Nelson Nery Oliveira Castro

Existence and asymptotic behaviour of solutions of a nonlinear evolution problem

Applications of Mathematics, Vol. 42 (1997), No. 6, 411–420

Persistent URL: http://dml.cz/dmlcz/134367

42 (1997) APPLICATIONS OF MATHEMATICS No. 6, 411–420

EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A NONLINEAR EVOLUTION PROBLEM

Nelson Nery de Oliveira Castro, João Pessoa

(Received August 20, 1996)

Abstract. We prove existence and asymptotic behaviour of a weak solutions of a mixed problem for

$$(*)\qquad \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2}\,|u|^{\varrho}\,u = f_1 \\ v'' + Av - \Delta v' + |u|^{\varrho+2}\,|v|^{\varrho}\,v = f_2 \end{cases}$$

where A is the pseudo-Laplacian operator.

Keywords: Nonlinear problem, existence of solutions, Galerkin method, compactness, pseudo-Laplacian, asymptotic behaviour

MSC 2000: 35L70

1. Introduction

In 1987 Medeiros-Miranda [7] proved the existence and uniqueness of weak solutions of the system

$$(**)\qquad \begin{cases} \Box\, u + |v|^{\varrho+2}\,|u|^{\varrho}\,u = f_1, \\ \Box\, v + |u|^{\varrho+2}\,|v|^{\varrho}\,v = f_2, \quad \varrho > -1, \end{cases}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ is the d'Alambertian operator. They proved existence of solutions for $n \geqslant 1$ (n: spatial dimension) and uniqueness for $n = 1, 2, 3$. We have studied the existence of solutions to a system analogous to that in $(**)$, namely

$$(***)\qquad \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2}\,|u|^{\varrho}\,u = f_1, \\ v'' + Av - \Delta v' + |u|^{\varrho+2}\,|v|^{\varrho}\,v = f_2, \end{cases}$$

where

$$A\omega = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \omega}{\partial x_i} \right|^{(p-2)} \frac{\partial \omega}{\partial x_i} \right), \quad p > 2.$$

Many significant variations of the (**) and (***)-problems had been studied by many authors. Tsutsumi [8] studied the differential equation

$$u'' + Au + Bu' = f$$

with initial conditions $u(0) = u_0$, $u'(0) = u_1$, where A is a nonlinear operator with some strong properties and B is a bounded linear operator associated with a bounded symmetric bilinear form. Biazutti [1] studied the existence of weak solutions of the initial boundary value problem for the system

$$\begin{aligned} u'' + Au - \Delta u' + G_1(u', v') &= f_1, \\ v'' + Av - \Delta v' + G_2(u', v') &= f_2, \end{aligned}$$

where A is as before, $p \geqslant 2$ and G_1, G_2 have some properties as functions of u' and v'.

(For other authors see references at the end).

2. Notation and main results

Let Ω be a regular bounded domain of $\mathbb{R}^n$. Let $T > 0$ be a real number and $Q = \Omega \times]0, T[$. The norm and inner product in $H_0^1(\Omega)$ and $L^2(\Omega)$ are denoted by $\|\cdot\|$, $((\cdot,\cdot))$ and $|\cdot|$, $(\cdot,\cdot)$, respectively.

Let X be a Banach space and $1 \leqslant p \leqslant \pm\infty$.

Then $L^p(0,T;X)$ is the Banach space of vector X-valued measurable functions $u\colon\ ]0,T[\to X$ such that $\|u(t)\|_X \in L^p(0,T)$.

If $1 \leqslant p < +\infty$, then $L^p(0,T;X)$ is normed by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t \right)^{\frac{1}{p}}.$$

In the case $p = +\infty$, we have

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_X.$$

Now we list some results and relations that will be used in the sequel.

2.1 Let $n \in \mathbb{N}$, $p \in \mathbb{R}$, with $n > p$, $p > 2$.

If $-1 < p \leqslant \frac{4(1-n+p)}{2(n-p-1)+np}$, then $\varrho < \frac{4}{np-2}$.

2.2 If $n \in \mathbb{N}$, e $\frac{4n+2}{4+n} < p < n$, then $\frac{4}{np-2} \leqslant \frac{1}{n-p}$.

2.3 Let n, p and ϱ be as before and let

$$\theta = \frac{2np(\varrho+2)}{(np-2)(\varrho+2)+2np(\varrho+1)} \quad \text{and} \quad \gamma = \frac{2np(\varrho+2)}{(np+2)(\varrho+2)-2np(\varrho+1)}.$$

Then

i) $1 < \theta < \frac{\varrho+2}{\varrho+1}$, ii) $1 < \gamma \leqslant \frac{np}{n-p}$, iii) $\frac{1}{\theta} + \frac{1}{\gamma} = 1$.

2.4 Let $\alpha = \frac{\varrho+2}{(\varrho+1)\theta}$, $\beta = \frac{\varrho+2}{(\varrho+2)-(\varrho+1)\theta}$.
Then

i) $\alpha > 1$, $\beta > 1$, ii) $\theta\beta = \frac{2np}{np-2}$, iii) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

2.5 Let $u, v \in W_0^{1,p}(\Omega)$. Then

i) $uv \in L^{\varrho+2}(\Omega)$, ii) $|v|^{\varrho+2}\,|u|^{\varrho}\,u,\ |u|^{\varrho+2}\,|v|^{\varrho}\,v \in L^{\theta}(\Omega)$.

The proofs of 2.1–2.5 are straightforward and can be found in Castro [3].

3. An existence theorem

Theorem 1. *Let n, p and ϱ be as before and suppose that*

$$\begin{aligned} &f_1, f_2 \in L^2(0,T;L^2(\Omega)),\\ &u_0, v_0 \in W_0^{1,p}(\Omega),\\ &u_1, v_1 \in L^2(\Omega). \end{aligned} \tag{1}$$

Then there exist functions $u, v\colon\ Q \to \mathbb{R}$ such that

$$\begin{aligned} &u, v \in L^\infty(0,T;W_0^{1,p}(\Omega)),\\ &u', v' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),\\ &\frac{\mathrm{d}}{\mathrm{d}t}(u'(t),w) + \langle Au(t),w\rangle + ((u'(t),w))\\ &\quad + \left\langle |v(t)|^{\varrho+2}\,|u(t)|^{\varrho}\,u(t),w\right\rangle = (f_1(t),w), \quad \forall\, w \in W_0^{1,p}(\Omega) \end{aligned}$$

in the sense of $D'(0,T)$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(v'(t),w) + \langle Av(t),w\rangle + ((v'(t),w))\\ &\quad + \left\langle |u(t)|^{\varrho+2}\,|v(t)|^{\varrho}\,u(t),w\right\rangle = (f_2(t),w), \quad \forall\, w \in W_0^{1,p}(\Omega) \end{aligned}$$

in the sense of $D'(0,T)$,

$$\begin{aligned} u(0) &= u_0, \quad u'(0) = u_1, \\ v(0) &= v_0, \quad v'(0) = v_1. \end{aligned}$$

Proof. Let $\{w_j\}_j$ be a spectral basis of $H_0^s(\Omega)$, $s > n\bigl(\frac{1}{2} - \frac{1}{p}\bigr) + 1$, which is an orthonormal complete system in $L^2(\Omega)$. Let $V_m = [w_1, \dots, w_m]$ be a subspace of $H_0^s(\Omega)$ generated by the first m vectors $w_1, \dots, w_m$. $\square$

Approximate problem. We consider the system

$$\begin{aligned} (2) \qquad & (u_m''(t), w) + \langle Au_m(t), w\rangle + ((u_m'(t), w)) + \bigl\langle |v_m(t)|^{\varrho+2}\, |u_m(t)|^{\varrho}\, u_m(t), w\bigr\rangle \\ & \qquad = (f_1(t), w), \\ & (v_m''(t), w) + \langle Av_m(t), w\rangle + ((v_m'(t), w)) + \bigl\langle |u_m(t)|^{\varrho+2}\, |v_m(t)|^{\varrho}\, v_m(t), w\bigr\rangle \\ & \qquad = (f_2(t), w) \qquad \forall\, w \in V_m, \\ & u_m(0) = u_{0m} \to u_0, \text{ in } W_0^{1,p}(\Omega); \\ & u_m'(0) = u_{1m}(0) \to u_1, \text{ in } L^2(\Omega), \\ & v_m(0) = v_{0m} \to v_0, \text{ in } W_0^{1,p}(\Omega); \\ & v_m'(0) = v_{1m}(0) \to v_1, \text{ in } L^2(\Omega). \end{aligned}$$

The system (2) is in the form required by the Caratheodory existence theorem, so there exists a solution $\{u_m(t), v_m(t)\}$ of (2) defined in $[0, t_m[$, $t_m > 0$. In what follows we will obtain some "a priori" estimates that will enable us to extend the solutions $u_m(t)$, $v_m(t)$ to the interval $[0,T]$.

Estimate I. In the system (2) we replace w by $u_m'(t)$ in equation $(2)_1$, and by $v_m'(t)$ in equation $(2)_2$.

Then adding both the expressions we get

$$\begin{aligned} (3) \qquad & \frac{\mathrm{d}}{\mathrm{d}t}\Bigl\{\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\ & \qquad + \frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\Bigr\} \\ & + \frac{1}{2}\left(\|u_m'(t)\|^2 + \|v_m'(t)\|^2\right) \leqslant |f_1(t)|\,|u_m'(t)| + |f_2(t)|\,|v_m(t)|. \end{aligned}$$

Now, integration from 0 to $t < t_m$, the inequality $ab \leqslant \frac{1}{2}(a^2+b^2)$ and the continuous immersion of $H_0^1(\Omega)$ in $L^2(\Omega)$, implies

$$\begin{aligned}(4)\qquad & \frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\ & + \frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \frac{1}{2}\int_0^t \left(\|u_m'(s)\|^2 + \|v_m'(s)\|^2\right) \, \mathrm{d}s \\ \leqslant\ & c\int_0^t \left(|f_1(s)|^2 + |f_2(s)|^2\right) \, \mathrm{d}s + \frac{1}{2}|u_m'(0)|^2 + \frac{1}{2}|v_m'(0)|^2 \\ & + \frac{1}{p}\|u_m(0)\|_0^p + \frac{1}{p}\|v_m(0)\|_0^p + \frac{1}{\varrho+2}\|x_m(0)v_m(0)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}.\end{aligned}$$

(∗) We remember that $\|\cdot\|_0$ means the norm in $W_0^{1,p}(\Omega)$.

Taking into account hypothesis (1) on f_1, f_2, $(2)_3$–$(2)_4$ and 2.5, from the above expression we get

$$\begin{aligned}(5)\qquad & \frac{1}{2}|v_m'(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\ & + \frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \frac{1}{2}\int_0^t \left(\|u_m'(s)\|^2 + \|v_m'(s)\|^2\right) \, \mathrm{d}s \leqslant C,\end{aligned}$$

where C is a constant independent of t and m.

So, we have:

$$u_m(t), v_m(t) \text{ may be extended to the interval } [0,T],$$

$$\begin{aligned}
&(6) && (u_m)_m, (v_m)_m && \text{are bounded in} && L^\infty(0,T;W_0^{1,p}(\Omega)),\\
&(7) && (u_m')_m, (v_m')_m && \text{are bounded in} && L^\infty(0,T;L^2(\Omega)),\\
&(8) && (u_m')_m, (v_m')_m && \text{are bounded in} && L^2(0,T;H_0^1(\Omega)),\\
&(9) && (u_m v_m)_m && \text{is bounded in} && L^\infty(0,T;L^{\varrho+2}(\Omega)).
\end{aligned}$$

Furthermore,

(10) $(Au_m)_m, (Av_m)_m$ are bounded in $L^\infty(0,T;W^{-1,p'}(\Omega))$, because A is a "bounded" operator, that is, it takes bounded sets into bounded sets.

Estimate II. Now we will obtain an estimate for u_m'', v_m''.

To this end we consider the projection operator given by

$$(11)\qquad \begin{aligned} P_m\colon\ & L^2(\Omega) \longrightarrow L^2(\Omega),\\ & h \longmapsto P_m h = \sum_{j=1}^m (h, w_j) w_j \end{aligned}$$

and suppose that $L^2(\Omega)$ is identified with its dual, so that we have the following sequence of continuous imbeddings:

$$(12)\quad H_0^s(\Omega)\subset W_0^{1,p}(\Omega)\subset H_0^1(\Omega)\subset L^2(\Omega)\subset H^{-1}(\Omega)\subset W^{-1,p1}(\Omega)\subset H^{-s}(\Omega)$$
$$\text{Also, } W_0^{1,p}(\Omega)\subset L^{\gamma}(\Omega) \text{ and } L^{\theta}(\Omega)\subset W^{-1,p'}(\Omega).$$

By using the imbeddings as in (12), 2.5 and the projection operator, we get from the approximate problem, in a standard way, that

$$(13)\qquad (u''_m)\_m,(v''_m)\_m \text{ are bounded in } L^2(0,T;H^{-s}(\Omega)).$$

Passage to the limit. As a consequence of (6)–(9) and (13) there exist subsequences denoted by $(u_v)\_v$, $(v_v)\_v$ such that

$$(14)\qquad \begin{aligned}
&u_\nu \overset{*}{\rightharpoonup} u,\ v_\nu \overset{*}{\rightharpoonup} \nu \ \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\
&u'_\nu \overset{*}{\rightharpoonup} u',\ v'_\nu \overset{*}{\rightharpoonup} v' \ \text{ in } L^\infty(0,T;L^2(\Omega)),\\
&u'_\nu \rightharpoonup u',\ v'_\nu \rightharpoonup v' \ \text{ in } L^\infty(0,T;H_0^1(\Omega)),\\
&u_\nu v_\nu \overset{*}{\rightharpoonup} z \ \text{ in } \ L^\infty(0,T;L^{\varrho+2}(\Omega)),\\
&u''_\nu \rightharpoonup u'',\ v''_\nu \rightharpoonup v'' \ \text{ in } \ L^2(0,T;H^{-s}(\Omega)),\\
&Au_\nu \overset{*}{\rightharpoonup} \eta,\ Av_\nu \overset{*}{\rightharpoonup} \xi \ \text{ in } L^\infty(0,T;W^{-1,p'}(\Omega)),\\
&|v_\nu|^{\varrho+2}\,|u_\nu|^{\varrho}\,u_\nu \overset{*}{\rightharpoonup} \lambda,\ |u_\nu|^{\varrho+2}\,|v_\nu|^{\varrho}\,v_\nu \overset{*}{\rightharpoonup} \mu \text{ in } L^\infty(0,T;L^{\theta}(\Omega)).
\end{aligned}$$

Now, by (14), Aubin-Lions Compactness Theorem and Lion's Lemma 1.3 (see [5], [7]), we get

$$(15)\qquad \begin{aligned}
&u_\nu v_\nu \overset{*}{\rightharpoonup} uv \text{ in } L^\infty(0,T;L^{\varrho+2}(\Omega)),\\
&|v_\nu|^{\varrho+2}\,|u_\nu|^{\varrho}\,u_v \overset{*}{\rightharpoonup} |v|^{\varrho+2}\,|u|^{\varrho}\,u \text{ in } L^\infty(0,T;L^{\theta}(\Omega)),\\
&|u_\nu|^{\varrho+2}\,|v_\nu|^{\varrho}\,v_v \overset{*}{\rightharpoonup} |u|^{\varrho+2}\,|v|^{\varrho}\,v \text{ in } L^\infty(0,T;L^{\theta}(\Omega)).
\end{aligned}$$

From now on we consider the equation $(2)_1$ in the form

$$(16)\qquad \begin{aligned}
&(u''_\nu(t),w)+\langle Au_\nu(t),w\rangle+((u'_\nu(t),w))\\
&\qquad +\big\langle |v_\nu(t)|^{\varrho+2}\,|u_\nu(t)|^{\varrho}\,u_\nu(t),w\big\rangle=(f_1(t),w)
\end{aligned}$$

where $w\in V_m$, $\nu\geqslant m$.

Multiplying (16) by $\varphi\in D(0,T)$, integrating from 0 to t and passing to the limit as $\nu\to\infty$, we deduce from the convergence in (14) and (15) that

$$(17)\qquad \begin{aligned}
&-\int_0^T (u'(t),w)\varphi'\,\mathrm{d}t+\int_0^T \langle \eta(t),w\rangle\,\varphi\,\mathrm{d}t+\int_0^T ((u'(t),w))\varphi\,\mathrm{d}t\\
&+\int_0^T \big\langle |v(t)|^{\varrho+2}\,|u(t)|^{\varrho}\,u(t),w\big\rangle\,\varphi\,\mathrm{d}t=\int_0^T (f_1(t),w)\varphi\,\mathrm{d}t,\ \ \forall\, w\in V_m,
\end{aligned}$$

$\forall\, \varphi \in D(T)$ and, by a density argument, $\forall\, w \in W_0^{1,p}(\Omega)$, $\forall\, \varphi \in D(0,T)$.

In a similar way it results from equation $(2)_2$ that

$$(17') \quad -\int_0^T (v'(t), w)\varphi' \,\mathrm{d}t + \int_0^T \langle \xi(t), w\rangle\, \varphi \,\mathrm{d}t + \int_0^T ((v'(t), w))\varphi \,\mathrm{d}t \\ + \int_0^T \big\langle |u(t)|^{\varrho+2}\, |v(t)|^{\varrho}\, v(t), w\big\rangle\, \varphi \,\mathrm{d}t = \int_0^T (f_2(t), w)\varphi \,\mathrm{d}t, \ \ \forall\, w \in W_0^{1,p}(\Omega).$$

In order to establish the theorem we next prove that $Au(t) = \eta(t)$, $Av(t) = \xi(t)$. To this end we suppose the initial conditions $u(0) = u_0$, $u'(0) = u_1$, $v(0) = v_0$ and $v'(0) = v_1$ are already proved.

Here, as it is known, it is essential to have the strong convergence

$$(18) \qquad u'_\nu \to u', \quad v'_\nu \to v' \quad \text{in} \quad L^2(0, T l L^2(\Omega)) \equiv L^2(Q).$$

So let us multiply the equation (16) by $\varphi \in C^1([0,T])$ and integrate from 0 to T obtaining

$$(19) (u'_\nu(T), w\varphi(T)) - (u'_\nu(0), w\varphi(0)) - \int_0^T (u'_\nu(t), w\varphi(t)) \,\mathrm{d}t \\ + \int_0^T \langle Au_\nu(t), w\varphi(t)\rangle \,\mathrm{d}t + \int_0^T ((u'_\nu(t), w\varphi(t)) \,\mathrm{d}t \\ + \int_0^T \big\langle |v'_\nu(t)|^{\varrho+2}\, |u_\nu(t)|^{\varrho}\, u_\nu(t), w\varphi(t)\big\rangle \,\mathrm{d}t = \int_0^T (f_1(t), w\varphi(t)) \,\mathrm{d}t, \ \forall\, w \in V_m.$$

Since the set of finite linear combinations of products of the type $w\,\varphi$, $w \in W_0^{1,p}(\Omega)$, $\varphi \in C^1([0,T])$, is dense in $V = \{v \in L^2(0,T; W_0^{1,p}(\Omega));\ v' \in L^2(0,T; L^2(\Omega))\}$ and since $u \in V$, by passing to the limit with $\nu \to \infty$ in the equation (19) we get

$$(u'(T), u(T)) - (u'(0), u(0)) - \int_0^T (u'(t), u'(t)) \,\mathrm{d}t + \int_0^T \langle \eta(t), u(t)\rangle \,\mathrm{d}t \\ + \int_0^T ((u'(t), u(t))) \,\mathrm{d}t + \int_0^T \|v(t)u(t)\|^{\varrho+2}_{L^{\varrho+2}(\Omega)} = \int_0^T (f_1(t), u(t)) \,\mathrm{d}t.$$

On the other hand, since A is a monotone operator we have

$$0 \leqslant \int_0^T \langle Au_\nu(t) - Aw, u_\nu(t) - w\rangle \,\mathrm{d}t, \quad \forall\, w \in W_0^{1,p}(\Omega),$$

and by a straightforward but lengthy calculation, we conclude

$$
\begin{aligned}
(21)\qquad & 0 \leqslant (u'(0), u(0)) - (u'(T), u(T)) + \int_0^T |u'(t)^2| \,\mathrm{d}t \\
& + \frac{1}{2}\|u(0)\|^2 - \frac{1}{2}\|u(T)\|^2 - \int_0^T \|u(t)v(t)\|^{\varrho+2}_{L^{\varrho+2}(\Omega)} \\
& + \int_0^T (f_1(t), u(t))\,\mathrm{d}t - \int_0^T \langle \eta(t), w\rangle \,\mathrm{d}t - \int_0^T \langle Aw, u(t) - w\rangle \,\mathrm{d}t, \\
& \forall\, w \in W_0^{1,p}(\Omega).
\end{aligned}
$$

Next we substitute (20) into (21) to obtain that

$$
(22)\qquad 0 \leqslant \int_0^T \langle \eta(t) - Aw, u(t) - w\rangle \,\mathrm{d}t, \quad \forall\, w \in W_0^{1,p}(\Omega).
$$

From this inequality, as A is a hemicontinuous operator, we have $Au(t) = \eta(t)$.

In an analogous way we show that $Av(t) = \xi(t)$.

The initial conditions are proved in a standard way (see [3]) and so the proof of Theorem 1 is complete.

Asymptotic behaviour

In what follows we will consider $f_1 = f_2 = 0$ and in this case we can extend the solution $\{u, v\}$, obtained in Theorem 1, to the interval $[0, +\infty)$. So in order to study the asymptotic behaviour of the solution of the problem $(***)$ with $f_1 = f_2 = 0$, we first consider the energy of the following approximate problem:

$$
(23)\qquad
\begin{aligned}
& (u''_m(t), w) + \langle Au_m(t), w\rangle + ((u'_m(t), w)) + \big\langle |v_m(t)|^{\varrho+2}\, |u_m(t)|^{\varrho}\, u_m(t), w\big\rangle = 0, \\
& (v''_m(t), w) + \langle Av_m(t), w\rangle + ((v'_m(t), w)) + \big\langle |u_m(t)|^{\varrho+2}\, |v_m(t)|^{\varrho}\, v_m(t), w\big\rangle = 0, \\
& u_m(0) = u_{0m} \to u_0, \ \text{ in } \ W_0^{1,p}(\Omega); \quad u'_m(0) = u_{1m} \to u_1, \ \text{ in } L^2(\Omega), \\
& v_m(0) = v_{0m} \to v_0, \ \text{ in } \ W_0^{1,p}(\Omega); \quad v'_m(0) = v_{1m} \to v_1, \ \text{ in } L^2(\Omega).
\end{aligned}
$$

We remember that in this case, $u_m(t)$, $v_m(t)$ may be extended to the whole interval $[0, \infty)$.

We define the energy of the system (23) by

$$
\begin{aligned}
(24)\qquad E_m(t) = {} & \frac{1}{2}|u'_m(t)|^2 + \frac{1}{2}|v'_m(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\
& + \frac{1}{\varrho+2}\|u_m(t)v_m(t)\|^{\varrho+2}_{L^{\varrho+2}(\Omega)}
\end{aligned}
$$

and it is simple to verify that $E_m(t)$ is a decreasing function for $t \geqslant 0$, with $0 \leqslant E_m(t) \leqslant E_m(0)$, $\forall\, t \geqslant 0$.

The study of the behaviour of the energy $E_m(t)$ of the system in (23) in the interval $[t, t+1]$ leads us after a rather lengthy calculation (see [3], [6] for details) to the inequality

$$E_m^{\frac{2}{p'}}(t) \leqslant c(E_m(t) - E_m(t+1)). \tag{25}$$

From (25) and by Nakao's Lemma [6] we obtain

$$\begin{aligned} & E_m(t) \leqslant c(1+t)^{-\frac{1}{\beta}}, \quad \forall\, t \geqslant 0, \quad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1, \\ & \frac{2}{p'} = 1 + \beta, \quad \beta > 0. \end{aligned} \tag{26}$$

The inequality (26) means that the energy of the approximate system (23) has an algebraic decay.

The next step is to take liminf ($m \to \infty$) in the expression of the approximate energy $E_m(t)$ to obtain

$$\begin{aligned} E(t) = {}& \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|v'(t)|^2 + \frac{1}{p}\|u(t)\|_0^p + \frac{1}{p}\|v(t)\|_0^p \\ & + \frac{1}{\varrho+2}\|u(t)v(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \leqslant c(1+t)^{-\frac{1}{\beta}}, \quad \forall\, t \geqslant 0, \end{aligned}$$

where β is as defined before and given by

$$\beta = \frac{2}{p'} - 1 > 0.$$

Therefore the energy associated to the system $(***)$ with $f_1 = f_2 = 0$ has an algebraic decay.

References

[1] *A. Biazutti*: Sobre uma Equação não Linear de Vibrações. Existência de Soluções Fracas e Comportamento Assintótico. Tese de Doutorado – IMUFRJ.

[2] *F.E. Browder, Bui An Ton.*: Nonlinear functional equations in Banach spaces and elliptic super regularization. Math. Zeitsch. *105* (1968), 177–195.

[3] *N.N. Castro*: Sobre um Problema de Evolução não Linear. Tese de Doutorado, IMUFRJ, 1995.

[4] *E. Coddington, N. Levinson*: Theory of Ordinary Differential Equations. New York.

[5] *J.L. Lions*: Quelques Méthodes de Résolutions des Problèmes aux Limites Non Linéaires. Dunod Paris, 1969.

[6] *M. Nakao*: A difference inequality and its applications to nonlinear evolution equations. J. Math. Soc. Jap. *(10)4* (1978), 747–762.
[7] *L.A. Medeiros, M.M. Miranda*: Weak solutions for a system of nonlinear Klein-Gordon equations. Annali Di Matemática Pura ed Applicata *IV (CXLVI)*, 173–183.
[8] *M. Tsutsumi*: Some nonlinear evolution equations of second order. Proc. Jap. Acad. *47* (1971), 950–955.

Author's address: *Nelson Nery de Oliveira Castro*, Departamento de Matemática, UFPB – CCEN, Cidade Universitária, CEP 58059-900, João Pessoa, Brasil.