $\mathscr{D}_h$ is rather artificial. We usually prefer to use either a division $\mathscr{D}_h^T$, which consists only of triangles, or a division $\mathscr{D}_h^K$, which consists only of quadrilaterals. When using $\mathscr{D}_h^K$ (or $\mathscr{D}_h^T$) the definition of the space X_h (see (12)) changes in a natural way. The formulation of Problem 4 remains formally without changes.

31. Theorem. *If we use divisions $\mathscr{D}_h^T$ (or divisions $\mathscr{D}_h^K$) for the definition of the spaces X_h then the assertions of Theorem 30 remain without changes.*

Proof. In the case of $\mathscr{D}_h^T$ Theorem 31 is evident. In the case of $\mathscr{D}_h^K$ let us consider the associated division $\mathscr{D}_h$ as an auxiliary division. Let V_h and V_h^A be the spaces defined on $\mathscr{D}_h^K$ and $\mathscr{D}_h$ by means of (13), respectively. Every function $w \in V_h$ uniquely determines a function $w_A \in V_h^A$. Both functions $w_A \in V_h^A$ and $w \in V_h$ have the same values at the nodal points of $\mathscr{D}_h^K$ (or, which is the same, at the nodal points of $\mathscr{D}_h$).

Using this notation we redefine the natural extension $\overline{w}$ of w by the relation

$$\overline{w} = w \quad \text{on } \Omega_h, \qquad \overline{w} = \overline{w}_A \quad \text{on } \omega_h.$$

Estimate (74) is replaced by

$$|\overline{w}|_{1,\omega_h} \leqslant Ch\sqrt{\frac{m}{\varrho}}|w_A|_{1,\Omega_h}$$

and estimate (75) by

$$\|\overline{w}\|_{0,\omega_h} \leqslant Ch\left(\frac{1}{\sqrt{\varrho}} + h^2\sqrt{\frac{m}{\varrho}}\right)\|w_A\|_{1,\Omega_h}.$$

Hence, w is replaced by w_A on the right-hand sides in Lemmas 26, 27, 29 and can be replaced by w_A on the right-hand side in Lemma 28. Thus, to prove Theorem 31 means to prove that

$$\|w_A\|_{1,\Omega_h} \leqslant C\|w\|_{1,\Omega_h}. \tag{86}$$

Let K_r $(r = 1, \ldots, n)$ be the quadrilaterals lying along Γ_{2h} and let T_{ri} $(i = 1, 2)$ be the triangles forming K_r. Let $p_{ri}\colon R^2 \to R^1$ be the linear polynomial satisfying

$$p_{ri}\big|_{T_{ri}} = w_A\big|_{T_{ri}}.$$

Let

$$x_1 = x_{1r}(\xi_1, \xi_2), \quad x_2 = x_{2r}(\xi_1, \xi_2)$$

be the transformation of type (33) which maps $\overline{K}_0$ one-to-one onto $\overline{K}_r$ and let

$$x_1 = x_{1ri}(\xi_1, \xi_2), \quad x_2 = x_{2ri}(\xi_1, \xi_2) \tag{87}$$

be a linear transformation which maps $\overline{T}_0$ one-to-one onto $\overline{T}_{ri}$. Then

$$p_{ri}^*(\xi_1, \xi_2) = p_{ri}(x_{1ri}(\xi_1, \xi_2), x_{2ri}(\xi_1, \xi_2))$$

is a linear polynomial in ξ_1, ξ_2,

$$(88) \qquad p_{ri}^*(\xi_1,\xi_2) = A_1(1-\xi_1-\xi_2) + A_2\xi_1 + A_3\xi_2,$$

and

$$w^*(\xi_1,\xi_2) = w(x_{1r}(\xi_1,\xi_2), x_{2r}(\xi_1,\xi_2))$$

is a bilinear polynomial in ξ_1, ξ_2,

$$(89) \qquad w^*(\xi_1,\xi_2) = B_1\xi_1(1-\xi_2) + B_2(1-\xi_1)(1-\xi_2) + B_3(1-\xi_1)\xi_2 + B_4\xi_1\xi_2,$$

where $B_i = w(P_i)$, $P_1,\dots,P_4$ being the vertices of K_r.

Using notation (31) we obtain by means of (87) (which is of the form [7, (9.1)] with $\overline{x}_2 = O(h)$, $\overline{y}_2 = O(h)$, $\overline{x}_3 = O(b)$, $\overline{y}_3 = O(b)$)

$$(90) \qquad |p_{ri}|^2_{1,T_{ri}} \leqslant \frac{C}{hb}\left(b^2\left\|\frac{\partial p_{ri}^*}{\partial\xi_1}\right\|^2_{0,T_0} + h^2\left\|\frac{\partial p_{ri}^*}{\partial\xi_2}\right\|^2_{0,T_0}\right).$$

According to (88) and (89), we have

$$\left\|\frac{\partial p_{ri}^*}{\partial\xi_1}\right\|^2_{0,T_0} = \frac{1}{2}(A_2-A_1)^2, \quad \left\|\frac{\partial p_{ri}^*}{\partial\xi_2}\right\|^2_{0,T_0} = \frac{1}{2}(A_3-A_1)^2,$$

$$\begin{aligned}\left\|\frac{\partial w^*}{\partial\xi_1}\right\|^2_{0,K_0} &= \{(B_2-B_1)^2+(B_3-B_4)^2\}/3 - (B_2-B_1)(B_3-B_4)/3\\ &= \{(B_2-B_1)^2+(B_3-B_4)^2\}/6 + [(B_2-B_1)-(B_3-B_4)]^2/6,\\ \left\|\frac{\partial w^*}{\partial\xi_2}\right\|^2_{0,K_0} &= \{(B_3-B_2)^2+(B_1-B_4)^2\}/6 + [(B_3-B_2)-(B_1-B_4)]^2/6.\end{aligned}$$

Let P_1P_3 be the common side of T_{r1} and T_{r2}. Then

$$\begin{aligned}&A_1 = B_2, \quad A_2 = B_1, \quad A_3 = B_3 \quad \text{in the case of } T_{r1},\\ &A_1 = B_4, \quad A_2 = B_3, \quad A_3 = B_1 \quad \text{in the case of } T_{r2}.\end{aligned}$$

Hence

$$\left\|\frac{\partial p_{ri}^*}{\partial\xi_1}\right\|^2_{0,T_0} \leqslant 3\left\|\frac{\partial w^*}{\partial\xi_1}\right\|^2_{0,K_0}, \quad \left\|\frac{\partial p_{ri}^*}{\partial\xi_2}\right\|^2_{0,T_0} \leqslant 3\left\|\frac{\partial w^*}{\partial\xi_2}\right\|^2_{0,K_0}.$$

Combining these estimates with (90) and (42) we arrive at

$$(91) \qquad |p_{ri}|_{1,T_{ri}} \leqslant C|w|_{1,K_r}.$$

Now we prove that

$$\|p_{ri}\|_{0,T_{ri}} \leqslant C\|w\|_{0,K_r}. \tag{92}$$

We have

$$\|p_{ri}\|_{0,T_{ri}} \leqslant C\sqrt{bh}\|p^*_{ri}\|_{0,T_0}. \tag{93}$$

Using (88) and (89) we obtain

$$\|p^*_{ri}\|^2_{0,T_0} = \frac{1}{12}(A_1^2 + A_2^2 + A_3^2 + A_1A_2 + A_1A_3 + A_2A_3),$$

$$\|w^*\|^2_{0,K_0} = \frac{2}{3}\|p^*_{ri}\|^2_{0,T_0} + \frac{1}{18}g(A_4) \tag{94}$$

where $A_4 = B_4$ in the case of T_{r1} and $A_4 = B_2$ in the case of T_{r2} and

$$g(t) = A_1^2 + A_2^2 + A_3^2 + A_1A_2 + A_2A_3 + (2A_1 + A_2 + 2A_3)t + 2t^2$$

in both cases. We have

$$\min g(t) + 6\|p^*_{ri}\|^2_{0,T_0} \geqslant \frac{1}{4}A_1^2 + \frac{3}{8}A_2^2 + \frac{1}{4}A_3^2.$$

This fact and (94) yield

$$\|p^*_{ri}\|^2_{0,T_0} \leqslant 3\|w^*\|^2_{0,K_0}. \tag{95}$$

Finally,

$$\|w^*\|_{0,K_0} \leqslant \frac{C}{\sqrt{bh}}\|w\|_{0,K_r}. \tag{96}$$

Estimate (92) now follows from (93), (95) and (96).

We have

$$w_A = w \quad \text{on } \Omega_h - \bigcup_{r=1}^{n} K_r.$$

Thus, using (91) and (92) we easily obtain (86). □

At the end we shall analyze the boundary value problem of equation (1) with boundary conditions opposite to conditions (2) and (3):

$$u = 0 \quad \text{on } \Gamma_2, \tag{97}$$

$$\sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} n_i(\Omega) = q \quad \text{on } \Gamma_1. \tag{98}$$

In this case we start again with divisions $\mathscr{D}_h$. Problem 4 and all results up to relation (71) remain without changes, except for Lemma 2, where (3) is replaced by (98), and except for the definition of $\mathscr{D}_h$: we divide into two triangles each quadrilateral lying at Γ_{h1}.

The natural extension $\overline{w}\colon \overline{\Omega}_h \cup \overline{\Omega} \to R^1$ of w is now defined by

$$\overline{w} = w \text{ on } \overline{\Omega}_h, \quad \overline{w} = 0 \text{ on } \omega_h.$$

We shall use again assumption (83). However, in this case we must specify the meaning of the constant C_1.

32. Proposition. *If we set* $C_1 = 1\,[\mathrm{m}^{-1}]$ *in the case* $R_2 = 1\,[\mathrm{m}]$ *then in the general case we have* $C_1 = 1/R_2$. *This means that* (83) *takes the form*

$$\frac{1}{R_2} h^2 \leqslant \frac{\varrho}{m}. \tag{99}$$

P r o o f. Let us set

$$\varrho_0 = \frac{\varrho}{R_2}.$$

Let m be arbitrary but fixed and let $\varepsilon \geqslant 0$ be the smallest number satisfying

$$h_m^2 = \frac{\varrho_0}{m + \varepsilon} \tag{100}$$

where h_m is the corresponding value of h in the case $R_2 = 1\,[\mathrm{m}]$. Multiplying (100) by R_2 we obtain

$$\frac{1}{R_2} h^2 \equiv \frac{1}{R_2}(R_2 h_m)^2 = \frac{\varrho}{m + \varepsilon} \leqslant \frac{\varrho}{m},$$

which proves (99). $\square$

The following lemma is important for our considerations.

33. Lemma. *The circle Γ_1 lies in the polygonal layer S_h with vertices P_i, A_i^1 $(i = 1, \ldots, n)$.*

P r o o f. Let $\overline{K} \subset S_h$ be arbitrary and let $P_j P_{j+1} \subset \overline{K}$. Let P_j^* be the mid-point of $P_j P_{j+1}$. Let us compute $\operatorname{dist}(P_j^*, \Gamma_1)$. We have

$$h^* := \operatorname{dist}(P_j, P_{j+1}) = \frac{R_1}{R_2} h.$$

Hence

$$\operatorname{dist}(P_j^*, \Gamma_1) = R_1 - \sqrt{R_1^2 - (h^*/2)^2} = R_1 - \sqrt{R_1^2 - \frac{1}{4}\left(\frac{R_1}{R_2} h\right)^2}$$

$$= R_1 \left(1 - \sqrt{1 - \frac{h^2}{4R_2^2}}\right) = R_1 \left(1 - \sqrt{1 - \frac{1}{4R_2} \frac{\varrho}{m + \varepsilon}}\right) \approx \frac{R_1}{8R_2} \frac{\varrho}{m + \varepsilon} < \frac{\varrho}{m},$$

which was to be proved. □

Lemma 24 is substituted by the following lemma:

34. Lemma. *For $w \in V_h$ we have*

$$\begin{aligned} |\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)| &\leqslant |L^\Gamma(\overline{w}) - \widetilde{L}_h^\Gamma(w)| \\ &+ \left| \iint_{\tau_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| + \left| \iint_{\tau_h} \widetilde{f} w \, \mathrm{d}x_1 \mathrm{d}x_2 \right|. \end{aligned} \tag{101}$$

P r o o f. Using the definitions of $\widetilde{a}_h(\widetilde{u}, w)$, $\widetilde{L}_h(w)$ and Green's theorem we obtain

$$\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = \iint_{\Omega_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 - \widetilde{L}_h^\Omega(w) - \widetilde{L}_h^\Gamma(w)$$

$$= \int_{\Gamma_{1h}} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} n_i(\Omega_h) w \, \mathrm{d}s - \iint_{\Omega_h} \left(\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \right) + \widetilde{f} \right) w \, \mathrm{d}x_1 \mathrm{d}x_2 - \widetilde{L}_h^\Gamma(w).$$

To the right-hand side let us add zero in the form

$$- \int_{\Gamma_1} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} n_i(\Omega) w \, \mathrm{d}s + L^\Gamma(w) = 0.$$

If we denote $\Delta = \overline{T} - T^{\mathrm{id}}$ and use Lemma 2, according to which equation (1) holds almost everywhere in Ω, then we can write

$$\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = \sum_{\Delta \subset \tau_h} \int_{\partial\Delta} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} n_i(\Delta) w \, \mathrm{d}s$$

$$- \iint_{\tau_h} \left(\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \right) + \widetilde{f} \right) \overline{w} \, \mathrm{d}x_1 \mathrm{d}x_2 + L^{\Gamma}(w) - \widetilde{L}_h^{\Gamma}(w).$$

Transforming the first term on the right-hand side by means of Green's theorem we obtain (101). □

Now we estimate the terms appearing on the right-hand side of (101).

35. Lemma. *Let $u \in H^2(\Omega)$ and $\widetilde{k}_i \in W^{1,\infty}(\tilde{\Omega})$ $(i = 1, 2)$. Then*

$$\left| \iint_{\tau_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant \frac{C}{\sqrt{\varrho}} h \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\tilde{\Omega}} \|u\|_{2,\Omega} \|w\|_{1,\Omega_h}. \tag{102}$$

If in addition

$$\widetilde{u} \in W^{1,\infty}(\widetilde{\Omega}) \tag{103}$$

then

$$\left| \iint_{\tau_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant Ch \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\tilde{\Omega}} |\tilde{u}|_{1,\infty,\tilde{\Omega}} \|w\|_{1,\Omega_h}. \tag{104}$$

P r o o f. We have

$$\left| \iint_{\tau_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant C \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\tilde{\Omega}} |\widetilde{u}|_{1,\tau_h} |w|_{1,\tau_h}.$$

By the relation analogous to (73), by (61) and by Lemma 6 we obtain

$$|\widetilde{u}|_{1,\tau_h} \leqslant C \frac{h}{\sqrt{\varrho}} \|u\|_{2,\Omega}.$$

This result implies (102).

Assumption (103) gives

$$|\widetilde{u}|_{1,\tau_h} \leqslant Ch |\widetilde{u}|_{1,\infty,\tilde{\Omega}}.$$

From here we obtain (104). □

36. Lemma. *Let $\tilde{f} \in W^{1,\infty}(\tilde{\Omega})$. Then*

$$\left| \iint_{\tau_h} \tilde{f} w \, \mathrm{d}x_1 \mathrm{d}x_2 \right| \leqslant Ch \|\tilde{f}\|_{0,\infty,\tilde{\Omega}} \|w\|_{1,\Omega_h}.$$

Proof. The assertion follows from $\|\tilde{f}\|_{0,\tau_h} \leqslant Ch \|\tilde{f}\|_{0,\infty,\tilde{\Omega}}$. □

37. Lemma. *Let assumption* (83) *be satisfied. Then*

$$|L^{\Gamma}(w) - \tilde{L}_h^{\Gamma}(w)| \leqslant Ch \|q\|_{0,\Gamma_1} \|w\|_{1,\Omega_h}.$$

Owing to Lemma 33 the proof is a slight modification of the proof of Lemma 29. Thus we omit it.

8. The second main result

In the case of (97) all preceding results yield the following theorem:

38. Theorem. *Let the assumptions of Theorem* 30 *be satisfied except for the additional assumption* (79) *which is substituted by* (103). *Then estimates* (84) *and* (85) *are again valid.*

The definition of the division $\mathscr{D}_h$ is again rather artificial. We usually prefer to use either a division $\mathscr{D}_h^T$, which consists only of triangles, or a division $\mathscr{D}_h^K$, which consists only of quadrilaterals. When using $\mathscr{D}_h^K$ (or $\mathscr{D}_h^T$) the definition of the space X_h (see (12)) changes in a natural way. The formulation of Problem 4 remains formally without changes.

39. Theorem. *If we use divisions $\mathscr{D}_h^T$ (or divisions $\mathscr{D}_h^K$) for the definition of the spaces X_h then the assertions of Theorem* 38 *remain without changes.*

Proof. The proof is a modification of the proof of Theorem 31. In the case of $\mathscr{D}_h^T$ Theorem 39 is evident. In the case of $\mathscr{D}_h^K$ let us consider the associated division $\mathscr{D}_h$ as an auxiliary division. Let V_h and V_h^A be the spaces defined on $\mathscr{D}_h^K$ and $\mathscr{D}_h$, respectively, by means of (13) where Γ_{1h} is substituted by Γ_{2h}. Every function $w \in V_h$ uniquely determines a function $w_A \in V_h^A$. Both functions $w_A \in V_h^A$ and $w \in V_h$ have the same values at the nodal points of $\mathscr{D}_h^K$ (or, which is the same, at the nodal points of $\mathscr{D}_h$).

It is evident that except for $|L^\Gamma(w) - \widetilde{L}_h^\Gamma(w)|$ all results remain true for the new meaning of w. For the remaining term we have

$$\widetilde{L}_h^\Gamma(w) = \widetilde{L}_h^\Gamma(w_A).$$

Thus

$$|L^\Gamma(w) - \widetilde{L}_h^\Gamma(w)| \leqslant |L^\Gamma(w) - L^\Gamma(w_A)| + |L^\Gamma(w_A) - \widetilde{L}_h^\Gamma(w_A)|.$$

According to Lemma 37 and relation (86),

$$|L^\Gamma(w_A) - \widetilde{L}_h^\Gamma(w_A)| \leqslant Ch\|q\|_{0,\Gamma_1}\|w\|_{1,\Omega_h}.$$

Further,

$$|L^\Gamma(w) - L^\Gamma(w_A)| = |L^\Gamma(w - w_A)| \leqslant C\|q\|_{0,\Gamma_1}\|w - w_A\|_{0,\Gamma_1}.$$

As $w - w_A = 0$ outside the layer S_h (for its definition see Lemma 33) we obtain in the same way as in the proof of [5, (1.1.10)] (where we set $\beta = h^2$)

$$\|w - w_A\|_{0,\Gamma_1}^2 \leqslant \frac{C}{h^2}\|w - w_A\|_{0,\tau_h}^2 + Ch^2|w - w_A|_{1,\tau_h}^2.$$

As $w - w_A = 0$ on Γ_{1h} we have, according to the proof of [7, Lemma 28.3],

$$\frac{C}{h^2}\|w - w_A\|_{0,\tau_h}^2 \leqslant Ch^2|w - w_A|_{1,\tau_h}^2.$$

Finally, by (86),

$$Ch^2|w - w_A|_{1,\tau_h}^2 \leqslant Ch^2(\|w\|_{1,\tau_h}^2 + \|w_A\|_{1,\tau_h}^2) \leqslant Ch^2\|w\|_{1,\Omega_h}^2.$$

This completes the proof of Theorem 39. □

Remark. According to [Kačur, personal communication], modifying the considerations of [5, Chapter 4] we can prove the following regularity results: Let $j \geqslant 1$. If $k_i \in C^{j-1,1}(\overline{\Omega})$, $f \in W_2^{j-1}(\Omega)$, $q \in C^{j-1,1}(\Gamma_r)$ ($r = 1$ or 2) then $u \in H^{j+1}(\Omega)$.

This means that the assumption guaranteeing (85) can be satisfied in both cases (2) and (97).

References

[1] *I. Babuška, and A.K. Aziz*: On the angle condition in the finite element method. SIAM J. Numer. Anal. *13* (1976), 214–226.

[2] *M. Feistauer, and A. Ženíšek*: Finite element solution of nonlinear elliptic problems. Numer. Math. *50* (1987), 451–475.

[3] *P. Jamet*: Estimations d'erreur pour des éléments finis presque dégénérés. RAIRO Anal. Numér. *10* (1976), 43–61.

[4] *M. Křížek*: On semiregular families of triangulations and linear interpolation. Appl. Math. *36* (1991), 223–232.

[5] *J. Nečas*: Les Méthodes Directes en Théorie des Equations Elliptiques. Academia-Masson, Prague-Paris, 1967.

[6] *L.A. Oganesian, and L.A. Rukhovec*: Variational-Difference Methods for the Solution of Elliptic Problems. Izd. Akad. Nauk ArSSR, Jerevan, 1979. (In Russian.)

[7] *A. Ženíšek*: Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations. Academic Press, London, 1990.

[8] *A. Ženíšek, and M. Vanmaele*: The interpolation theorem for narrow quadrilateral isoparametric finite elements. Numer. Math. *72* (1995), 123–141.

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