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LINEAR MODEL WITH INACCURATE VARIANCE COMPONENTS

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Summary. A linear model with approximate variance components is considered. Differences among approximate and actual values of variance components influence the proper position and the shape of confidence ellipsoids, the level of statistical tests and their power function. A procedure how to recognize whether these diferences can be neglected is given in the paper.

Keywords: mixed linear model, linear model with variance components

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INTRODUCTION

Let a linear model $(Y, X\beta, \Sigma(\vartheta))$, $\beta \in \mathbb{R}^k$, $\vartheta \in \underline{\vartheta}$, be under consideration. Here Y is an n-dimensional random vector (observation vector), X a known $n \times k$ matrix (design matrix), β an unknown k-dimensional parameter (parameter of the first order), $\Sigma(\vartheta)$ a covariance matrix parametrized by a p-dimensional vector ϑ (parameter of the second order), $\mathbb{R}^k$ a k-dimensional real linear space and $\underline{\vartheta} \subset \mathbb{R}^p$ a parametric space of the second order parameters.

It is well known that the ϑ_0-locally best linear estimator $\hat{\beta}(Y, \vartheta_0)$ of β (if it exists) is

$$\hat{\beta}(Y, \vartheta_0) = [X'\Sigma^{-1}(\vartheta_0)X]^{-1}X'\Sigma^{-1}(\vartheta_0)Y;$$

further,

$$\operatorname{Var}[\hat{\beta}(Y, \vartheta_0)|\vartheta^*] - \operatorname{Var}[\hat{\beta}(Y, \vartheta^*)|\vartheta^*]$$

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is positive semidefinite, if ϑ^* is an actual value of the vector ϑ, $\vartheta^* \neq \vartheta_0$. (Here $\operatorname{Var}[\hat{\beta}(Y,\vartheta)|\vartheta^*]$ is the covariance matrix of the ϑ-locally best linear estimator at the point $\vartheta^* \in \underline{\vartheta}$.) Therefore statisticians try to use such a ϑ_0 which is as near to the actual value ϑ^* as possible, since the actual value ϑ^* is usually unknown. Therefore it is of some importance to investigate the effect of the inequality $\vartheta_0 \neq \vartheta^*$ on basic statistical inferences.

The aim of the paper is to give a criterion which enables us to decide whether or not the difference $\vartheta_0 - \vartheta^*$ can be neglected in the above mentioned statistical inference.

A starting point for further consideration are papers [1], [3], [4] and [5].

1. Definitions and auxiliary statements

Let $Y \sim N_n(X\beta, \Sigma(\vartheta))$, i.e., Y is normally distributed with the mean value $E(Y|\beta) = X\beta$, $\beta \in \mathbb{R}^k$, and with the covariance matrix $\Sigma(\vartheta)$, $\vartheta \in \underline{\vartheta}$.

Definition 1.1. The model $Y \sim N_n(X\beta, \Sigma(\vartheta))$, $\beta \in \mathbb{R}^k$, $\vartheta \in \underline{\vartheta} \subset \mathbb{R}^k$, is regular, if the rank $r(X)$ of the $n \times k$ matrix X is $k < n$, $\vartheta \in \underline{\vartheta} \Rightarrow \Sigma(\vartheta)$ is positive definite and $\underline{\vartheta}$ contains an open sphere.

Assumption 1.2. The covariance matrix $\Sigma(\vartheta)$ is of the form $\sum\limits_{i=1}^{p} \vartheta_i V_i$, where $V_1, \ldots, V_p$ are known symmetric matrices.

In what follows the regular model from Definition 1.1 together with Assumption 1.2 is under consideration.

Let G be an $s \times k$ matrix with the rank $r(G) = s \leqslant k$.

Lemma 1.3. *Let $\chi_s^2(1-\alpha)$ be the $(1-\alpha)$ quantile of the chi-square distribution with s degrees of freedom. Let β^* be the actual value of the parameter β and let ϑ^* be the actual value of the parameter ϑ. Then*

$$P\{[\beta^* - \hat{\beta}(Y,\vartheta^*)]'G'(GC^{-1}G')^{-1}G[\beta^* - \hat{\beta}(Y,\vartheta^*)] \leqslant \chi_s^2(1-\alpha)\} = 1-\alpha,$$

where $C = X'\Sigma^{-1}(\vartheta^*)X$.

Proof. Cf. [2], p. 212. □

The notation

$$v = Y - X\hat{\beta}(Y,\vartheta^*),$$
$$\delta\vartheta = \vartheta - \vartheta^*,$$
$$\Delta\hat{\beta}_i(Y,\vartheta^*) = [(\partial\hat{\beta}_i(Y,\vartheta)/\partial\vartheta_1, \ldots, \partial\hat{\beta}_i(Y,\vartheta)/\partial\vartheta_p)|_{\vartheta=\vartheta^*}]'$$

(′ denotes the transposition),

$$L'_f = f'C^{-1}X'\Sigma^{-1}(\vartheta^*),\ f \in \mathbb{R}^k,$$
$$\Delta\hat{\beta}(Y,\vartheta^*) = [\Delta\hat{\beta}_1(Y,\vartheta^*),\dots,\Delta\hat{\beta}_k(Y,\vartheta^*)]',$$
$$\delta\hat{\beta} = [\Delta\hat{\beta}(Y,\vartheta^*)]\delta\vartheta$$

will be used in the sequel.

Lemma 1.4.
(i)
$$\delta\hat{\beta} = [\Delta\hat{\beta}(Y,\vartheta^*)]\delta\vartheta = -C^{-1}X'\Sigma^{-1}(\vartheta^*)\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)v.$$

(ii) *The random vectors $G\hat{\beta}(Y,\vartheta^*)$ and $G\delta\hat{\beta}$ are stochastically independent.*
(iii)
$$G\delta\hat{\beta} \sim N_s(0, G\,\mathrm{Var}\{[\Delta\hat{\beta}(Y,\vartheta^*)]\delta\vartheta|\vartheta^*\}G'),$$

where

$$\mathrm{Var}\{[\Delta\hat{\beta}(Y,\vartheta^*)]\delta\vartheta|\vartheta^*\} = C^{-1}X'\Sigma^{-1}(\vartheta^*)\Sigma(\delta\vartheta)$$
$$\times[M_X\Sigma(\vartheta^*)M_X]^+\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)XC^{-1},$$
$$\Sigma(\delta\vartheta) = \sum_{i=1}^{p}\delta\vartheta_i V_i,\ M_X = I - X(X'X)^{-1}X'$$

and

$$[M_X\Sigma(\vartheta^*)M_X]^+ = \Sigma^{-1}(\vartheta^*) - \Sigma^{-1}(\vartheta^*)X[X'\Sigma^{-1}(\vartheta^*)X]^{-1}X'\Sigma^{-1}(\vartheta^*).$$

Proof. Cf. [5] □

Remark 1.5. The confidence ellipsoid for the function $G\beta, \beta \in \mathbb{R}^k$, which can be constructed from Lemma 1.3, has its center at the point $G\hat{\beta}(Y,\vartheta^*)$. If ϑ^* is changed into $\vartheta^* + \delta\vartheta$ ($\delta\vartheta$ sufficiently small), then the center is changed into $G\hat{\beta}(Y,\vartheta^*) + G\delta\hat{\beta}$. Thus Lemma 1.4 characterizes the behaviour of the center of the confidence ellipsoid, when an approximate value $\vartheta^* + \delta\vartheta = \vartheta$ is used instead of the actual value ϑ^*.

Lemma 1.5. *Let $f(\beta) = f'\beta$, $\beta \in \mathbb{R}^k$. Then*

$$f'\delta\hat{\beta} \sim N_1(0, \delta\vartheta' W_f\delta\vartheta = L'_f\Sigma(\delta\vartheta)[M_X\Sigma(\vartheta^*)M_X]^+\Sigma(\delta\vartheta)L_f),$$

where
$$\{W_f\}_{i,j} = L'_f V_i[M_X\Sigma(\vartheta^*)M_X]^+V_jL_f,\ i,j = 1,\dots,p.$$

Proof. It is a consequence of Lemma 1.4. □

The notation $\mathcal{M}(A_{m,n})$ is used in the sequel for the subspace $\{Au\colon u \in \mathbb{R}^n\}$.

Lemma 1.7. *Let A and B be positive semidefinite $n \times n$ matrices. Then $\mathcal{M}(A, B) = \mathcal{M}(A + B)$.*

Proof. It is a consequence of Theorem 6.2.3 in [7]. $\square$

2. Confidence ellipsoid

Let the random variable

$$[\beta^* - \hat{\beta}(Y, \vartheta)]'G'\{G[X'\Sigma^{-1}(\vartheta)X]^{-1}G'\}^{-1}G[\beta^* - \hat{\beta}(Y, \vartheta)],$$

where $\vartheta = \vartheta^* + \delta\vartheta$, be denoted as $k_G(Y, \vartheta)$. Thus (Lemma 1.3) $k_G(Y, \vartheta^*) \sim \chi^2_s$.

Theorem 2.1. *Let*

$$\delta k_G = \delta\vartheta' \partial k_G(Y, \vartheta)/\partial\vartheta|_{\vartheta=\vartheta^*}.$$

Then

$$\begin{aligned}\delta k_G = & -2[\hat{\beta}(Y, \vartheta^*) - \beta^*]'X'U_G\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)v \\ & - [\hat{\beta}(Y, \vartheta^*) - \beta^*]'X'U_G\Sigma(\delta\vartheta)U_GX[\hat{\beta}(Y, \vartheta^*) - \beta^*],\end{aligned}$$

where

$$U_G = \Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*).$$

Further,

$$E(\delta k_G|\beta^*, \vartheta^*) = -\operatorname{Tr}[U_G\Sigma(\delta\vartheta)] = -\delta\vartheta'[\operatorname{Tr}(U_GV_1), \ldots, \operatorname{Tr}(U_GV_p)]'$$

and

$$\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*) = \delta\vartheta'(2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+})\delta\vartheta,$$

where

$$\{S_{U_G}\}_{i,j} = \operatorname{Tr}(U_GV_iU_GV_j),\ i, j = 1, \ldots, p,$$

$$\{C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}\}_{i,j} = \operatorname{Tr}\{U_GV_i[M_X\Sigma(\vartheta^*)M_X]^+V_j\},\ i, j = 1, \ldots, p.$$

Proof. Obviously

$$\begin{aligned}&\partial k_G(Y, \vartheta)/\partial\vartheta_i \\ &= 2\{\partial[\beta^* - \hat{\beta}(Y, \vartheta)]'/\partial\vartheta_i\}G'\{G[X'\Sigma^{-1}(\vartheta)X]^{-1}G'\}^{-1} \\ &\quad \times G[\beta^* - \hat{\beta}(Y, \vartheta)] + [\beta^* - \hat{\beta}(Y, \vartheta)]'G' \\ &\quad \times \left((\partial/\partial\vartheta_i)\{G[X'\Sigma^{-1}(\vartheta)X]^{-1}G'\}^{-1}\right) G[\beta^* - \hat{\beta}(Y, \vartheta)].\end{aligned}$$

Now the relations

$$\begin{aligned}\partial[\beta^* - \hat{\beta}(Y,\vartheta)]'/\partial\vartheta_i|_{\vartheta=\vartheta^*} &= -\,\partial Y'\Sigma^{-1}(\vartheta)X[X'\Sigma^{-1}(\vartheta)X]^{-1}/\partial\vartheta_i|_{\vartheta=\vartheta^*}\\ &= v'\Sigma^{-1}(\vartheta^*)V_i\Sigma^{-1}(\vartheta^*)XC^{-1}\end{aligned}$$

and

$$\begin{aligned}&\partial\{G[X'\Sigma^{-1}(\vartheta)X]^{-1}G'\}^{-1}/\partial\vartheta_i|_{\vartheta=\vartheta^*}\\ &= (GC^{-1}G')^{-1}GC^{-1}[\partial X'\Sigma^{-1}(\vartheta)X/\partial\vartheta_i]C^{-1}G'(GC^{-1}G')^{-1}|_{\vartheta=\vartheta^*}\\ &= -(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*)V_i\Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}\end{aligned}$$

can be used. Thus we obtain

$$\begin{gathered}\partial k_G(Y,\vartheta)/\partial\vartheta_i|_{\vartheta=\vartheta^*}\\ 2v'\Sigma^{-1}(\vartheta^*)V_i\Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}G[\beta^* - \hat{\beta}(Y,\vartheta^*)]\\ -[\beta^* - \hat{\beta}(Y,\vartheta^*)]'G'(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*)V_i\\ \times\Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}G[\beta^* - \hat{\beta}(Y,\vartheta^*)].\end{gathered}$$

Since

$$\begin{aligned}X'U_G &= X'\Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*)\\ &= G'(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*),\end{aligned}$$

the first statement is proved.

In the next step we use the notation

$$X'U_G\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*) = A, \quad X'U_G\Sigma(\delta\vartheta)U_GX = B, \quad \xi = \hat{\beta}(Y,\vartheta^*) - \beta^*.$$

Since $Y \sim N_n[X\beta^*, \Sigma(\vartheta^*)]$, we have

$$\xi \sim N_k(0, C^{-1}), \quad v \sim N_n[0, \Sigma(\vartheta^*) - XC^{-1}X']$$

and ξ and v are stochastically independent.

$$\begin{gathered}E(\delta k_G|\beta^*,\vartheta^*) = E(-2\xi'Av - \xi'B\xi|\beta^*,\vartheta^*)\\ = -2E(\xi'|\beta^*,\vartheta^*)AE(v|\beta^*,\vartheta^*) - \mathrm{Tr}[B\,\mathrm{Var}(\xi|\beta^*,\vartheta^*)]\\ -E(\xi'|\beta^*,\vartheta^*)BE(\xi|\beta^*,\vartheta^*) = -\,\mathrm{Tr}(BC^{-1});\\ BC^{-1} = X'U_G\Sigma(\delta\vartheta)U_GXC^{-1} \Rightarrow \mathrm{Tr}(BC^{-1}) = \mathrm{Tr}[U_GXC^{-1}X'U_G\Sigma(\delta\vartheta)].\end{gathered}$$

However, $U_G X C^{-1} X' U_G = U_G$ which implies the second statement.

In the last step we use the relations

$$E[(\xi' B \xi)^2 | \beta^*, \vartheta^*] = 2\,\mathrm{Tr}[B\,\mathrm{Var}(\xi|\beta^*, \vartheta^*) B\,\mathrm{Var}(\xi|\beta^*, \vartheta^*)] + \{\mathrm{Tr}[B\,\mathrm{Var}(\xi|\beta^*, \vartheta^*)]\}^2,$$

$$\begin{aligned}\mathrm{Var}(\delta k_G|\beta^*, \vartheta^*) &= E[(2\xi' A v + \xi' B\xi)^2 |\beta^*, \vartheta^*] - [E(2\xi' A v + \xi' B \xi|\beta^*, \vartheta^*)]^2; \\ E[(2\xi' A v + \xi' B \xi)^2|\beta^*, \vartheta^*] &= E[\mathrm{Tr}(4A'\xi\xi' A v v')|\beta^*, \vartheta^*] \\ &\quad + E(\xi' B \xi \xi' B \xi|\beta^*, \vartheta^*) \\ &= 4\,\mathrm{Tr}\{\Sigma^{-1}(\vartheta^*)\Sigma(\delta\vartheta) U_G X C^{-1} X' U_G \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)[\Sigma(\vartheta^*) - XC^{-1}X']\} \\ &\quad + 2\{\mathrm{Tr}[X' U_G \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*) U_G X C^{-1} X' U_G \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*) U_G X C^{-1}]\} \\ &\quad + \{\mathrm{Tr}[U_G \Sigma(\delta\vartheta)]\}^2.\end{aligned}$$

Since

$$\begin{aligned}&\mathrm{Tr}[\Sigma^{-1}(\vartheta^*)\Sigma(\delta\vartheta) U_G X C^{-1} X' U_G \Sigma(\delta\vartheta)(I - \Sigma^{-1}(\vartheta^*) X C^{-1} X')] \\ &= \mathrm{Tr}[U_G \Sigma(\delta\vartheta)[M_X \Sigma(\vartheta^*) M_X]^+ \Sigma(\delta\vartheta)] = \delta\vartheta' C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}\delta\vartheta\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Tr}[X' U_G \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*) U_G X C^{-1} X' U_G \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*) U_G X C^{-1}] \\ &= \mathrm{Tr}[U_G \Sigma(\delta\vartheta) U_G \Sigma(\delta\vartheta)] = \delta\vartheta' S_{U_G}\delta\vartheta,\end{aligned}$$

the proof can be easily completed. □

Lemma 2.2. *Let* $\alpha' = P\{\chi_s^2 + \delta k_G \geqslant \chi_s^2(1-\alpha)\}$. *Then*

$$\begin{aligned}\alpha' \leqslant\, & P\{\chi_s^2 > \chi_s^2(1-\alpha) - \nu - \varepsilon \big| |\delta k_G - \nu| < \varepsilon\} P\{|\delta k_G - \nu| < \varepsilon\} \\ &+ P\{\chi_s^2 + \delta k_G > \chi_s^2(1-\alpha) \big| |\delta k_G - \nu| \geqslant \varepsilon\}\,\mathrm{Var}(\delta k_G|\beta^*, \vartheta^*)/\varepsilon^2.\end{aligned}$$

Here $\nu = -\,\mathrm{Tr}[U_G \Sigma(\delta\vartheta)]$.

P r o o f. Obviously

$$\begin{aligned}\alpha' &= P\{\chi_s^2 + \delta k_G \geqslant \chi_s^2(1-\alpha)\} \\ &= P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) - \delta k_G \big| |\delta k_G - \nu| < \varepsilon\} P\{|\delta k_G - \nu| < \varepsilon\} \\ &\quad + P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) - \delta k_G \big| |\delta k_G - \nu| \geqslant \varepsilon\} P\{|\delta k_G - \nu| \geqslant \varepsilon\}.\end{aligned}$$

With respect to the Chebyshev inequality

$$P\{|\delta k_G - \nu| \geqslant \varepsilon\} \leqslant \operatorname{Var}(\delta k_G)/\varepsilon^2$$

and the obvious relationship

$$P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) - \delta k_G \big| |\delta k_G - \nu| < \varepsilon\} \leqslant$$

$$\leqslant P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) - \nu - \varepsilon \big| |\delta k_G - \nu| < \varepsilon\},$$

the proof can be finished. □

Remark 2.3. If $\varepsilon = t\sqrt{\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*)}$, where t is sufficiently large, then $P\{|\delta k_G - \nu| < \varepsilon\}$ is sufficiently near to 1 and $\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*)/\varepsilon^2 = 1/t^2$ is sufficiently near to 0. Thus the value α' can be majorized by the value

$$P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) + \operatorname{Tr}[U_G\Sigma(\delta\vartheta)] - t\sqrt{\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*)}\},$$

where

$$\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*) = \delta\vartheta'(2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+})\delta\vartheta.$$

The term $\nu = -\operatorname{Tr}[U_G\Sigma(\delta\vartheta)] = -\delta\vartheta' \operatorname{Tr}[(U_GV_1), \dots, \operatorname{Tr}(U_GV_p)]'$ depends on $\delta\vartheta$ linearly and the term $t\sqrt{\operatorname{Var}(\delta k_G|\beta^*, \vartheta^*)}\}$ depends linearly on the norm $\|\delta\vartheta\| = \sqrt{\delta\vartheta\delta\vartheta'}$.

Let the function $\Phi(x)$, $x \in \mathbb{R}^k$, be defined as follows:

$$\Phi(x) = -x'a + t\sqrt{x'Ax},$$

where $a = [\operatorname{Tr}(U_GV_1), \dots \operatorname{Tr}(U_GV_p)]'$ and $A = 2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$.

Definition 2.4. Let

$$\mathcal{K}_\varepsilon = \{x \colon x \in \mathbb{R}^k, \Phi(x) \leqslant \delta_\varepsilon\},$$

where δ_ε is given by the relationship

$$P\{\chi_s^2 \geqslant \chi_s^2(1-\alpha) - \delta_\varepsilon\} = \alpha + \varepsilon.$$

Lemma 2.5. *The matrices S_{U_G} and $C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$ are at least p.s.d.*

Proof. The matrix U_G is p.s.d.; thus there exists a matrix J such that $JJ' = U_G$. The matrix S_{U_G} is the Gram matrix of the p-tuple

$$\{J'V_1J, \dots, J'V_pJ\}$$

in the Hilbert space $\mathcal{S}$ of symmetric matrices with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(AB), \ \ A, B \in \mathcal{S}.$$

The matrix $[M_X\Sigma(\vartheta^*)M_X]^+$ is also p.s.d.; thus there exists a matrix K such that $[M_X\Sigma(\vartheta^*)M_X]^+ = KK'$. Let us consider a Hilbert space $\mathcal{M}$ of matrices with given dimensions; the inner product is given by the relation $\langle A, B\rangle = \mathrm{Tr}(A'B)$, $A, B \in \mathcal{M}$. The matrix $C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$ is the Gram matrix of the p-tuple $\{J'V_1K, \dots, J'V_pK\}$ in such a space.

Since any Gram matrix is at least p.s.d., the proof is complete. □

Lemma 2.6. *Let $A_1, \dots, A_p$ be any p-tuple of $n \times n$ symmetric matrices. If G is the Gram matrix of this p-tuple, i.e.*

$$\{G\}_{i,j} = \mathrm{Tr}(A_iA_j), \ \ i, j = 1, \dots, p,$$

then

$$[\mathrm{Tr}(A_1), \dots, \mathrm{Tr}(A_p)]' \in \mathcal{M}(G).$$

P r o o f. Let $\mathcal{S}_n$ be the Hilbert space of $n \times n$ symmetric matrices with the inner product $\langle A, B\rangle = \mathrm{Tr}(AB)$, $A, B \in \mathcal{S}_n$. Let $\mathcal{P}(U)$ denote the projection of the matrix $U \in \mathcal{S}_n$ onto the subspace generated by the matrices $A_1, \dots, A_p$. Then there exist numbers $c_1(U), \dots, c_p(U)$, such that $\mathcal{P}(U) = \sum_{j=1}^{p} c_j(U)A_j$. Let $\mathcal{P}(I) = \sum_{j=1}^{p} c_j(I)A_j$. Then

$$\forall\{i = 1, \dots, p\}\, \mathrm{Tr}(A_i) = \mathrm{Tr}(A_iI) = \sum_{j=1}^{p} \mathrm{Tr}(A_ic_j(I)A_j) = \{G\}_{i,.}c,$$

where $c = (c_1(I), \dots, c_p(I))'$.

Thus

$$[\mathrm{Tr}(A_1), \dots, \mathrm{Tr}(A_p)]' = Gc.$$

□

Corollary 2.7. *Let $A = S_{U_G} + C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$. Then $\mathcal{M}(S_{U_G}) \subset \mathcal{M}(A)$ (which follows by Lemma 1.7). If $a = [\mathrm{Tr}(U_GV_1), \dots, \mathrm{Tr}(U_GV_p)]'$, then, by virtue of Lemma 2.6, $a \in \mathcal{M}(S_{U_G}) \subset \mathcal{M}(A)$ and the equation $(t^2A - aa')x_0 = a\delta_\varepsilon$ (with respect to x_0) is consistent.*

P r o o f. If $a \in \mathcal{M}(A)$, then $\exists\{u \in \mathbb{R}^n\}a = Au$. Let $x_0 = ku$; now the equation

$$(t^2A - Auu'A)ku = Au\delta_\vartheta$$

implies

$$k(t^2 - u'Au)Au = Au\delta_\varepsilon.$$

Since the number $k = \delta_\varepsilon/(t^2 - u'Au)$ always exists (the number t can be chosen), the solution exists as well. □

Lemma 2.8. *Let*

$$A = 2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$$

and

$$a = [\mathrm{Tr}(U_G V_1), \dots, \mathrm{Tr}(U_G V_p)]'.$$

Then the boundary of the domain $\mathcal{K}_\varepsilon$ from Definition 2.4 is given by the set

$$\overline{\mathcal{K}}_\varepsilon = \left\{ u\colon u \in \mathbb{R}^k, (u - u_0)'(t^2 A - aa')(u - u_0) = \delta_\varepsilon^2 \frac{t^2}{t^2 - a'A^- a} \right\},$$

where $u_0 = \frac{\delta_\varepsilon}{t^2 - a'A^- a} A^- a$.

Proof. $\Phi(u) = \delta_\varepsilon \Leftrightarrow u'a + \delta_\varepsilon = t\sqrt{u'Au} \Leftrightarrow (u'a + \delta_\varepsilon)^2 = t^2 u'Au$. The last equality can be rewritten as

$$u'(t^2 A - aa')u - 2u'a\delta_\varepsilon = \delta_\varepsilon^2.$$

Let u_0 be such that $(t^2 A - aa')u_0 = a\delta_\varepsilon (\Rightarrow -2u'(t^2 A - aa')u_0 = -2u'a\delta_\varepsilon)$. The vector u_0 exists by virtue of Corollary 2.7. Thus

$$\begin{aligned} u'(t^2 A - aa')u - 2u'a\delta_\varepsilon = \delta_\varepsilon^2 \Leftrightarrow \\ [u - (t^2 A - aa')^- a\delta_\varepsilon]'(t^2 A - aa')[u - (t^2 A - aa')^- a\delta_\varepsilon] \\ = a'(t^2 A - aa')^- a\delta_\varepsilon^2 + \delta_\varepsilon^2. \end{aligned}$$

(The l.h.s. and also the r.h.s. of the last equality are invariant with respect to a g-inverse of the matrix $t^2 A - aa'$.) Now we use the equality

$$(t^2 A - aa')^- = \frac{1}{t^2(t^2 - a'A^- a)} \left[(t^2 - a'A^- a)A^- + A^- aa'A^- \right],$$

which can be easily proved. Thus we obtain

$$\begin{aligned} (t^2 A - aa')^- a\delta_\varepsilon &= \frac{\delta_\varepsilon}{t^2 - a'A^- a} A^- a, \\ a'(t^2 A - aa')^- a\delta_\varepsilon^2 + \delta_\varepsilon^2 &= \delta_\varepsilon^2 \frac{t^2}{t^2 - a'A^- a} \end{aligned}$$

and the expression for $\overline{\mathcal{K}}_\varepsilon$. □

Theorem 2.9. *Let β^* and ϑ^* be the actual values of β and ϑ, respectively. Let G be an $s \times k$ matrix with the rank $r(G) = s \leqslant k$. Then*

$$\delta\vartheta \in \mathcal{K}_\varepsilon \quad \Rightarrow$$
$$P\{\beta^* \in \{u\colon [u - G\hat{\beta}(Y, \vartheta^* + \delta\vartheta)]'\{G[X'\Sigma^{-1}(\vartheta^* + \delta\vartheta)X]^{-1}G'\}^{-1}$$
$$\times [u - G\hat{\beta}(Y, \vartheta^* + \delta\vartheta)] \leqslant \chi_s^2(1-\alpha)\}\} \geqslant 1 - \alpha - \varepsilon.$$

Proof. It is an obvious consequence of Lemma 2.2, Definition 2.4 and Lemma 2.8. □

Remark 2.10. If the set $\overline{\mathcal{K}}_\varepsilon$ is the surface of an ellipsoide, then $\mathcal{K}_\varepsilon$ is the union of $\overline{\mathcal{K}}_\varepsilon$ and its interior. If $\overline{\mathcal{K}}_\varepsilon$ is not characterized by an ellipsoide, then it is necessary to find the proper part $\mathcal{K}_\varepsilon$ of a set with the boundary $\overline{\mathcal{K}}_\varepsilon$.

3. Test of linear hypothesis

Let $Y \sim N_n(X\beta^*, \sum_{i=1}^{p} \vartheta_i^* V_i)$. Let the null-hypothesis concerning β^* be $H_0\colon H\beta^* + h = 0$, where H is a $q \times k$ matrix with the rank $r(H) = q$, and let the alternative hypothesis be $H_a\colon H\beta^* + h \neq 0$.

Lemma 3.1. (i) *If H_0 is true, then the statistic*

$$T_H(Y, \vartheta^*) = [H\hat{\beta}(Y, \vartheta^*) + h]'\{H[X'\Sigma^{-1}(\vartheta^*)X]^{-1}H'\}^{-1}[H\hat{\beta}(Y, \vartheta^*) + h]$$

possesses the central chi-square distribution with q degrees of freedom.

(ii) *If $H\beta^* + h = \xi \neq 0$, then $T(Y, \vartheta^*)$ possesses a noncentral chi-square distribution with q degrees of freedom and $\xi'(HC^{-1}H')^{-1}\xi$ is the parameter of its noncentrality.*

Proof. Both statements follow from the second fundamental theorem of the least squares theory given in [6], p. 155. □

Remark 3.2. The statistic $T(Y, \vartheta^*)$ has been used for testing the hypothesis H_0 against H_a. If $T(Y, \vartheta^*) \geqslant \chi_q^2(1-\alpha)$, then H_0 is rejected with the risk α. The power function of this test is

$$\beta(\xi) = P\{\chi_q^2(\xi'[HC^{-1}H']^{-1}\xi) \geqslant \chi_q^2(1-\alpha)\}, \quad \xi \in \mathbb{R}^q.$$

Theorem 3.3. *Let*

$$T(Y, \vartheta) = [H\hat{\beta}(Y, \vartheta) + h]'\{H[X'\Sigma^{-1}(\vartheta)X]^{-1}H'\}^{-1}[H\hat{\beta}(Y, \vartheta) + h]$$

and

$$\delta T_H = \delta\vartheta' \partial T_H(Y,\vartheta)/\partial\vartheta|_{\vartheta=\vartheta^*}.$$

Then

(i)

$$\begin{aligned}\delta T_H = & -2[H\hat{\beta}(Y,\vartheta^*)+h]' C_H F_H \Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)v \\ & -[H\hat{\beta}(Y,\vartheta^*)+h]' C_H F_H \Sigma(\delta\vartheta) F_H' C_H [H\hat{\beta}(Y,\vartheta^*)+h],\end{aligned}$$

where $F_H = HC^{-1}X'\Sigma^{-1}(\vartheta^*)$ *and* $C_H = (HC^{-1}H')^{-1}$.

(ii)

$$\begin{aligned}E(\delta T_H|\beta^*,\vartheta^*) = & -\delta\vartheta'[\mathrm{Tr}(U_H V_1),\dots,\mathrm{Tr}(U_H V_p)] \\ & -\delta\vartheta'[\xi' Z_1 \xi,\dots,\xi' Z_p \xi],\end{aligned}$$

where $U_H = F_H' C_H F_H$, $Z_i = C_H F_H V_i F_H' C_H$, $i=1,\dots,p$, *and* $\xi = H\beta^* + h$.

(iii)

$$\begin{aligned}\mathrm{Var}(\delta T_H|\beta^*,\vartheta^*) = & \, 4\,\mathrm{Tr}\left\{U_H \Sigma(\delta\vartheta)[M_X\Sigma(\vartheta^*)M_X]^+\Sigma(\delta\vartheta)\right\} \\ & +2\,\mathrm{Tr}[U_H\Sigma(\delta\vartheta)U_H\Sigma(\delta\vartheta)] \\ & +4\xi' C_H F_H \Sigma(\delta\vartheta)[U_H + [M_X\Sigma(\vartheta^*)M_X]^+]\Sigma(\delta\vartheta)F_H' C_H \xi.\end{aligned}$$

Proof. (i) It follows by the relations

$$\begin{aligned}\delta T_H &= \delta\vartheta' \partial T_H(Y,\vartheta)/\partial\vartheta|_{\vartheta=\vartheta^*}, \\ \partial\hat{\beta}(Y,\vartheta)/\partial\vartheta_i|_{\vartheta=\vartheta^*} &= -C^{-1}X'\Sigma^{-1}(\vartheta^*)V_i\Sigma^{-1}(\vartheta^*)v, \\ & \qquad i=1,\dots,p, \\ \partial\{H[X'\Sigma^{-1}(\vartheta)X]^{-1}H'\}^{-1}/\partial\vartheta_i|_{\vartheta=\vartheta^*} &= -C_H F_H V_i F_H' C_H;\end{aligned}$$

further we continue analogously to the proof of Theorem 2.1.

(ii) and (iii) can be proved in a similar way as in Theorem 2.1; since the procedure is rather tedious, it is omitted. □

In the sequel the notation

$$\begin{aligned}\varphi(x) &= -x'a_0 + t\sqrt{x'A_0x}, \\ a_0 &= [\mathrm{Tr}(U_H V_1),\dots,\mathrm{Tr}(U_H V_p)]', \\ A_0 &= 2S_{U_H} + 4C_{U_H,[M_X\Sigma(\vartheta^*)M_X]^+},\end{aligned}$$

$$
\begin{aligned}
\{A_0\}_{i,j} &= 2\,\mathrm{Tr}(U_H V_i U_H V_j) + 4\,\mathrm{Tr}\{U_H V_i [M_X \Sigma(\vartheta^*) M_X]^+ V_j\}, \\
&\quad i, j = 1, \dots, p, \\
\lambda_\xi(x) &= -x' a_\xi - t\sqrt{x' A_\xi x}, \\
a_\xi &= a_0 + (\xi' Z_1 \xi, \dots, \xi' Z_p \xi)', \\
A_\xi &= A_0 + D_\xi, \\
\{D_\xi\}_{i,j} &= \xi' C_H F_H V_i \{U_H + [M_X \Sigma(\vartheta^*) M_X]^+\} V_j F_H' C_H \xi, \\
&\quad i, j = 1, \dots, p, \\
\mathcal{R}_\varepsilon &= \{x \colon \varphi(x) \leqslant \delta_\varepsilon\}, \quad \text{where} \\
&\quad P\{\chi_q^2 \geqslant \chi_q^2(1-\alpha) - \delta_\varepsilon\} = \alpha + \varepsilon, \\
\mathcal{H}_{\varepsilon,\xi} &= \{x \colon \lambda_\xi(x) \geqslant -\delta_{\varepsilon,\xi}\}, \quad \text{where} \\
&\quad P\{\chi_q^2(\xi'[HC^{-1}H']^{-1}\xi) \geqslant \chi_q^2(1-\alpha) + \delta_{\varepsilon,\xi}\} = \beta(\xi) - \varepsilon, \\
\beta(\xi) &= P\{\chi_q^2(\xi'[HC^{-1}H']^{-1}\xi) \geqslant \chi_q^2(1-\alpha)\}, \xi \in \mathbb{R}^q,
\end{aligned}
$$

will be used.

Lemma 3.4. *The matrices S_{U_H}, $C_{U_H,[M_X\Sigma(\vartheta^*)M_X]^+}$ and D_ξ are at least positive semidefinite.*

P r o o f. With respect to Lemma 2.5 it suffices to prove that D_ξ is p.s.d. Since

$$
\begin{aligned}
&\xi' C_H F_H V_i \{U_H + [M_X \Sigma(\vartheta^*) M_X]^+\} V_j F_H' C_H \xi \\
&= \mathrm{Tr}\left(\{U_H + [M_X \Sigma(\vartheta^*) M_X]^+\} V_j F_H' C_H \xi \xi' C_H F_H V_i\right)
\end{aligned}
$$

and the matrices $U_H + [M_X\Sigma(\vartheta^*)M_X]^+$ (cf. Lemma 1.7) and $F_H' C_H \xi\xi' C_H F_H$ are p.s.d., the proof can be completed in a similar way as the proof that $C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$ is p.s.d. in Lemma 2.5. □

Lemma 3.5.

(i) *The boundary of the set $\mathcal{R}_\varepsilon$ is*

$$
\overline{\mathcal{R}}_\varepsilon = \left\{x \colon (x - x_0)'(t^2 A_0 - a_0 a_0')(x - x_0) = \frac{\delta_\varepsilon^2 t^2}{t^2 - a_0' A_0^- a_0}\right\},
$$

where $x_0 = \frac{\delta_\varepsilon}{t^2 - a_0' A_0^- a_0} A_0^- a_0$.

(ii) *The boundary of the set $\mathcal{H}_{\varepsilon,\xi}$ is*

$$
\overline{\mathcal{H}}_{\varepsilon,\xi} = \left\{y \colon (y + y_0)'(t_\xi^2 A_\xi - a_\xi a_\xi')(y + y_0) = \frac{\delta_{\varepsilon,\xi}^2 t^2}{t^2 - a_\xi' A_\xi^- a_\xi}\right\}
$$

where $y_0 = \frac{\delta_{\varepsilon,\xi}}{t^2 - a_\xi' A_\xi^- a_\xi} A_\xi^- a_\xi$.

P r o o f. It can proceed analogously to the proof of Lemma 2.8. □

Theorem 3.6. (i) *If H_0 is true, i.e. $\xi = 0$, then*

$$\delta\vartheta \in \mathcal{R}_\vartheta \quad \Rightarrow \quad P\{T_H(Y, \vartheta^* + \delta\vartheta) \geqslant \chi_q^2(1-\alpha)\} \leqslant \alpha + \varepsilon.$$

(ii) *If $\xi \neq 0$, then*

$$\delta\vartheta \in \mathcal{H}_{\varepsilon,\xi} \quad \Rightarrow \quad P\{T_H(Y, \vartheta^* + \delta\vartheta) \geqslant \chi_q^2(1-\alpha)\} \geqslant \beta(\xi) - \varepsilon.$$

Proof. It is sufficient to modify properly the procedures given in Section 2 and to use arguments analogous to those given in Remark 2.3. □

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